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Glen E. Bredon

Sheaf Theory

Second Edition

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Sheaf Theory

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Preface

This book is primarily concerned with the study of cohomology theories of general topological spaces with "general coefficient systems." Sheaves play several roles in this study. For example, they provide a suitable notion of "general coefficient systems." Moreover, they furnish us with a common method of defining various cohomology theories and of comparison between different cohomology theories.

The parts of the theory of sheaves covered here are those areas important to algebraic topology. Sheaf theory is also important in other fields of mathematics, notably algebraic geometry, but that is outside the scope of the present book. Thus a more descriptive title for this book might have been *Algebraic Topology from the Point of View of Sheaf Theory.*

Several innovations will be found in this book. Notably, the concept of the "tautness" of a subspace (an adaptation of an analogous notion of Spanier to sheaf-theoretic cohomology) is introduced and exploited throughout the book. The fact that sheaf-theoretic cohomology satisfies the homotopy property is proved for general topological spaces.[1] Also, relative cohomology is introduced into sheaf theory. Concerning relative cohomology, it should be noted that sheaf-theoretic cohomology is usually considered as a "single space" theory. This is not without reason, since cohomology relative to a closed subspace can be obtained by taking coefficients in a certain type of sheaf, while that relative to an open subspace (or, more generally, to a taut subspace) can be obtained by taking cohomology with respect to a special family of supports. However, even in these cases, it is sometimes of notational advantage to have a relative cohomology theory. For example, in our treatment of characteristic classes in Chapter IV the use of relative cohomology enables us to develop the theory in full generality and with relatively simple notation. Our definition of relative cohomology in sheaf theory is the first fully satisfactory one to be given. It is of interest to note that, unlike absolute cohomology, the relative cohomology groups are not the derived functors of the relative cohomology group in degree zero (but they usually are so in most cases of interest).

The reader should be familiar with elementary homological algebra. Specifically, he should be at home with the concepts of category and functor, with the algebraic theory of chain complexes, and with tensor products and direct limits. A thorough background in algebraic topology is also nec-

[1]This is not even restricted to Hausdorff spaces. This result was previously known only for paracompact spaces. The proof uses the notion of a "relatively Hausdorff subspace" introduced here. Although it might be thought that such generality is of no use, it (or rather its mother theorem II-11.1) is employed to advantage when dealing with the derived functor of the inverse limit functor.

essary. In Chapters IV, V and VI it is assumed that the reader is familiar with the theory of spectral sequences and specifically with the spectral sequence of a double complex. In Appendix A we give an outline of this theory for the convenience of the reader and to fix our notation.

In Chapter I we give the basic definitions in sheaf theory, develop some basic properties, and discuss the various methods of constructing new sheaves out of old ones. Chapter II, which is the backbone of the book, develops the sheaf-theoretic cohomology theory and many of its properties.

Chapter III is a short chapter in which we discuss the Alexander-Spanier, singular, de Rham, and Čech cohomology theories. The methods of sheaf theory are used to prove the isomorphisms, under suitable restrictions, of these cohomology theories to sheaf-theoretic cohomology. In particular, the de Rham theorem is discussed at some length. Most of this chapter can be read after Section 9 of Chapter II and all of it can be read after Section 12 of Chapter II.

In Chapter IV the theory of spectral sequences is applied to sheaf cohomology and the spectral sequences of Leray, Borel, Cartan, and Fary are derived. Several applications of these spectral sequences are also discussed. These results, particularly the Leray spectral sequence, are among the most important and useful areas of the theory of sheaves. For example, in the theory of transformation groups the Leray spectral sequence of the map to the orbit space is of great interest, as are the Leray spectral sequences of some related mappings; see [15].

Chapter V is an exposition of the homology theory of locally compact spaces with coefficients in a sheaf introduced by A. Borel and J. C. Moore. Several innovations are to be found in this chapter. Notably, we give a definition, in full generality, of the homomorphism induced by a map of spaces, and a theorem of the Vietoris type is proved. Several applications of the homology theory are discussed, notably the generalized Poincaré duality theorem for which this homology theory was developed. Other applications are found in the last few sections of this chapter. Notably, three sections are devoted to a fairly complete discussion of generalized manifolds. Because of the depth of our treatment of Borel-Moore homology, the first two sections of the chapter are devoted to technical development of some general concepts, such as the notion and simple properties of a cosheaf and of the operation of dualization between sheaves and cosheaves. This development is not really needed for the definition of the homology theory in the third section, but is needed in the treatment of the deeper properties of the theory in later sections of the chapter. For this reason, our development of the theory may seem a bit wordy and overcomplicated to the neophyte, in comparison to treatments with minimal depth.

In Chapter VI we investigate the theory of cosheaves (on general spaces) somewhat more deeply than in Chapter V. This is applied to Čech homology, enabling us to obtain some uniqueness results not contained in those of Chaper V.

At the end of each chapter is a list of exercises by which the student

may check his understanding of the material. The results of a few of the easier exercises are also used in the text. Solutions to many of the exercises are given in Appendix B. Those exercises having solutions in Appendix B are marked with the symbol Ⓢ.

The author owes an obvious debt to the book of Godement [40] and to the article of Grothendieck [41], as well as to numerous other works. The book was born as a private set of lecture notes for a course in the theory of sheaves that the author gave at the University of California in the spring of 1964. Portions of the manuscript for the first edition were read by A. Borel, M. Herrera, and E. Spanier, who made some useful suggestions. Special thanks are owed to Per Holm, who read the entire manuscript of that edition and whose perceptive criticism led to several improvements.

This book was originally published by McGraw-Hill in 1967. For this second edition, it has been substantially rewritten with the addition of over eighty examples and of further explanatory material, and, of course, the correction of the few errors known to the author. Some more recent discoveries have been incorporated, particularly in Sections II-16 and IV-8 regarding cohomology dimension, in Chapter IV regarding the Oliver transfer and the Conner conjecture, and in Chapter V regarding generalized manifolds. The Appendix B of solutions to selected exercises is also a new feature of this edition, one that should greatly aid the student in learning the theory of sheaves. Exercises were chosen for solution on the basis of their difficulty, or because of an interesting solution, or because of the usage of the result in the main text.

Among the items added for this edition are new sections on Čech cohomology, the Oliver transfer, intersection theory, generalized manifolds, locally homogeneous spaces, homological fibrations and p-adic transformation groups. Also, Chapter VI on cosheaves and Čech homology is new to this edition. It is based on [12].

Several of the added examples assume some items yet to be proved, such as the acyclicity of a contractible space or that sheaf cohomology and singular cohomology agree on nice spaces. Disallowing such forward references would have impoverished our options for the examples.

As well as the common use of the symbol $\square$ to signal the end, or absence, of a proof, we use the symbol $\diamond$ to indicate the end of an example, although that is usually obvious.

Throughout the book the word "map" means a *morphism* in the particular category being discussed. Thus for spaces "map" means "continuous function" and for groups "map" means "homomorphism."

Occasionally we use the equal sign to mean a "canonical" isomorphism, perhaps not, strictly speaking, an equality. The word "canonical" is often used for the concept for which the word "natural" was used before category theory gave that word a precise meaning. That is, "canonical" certainly means natural when the latter has meaning, but it means more: that which might be termed "God-given." We shall make no attempt to define that concept precisely. (Thanks to Dennis Sullivan for a theological discussion

in 1969.)

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Contents

Chapter I

Sheaves and Presheaves

In this chapter we shall develop the basic properties of sheaves and presheaves and shall give many of the fundamental definitions to be used throughout the book. In Sections 2 and 5 various algebraic operations on sheaves are introduced. If we are given a map between two topological spaces, then a sheaf on either space induces, in a natural way, a sheaf on the other space, and this is the topic of Section 3. Sheaves on a fixed space form a category whose morphisms are called homomorphisms. In Section 4, this fact is extended to the collection of sheaves on all topological spaces with morphisms now being maps f of spaces together with so-called f-cohomomorphisms of sheaves on these spaces. In Section 6 the basic notion of a family of supports is defined and a fundamental theorem is proved concerning the relationship between a certain type of presheaf and the cross-sections of the associated sheaf. This theorem is applied in Section 7 to show how, in certain circumstances, the classical singular, Alexander-Spanier, and de Rham cohomology theories can be described in terms of sheaves.

1 Definitions

Of central importance in this book is the notion of a presheaf (of abelian groups) on a topological space X. A presheaf A on X is a function that assigns, to each open set $U \subset X$, an abelian group $A(U)$ and that assigns, to each pair $U \subset V$ of open sets, a homomorphism (called the restriction)

$$r_{U,V} : A(V) \to A(U)$$

in such a way that

$$r_{U,U} = 1$$

and

$$r_{U,V}r_{V,W} = r_{U,W} \qquad \text{when} \qquad U \subset V \subset W.$$

Thus, using functorial terminology, we have the following definition:

1.1. Definition. *Let X be a topological space. A "presheaf" A (of abelian groups) on X is a contravariant functor from the category of open subsets of X and inclusions to the category of abelian groups.*

In general, one may define a presheaf with values in an arbitrary category. Thus, if each $A(U)$ is a ring and the $r_{U,V}$ are ring homomorphisms, then A is called a presheaf of rings. Similarly, let A be a presheaf of rings on X and suppose that B is a presheaf on X such that each $B(U)$ is an $A(U)$-module and the $r_{U,V} : B(V) \to B(U)$ are module homomorphisms [that is, if $\alpha \in A(V), \beta \in B(V)$ then $r_{U,V}(\alpha\beta) = r_{U,V}(\alpha)r_{U,V}(\beta)$]. Then B is said to be an A-module.

Occasionally, for reasons to be explained later, we refer to elements of $A(U)$ as "sections of A over U." If $s \in A(V)$ and $U \subset V$ then we use the notation $s|U$ for $r_{U,V}(s)$ and call it the "restriction of s to U."

Examples of presheaves are abundant in mathematics. For instance, if M is an abelian group, then there is the "constant presheaf" A with $A(U) = M$ for all U and $r_{U,V} = 1$ for all $U \subset V$. We also have the presheaf B assigning to U the group (under pointwise addition) $B(U)$ of all functions from U to M, where $r_{U,V}$ is the canonical restriction. If M is the group of real numbers, we also have the presheaf C, with $C(U)$ being the group of all *continuous* real-valued functions on U. Similarly, one has the presheaves of differentiable functions on (open subsets of) a differentiable manifold X; of differential p-forms on X; of vector fields on X; and so on. In algebraic topology one has, for example, the presheaf of singular p-cochains of open subsets $U \subset X$; the presheaf assigning to U its pth singular cohomology group; the presheaf assigning to U the pth singular chain group of X mod $X - U$; and so on.

It is often the case that a presheaf A on X will have a relatively simple structure "locally about a point $x \in X$." To make precise what is meant by this, one introduces the notion of a "germ" of A at the point $x \in X$. Consider the set $\mathfrak{M}$ of all elements $s \in A(U)$ for all open sets $U \subset X$ with $x \in U$. We say that the elements $s \in A(U)$ and $t \in A(V)$ of $\mathfrak{M}$ are equivalent if there is a neighborhood $W \subset U \cap V$ of x in X with $r_{W,U}(s) = r_{W,V}(t)$. The equivalence classes of $\mathfrak{M}$ under this equivalence relation are called the germs of A at x. The equivalence class containing $s \in A(U)$ is called the *germ* of s at $x \in U$. Thus, for example, one has the notion of the germ of a continuous real-valued function f at any point of the domain of f.

Of course, the set $\mathscr{A}_x$ of germs of A at x that we have constructed is none other than the direct limit

$$\mathscr{A}_x = \varinjlim A(U),$$

where U ranges over the open neighborhoods of x in X. The set $\mathscr{A}_x$ inherits a canonical group structure from the groups $A(U)$. The disjoint union $\mathscr{A}$ of the $\mathscr{A}_x$ for $x \in X$ provides information about the local structure of A, but most global structure has been lost, since we have discarded all relationships between the $\mathscr{A}_x$ for x varying. In order to retrieve some global structure, a topology is introduced into the set $\mathscr{A}$ of germs of A, as follows. Fix an element $s \in A(U)$. Then for each $x \in U$ we have the germ s_x of s at x. For s fixed, the set of all germs $s_x \in \mathscr{A}_x$ for $x \in U$ is taken

to be an open set in $\mathscr{A}$. The topology of $\mathscr{A}$ is taken to be the topology generated by these open sets. (We shall describe this more precisely later in this section.) With this topology, $\mathscr{A}$ is called "the sheaf generated by the presheaf A" or "the sheaf of germs of A," and we denote this by

$$\boxed{\mathscr{A} = \mathscr{Sheaf}(A) \quad \text{or} \quad \mathscr{A} = \mathscr{Sheaf}(U \mapsto A(U)).}$$

In general, the topology of $\mathscr{A}$ is highly non-Hausdorff.

There is a natural map $\pi : \mathscr{A} \to X$ taking $\mathscr{A}_x$ into the point x. It will be verified later in this section that π is a local homeomorphism. That is, each point $t \in \mathscr{A}$ has a neighborhood N such that the restriction $\pi|N$ is a homeomorphism onto a neighborhood of $\pi(t)$. (The set $\{s_x \mid x \in U\}$ for $s \in A(U)$ is such a set N.) Also it is the case that in a certain natural sense, the group operations in $\mathscr{A}_x$, for x varying, are continuous in x. These facts lead us to the basic definition of a sheaf on X:

1.2. Definition. *A "sheaf" (of abelian groups) on X is a pair $(\mathscr{A}, \pi)$ where:*

(i) *$\mathscr{A}$ is a topological space (not Hausdorff in general);*

(ii) *$\pi : \mathscr{A} \to X$ is a local homeomorphism onto X;*

(iii) *each $\mathscr{A}_x = \pi^{-1}(x)$, for $x \in X$, is an abelian group (and is called the "stalk" of $\mathscr{A}$ at x);*

(iv) *the group operations are continuous.*

(In practice, we always regard the map π as being understood and we speak of the sheaf $\mathscr{A}$.) The meaning of (iv) is as follows: Let $\mathscr{A}\triangle\mathscr{A}$ be the subspace of $\mathscr{A} \times \mathscr{A}$ consisting of those pairs $\langle\alpha,\beta\rangle$ with $\pi(\alpha) = \pi(\beta)$. Then the function $\mathscr{A}\triangle\mathscr{A} \to \mathscr{A}$ taking $\langle\alpha,\beta\rangle \mapsto \alpha - \beta$ is continuous. [Equivalently, $\alpha \mapsto -\alpha$ of $\mathscr{A} \to \mathscr{A}$ is continuous and $\langle\alpha,\beta\rangle \mapsto \alpha + \beta$ of $\mathscr{A}\triangle\mathscr{A} \to \mathscr{A}$ is continuous.]

Similarly one may define, for example, a sheaf of rings or a module (sheaf of modules) over a sheaf of rings.

Thus, for a sheaf $\mathscr{R}$ of abelian groups to be a *sheaf of rings*, each stalk is assumed to have the (given) structure of a ring, and the map $\langle\alpha,\beta\rangle \mapsto \alpha\beta$ of $\mathscr{R}\triangle\mathscr{R} \to \mathscr{R}$ is assumed to be continuous (in addition to (iv)). By a *sheaf of rings with unit* we mean a sheaf of rings in which each stalk has a unit and the assignment to each $x \in X$ of the unit $1_x \in \mathscr{R}_x$ is continuous.[1]

If $\mathscr{R}$ is a sheaf of rings and if $\mathscr{A}$ is a sheaf in which each stalk $\mathscr{A}_x$ has a given $\mathscr{R}_x$-module structure, then $\mathscr{A}$ is called an $\mathscr{R}$-module (or a module over $\mathscr{R}$) if the map $\mathscr{R}\triangle\mathscr{A} \to \mathscr{A}$ given by $\langle\rho,\alpha\rangle \mapsto \rho\alpha$ is continuous, where, of course, $\mathscr{R}\triangle\mathscr{A} = \{\langle\rho,\alpha\rangle \in \mathscr{R} \times \mathscr{A} \mid \pi(\rho) = \pi(\alpha)\}$.

[1] Example 1.13 shows that this latter condition is not superfluous.

For example, the sheaf Ω^0 of germs of smooth real-valued functions on a differentiable manifold M^n is a sheaf of rings with unit, and the sheaf Ω^p of germs of differential p-forms on M^n is an Ω^0-module; see Section 7.

If $\mathscr{A}$ is a sheaf on X with projection $\pi : \mathscr{A} \to X$ and if $Y \subset X$, then the *restriction* $\mathscr{A}|Y$ of $\mathscr{A}$ is defined to be

$$\boxed{\mathscr{A}|Y = \pi^{-1}(Y),}$$

which is a sheaf on Y.

If $\mathscr{A}$ is a sheaf on X and if $Y \subset X$, then a *section* (or *cross section*) of $\mathscr{A}$ over Y is a map $s : Y \to \mathscr{A}$ such that $\pi \circ s$ is the identity. Clearly the pointwise sum or difference (or product in a sheaf of rings, and so on) of two sections over Y is a section over Y.

Every point $x \in Y$ admits a section s over some neighborhood U of x by (ii). It follows that $s - s$ is a section over U taking the value 0 in each stalk. This shows that the *zero section* $0 : X \to \mathscr{A}$, is indeed a section. It follows that for any $Y \subset X$, the set $\mathscr{A}(Y)$ of sections over Y forms an abelian group. Similarly, $\mathscr{R}(Y)$ is a ring (with unit) if $\mathscr{R}$ is a sheaf of rings (with unit), and moreover, $\mathscr{A}(Y)$ is an $\mathscr{R}(Y)$-module if $\mathscr{A}$ is an $\mathscr{R}$-module.

Clearly, the restriction $\mathscr{A}(Y) \to \mathscr{A}(Y')$, for $Y' \subset Y$, is a homomorphism. Thus, in particular, the assignment $U \mapsto \mathscr{A}(U)$, for open sets $U \subset X$, defines a presheaf on X. This presheaf is called the *presheaf of sections* of $\mathscr{A}$.

Another common notation for the group of all sections of $\mathscr{A}$ is

$$\boxed{\Gamma(\mathscr{A}) = \mathscr{A}(X).}$$

See Section 6 for an elaboration on this notation.

We shall now list some elementary consequences of Definition 1.2. The reader may supply any needed argument.

(a) π is an open map.

(b) Any section of $\mathscr{A}$ over an open set is an open map.

(c) Any element of $\mathscr{A}$ is in the range of some section over some open set.

(d) The set of all images of sections over open sets is a base for the topology of $\mathscr{A}$.

(e) For any two sections $s \in \mathscr{A}(U)$ and $t \in \mathscr{A}(V)$, U and V open, the set W of points $x \in U \cap V$ such that $s(x) = t(x)$ is *open*.

Note that if $\mathscr{A}$ were Hausdorff then the set W of (e) would also be closed in $U \cap V$. That is generally false for sheaves. Thus (e) indicates the "strangeness" of the topology of $\mathscr{A}$. It is a consequence of part (ii) of Definition 1.2.

1.3. Example. A simple example of a non-Hausdorff sheaf is the sheaf on the real line that has zero stalk everywhere but at 0, and has stalk $\mathbb{Z}_2$ at 0. There is only one topology consistent with Definition 1.2, and the two points in the stalk at 0 cannot be separated by open sets (sections over open sets in $\mathbb{R}$). As a topological space, this is the standard example of a non-Hausdorff 1-manifold. ◇

1.4. Example. Perhaps a more illuminating and more important example of a non-Hausdorff sheaf is the sheaf $\mathscr{C}$ of germs of continuous real-valued functions on $\mathbb{R}$. The function $f(x) = x$ for $x \geq 0$ and $f(x) = 0$ for $x \leq 0$ has a germ f_0 at $0 \in \mathbb{R}$ that does not equal the germ 0_0 of the zero function, but a section through f_0 takes value 0 in the stalk at x for all $x < 0$ sufficiently near 0. Thus f_0 and 0_0 cannot be separated by open sets in $\mathscr{C}$. The sheaf of germs of differentiable functions gives a similar example, but the sheaf of germs of real analytic functions is Hausdorff. ◇

1.5. We now describe more precisely the construction of the sheaf generated by a given presheaf.

Let A be a presheaf on X. For each open set $U \subset X$ consider the space $U \times A(U)$, where U has the subspace topology and $A(U)$ has the *discrete* topology. Form the topological sum

$$E = \mathop{+}_{U \subset X} (U \times A(U)).$$

Consider the following equivalence relation R on E: If $\langle x, s\rangle \in U \times A(U)$ and $\langle y, t\rangle \in V \times A(V)$ then $\langle x, s\rangle R \langle y, t\rangle \Leftrightarrow (x = y$ and there exists an open neighborhood W of x with $W \subset U \cap V$ and $s|W = t|W)$.

Let $\mathscr{A}$ be the quotient space E/R and let $\pi : \mathscr{A} \to X$ be the projection induced by the map $p : E \to X$ taking $\langle x, s\rangle \mapsto x$. We have the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\;q\;} & \mathscr{A} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X. & \end{array}$$

Recall that the topology of $\mathscr{A} = E/R$ is defined by: $Y \subset \mathscr{A}$ is open $\Leftrightarrow q^{-1}(Y)$ is open in E. Note also that for any open subset E' of E, the saturation $R(E') = q^{-1}q(E')$ of E' is open. Thus q is an open map. Now π is continuous, since p is open and q is continuous; π is locally one-to-one, since p is locally one-to-one and q is onto. Thus π is a local homeomorphism.

Clearly $\mathscr{A}_x = \pi^{-1}(x)$ is the direct limit of $A(U)$ for U ranging over the open neighborhoods of x. Thus the stalk $\mathscr{A}_x$ has a canonical group structure. It is easy to see that the group operations in $\mathscr{A}$ are continuous since they are so in E. Therefore $\mathscr{A}$ is a sheaf.

$\mathscr{A}$ is called the *sheaf generated by the presheaf* A. As we have noted, this is denoted by $\mathscr{A} = \mathscr{Sheaf}(A)$ or $\mathscr{A} = \mathscr{Sheaf}(U \mapsto A(U))$.

1.6. Let $\mathscr{A}_0$ be a sheaf, A the presheaf of sections of $\mathscr{A}_0$, and $\mathscr{A} = \mathscr{Sheaf}(A)$. Any element of $\mathscr{A}_0$ lying over $x \in X$ has a local section about it, and this determines an element of $\mathscr{A}$ over x. This gives a canonical function $\lambda : \mathscr{A}_0 \to \mathscr{A}$. By the definition of the topology of $\mathscr{A}$, λ is open and continuous. It is also bijective on each stalk, and hence globally. Therefore λ is a homeomorphism. It also preserves group operations. Thus $\mathscr{A}_0$ and $\mathscr{A}$ are essentially the same. For this reason we shall usually not distinguish between a sheaf and its presheaf of sections and shall denote them by the same symbol.

1.7. Let A be a presheaf and $\mathscr{A}$ the sheaf that it generates. For any open set $U \subset X$ there is a natural map $\theta_U : A(U) \to \mathscr{A}(U)$ (recall the construction of $\mathscr{A}$) that is a homomorphism and commutes with restrictions (which is the meaning of "natural"). When is θ_U an isomorphism for all U? Recalling that $\mathscr{A}_x = \varinjlim_{x \in U} A(U)$, it follows that an element $s \in A(U)$ is in $\operatorname{Ker}\theta_U \iff s$ is "locally trivial" (that is, for every $x \in U$ there is a neighborhood V of x such that $s|V = 0$).

Thus θ_U is a monomorphism for all $U \subset X \iff$ the following condition holds:

(S1) *If* $U = \bigcup_\alpha U_\alpha$, *with* U_α *open in* X, *and* $s, t \in A(U)$ *are such that* $s|U_\alpha = t|U_\alpha$ *for all* α, *then* $s = t$.[2]

A presheaf satisfying condition (S1) is called a *monopresheaf.*

Similarly, let $t \in \mathscr{A}(U)$. For each $x \in U$ there is a neighborhood U_x of x and an element $s_x \in A(U_x)$ with $\theta_{U_x}(s_x)(x) = t(x)$. Since $\pi : \mathscr{A} \to X$ is a local homeomorphism, $\theta(s_x)$ and t coincide in some neighborhood V_x of x. We may assume that $V_x = U_x$. Now $\theta(s_x|U_x \cap U_y) = \theta(s_y|U_x \cap U_y)$ so that *if* (S1) *holds*, we obtain that $s_x|U_x \cap U_y = s_y|U_x \cap U_y$. If A were a presheaf of sections (of any map), then this condition would imply that the s_x are restrictions to U_x of a section $s \in A(U)$. Conversely, if there is an element $s \in A(U)$ with $s|U_x = s_x$ for all x, then $\theta(s) = t$.

We have shown that if (S1) holds, then θ_U is surjective for all U (and hence is an isomorphism) $\iff$ the following condition is satisfied:

(S2) *Let* $\{U_\alpha\}$ *be a collection of open sets in* X *and let* $U = \bigcup U_\alpha$. *If* $s_\alpha \in A(U_\alpha)$ *are given such that* $s_\alpha|U_\alpha \cap U_\beta = s_\beta|U_\alpha \cap U_\beta$ *for all* α, β, *then there exists an element* $s \in A(U)$ *with* $s|U_\alpha = s_\alpha$ *for all* α.

A presheaf satisfying (S2) is called *conjunctive.* If it only satisfies (S2) for a particular collection $\{U_\alpha\}$, then it is said to be *conjunctive for* $\{U_\alpha\}$.

Thus, sheaves are in one-to-one correspondence with presheaves satisfying (S1) and (S2), that is, with conjunctive monopresheaves. For this

[2] Clearly, we could take $t = 0$ here, i.e., replace (s, t) with $(s - t, 0)$. However, the condition is phrased so that it applies to presheaves of *sets*.

reason it is common practice not to distinguish between sheaves and conjunctive monopresheaves.[3]

Note that with the notation $U_{\alpha,\beta} = U_\alpha \cap U_\beta$, (S1) and (S2) are equivalent to the hypothesis that the sequence

$$\boxed{0 \longrightarrow A(U) \xrightarrow{f} \prod_\alpha A(U_\alpha) \xrightarrow{g} \prod_{\langle\alpha,\beta\rangle} A(U_{\alpha,\beta})}$$

is exact, where $f(s) = \prod_\alpha (s|U_\alpha)$ and

$$g\left(\prod_\alpha s_\alpha\right) = \prod_{\langle\alpha,\beta\rangle} (s_\alpha|U_{\alpha,\beta} - s_\beta|U_{\alpha,\beta}),$$

where $\langle\alpha,\beta\rangle$ denotes ordered pairs of indices.

1.8. Definition. *Let $\mathscr{A}$ be a sheaf on X and let $Y \subset X$. Then $\mathscr{A}|Y = \pi^{-1}(Y)$ is a sheaf on Y called the "restriction" of $\mathscr{A}$ to Y.*

1.9. Definition. *Let G be an abelian group. The "constant" sheaf on X with stalk G is the sheaf $X \times G$ (giving G the discrete topology). It is also denoted by G when the context indicates this as a sheaf. A sheaf $\mathscr{A}$ on X is said to be "locally constant" if every point of X has a neighborhood U such that $\mathscr{A}|U$ is constant.*

1.10. Definition. *If $\mathscr{A}$ is a sheaf on X and $s \in \mathscr{A}(X)$ is a section, then the "support" of s is defined to be the closed set $|s| = \{x \in X \mid s(x) \neq 0\}$.*

The set $|s|$ is closed since its complement is the set of points at which s coincides with the zero section, and that is open by item (e) on page 4.

1.11. Example. An important example of a sheaf is the *orientation sheaf* on an n-manifold M^n. Using singular homology, this can be defined as the sheaf $\mathscr{O}_n = \mathscr{S}\!\mathit{heaf}(U \mapsto H_n(M^n, M^n - U; \mathbb{Z}))$. It is easy to see that this is a locally constant sheaf with stalks $\mathbb{Z}$. It is constant if M^n is orientable. If M^n has a boundary then $\mathscr{O}_n$ is no longer locally constant since its stalks are zero over points of the boundary.

More generally, for any space X and index p there is the sheaf $\mathscr{H}_p(X) = \mathscr{S}\!\mathit{heaf}(U \mapsto H_p(X, X - U; \mathbb{Z}))$, which is called the "$p$-th local homology sheaf" of X. Generally, it has a rather complicated structure. The reader would benefit by studying it for some simple spaces. For example, the sheaf $\mathscr{H}_1(\perp)$ has stalk $\mathbb{Z} \oplus \mathbb{Z}$ at the triple point, stalks 0 at the three end points, and stalks $\mathbb{Z}$ elsewhere. How do these stalks fit together? ◇

[3]Indeed, in certain generalizations of the theory, Definition 1.2 is not available and the other notion is used. This will not be of concern to us in this book.

1.12. Example. Consider the presheaf P on the real line $\mathbb{R}$ that assigns to an open set $U \subset \mathbb{R}$, the group $P(U)$ of all real-valued polynomial functions on U. Then P is a monopresheaf that is conjunctive for *coverings* of $\mathbb{R}$, but it is not conjunctive for arbitrary collections of open sets. For example, the element $1 \in P((0,1))$ (the constant function with value 1) and the element $x \in P((2,3))$ do not come from any single polynomial on $(0,1) \cup (2,3)$. The sheaf $\mathscr{P} = \mathscr{Sheaf}(P)$ has for $\mathscr{P}(U)$ the functions that are "locally polynomials"; e.g., 1 and x, as before, do combine to give an element of $\mathscr{P}((0,1) \cup (2,3))$. Important examples of this type of behavior are given in Section 7 and Exercise 12. $\diamond$

1.13. Example. Consider the presheaf A on $X = [0,1]$ with $A(U) = \mathbb{Z}$ for all $U \neq \varnothing$ and with $r_{V,U} : A(U) \to A(V)$ equal to the identity if $0 \in V$ or if $0 \notin U$ but $r_{V,U} = 0$ if $0 \in U - V$. Let $\mathscr{A} = \mathscr{Sheaf}(A)$. Then $\mathscr{A}_x \approx \mathbb{Z}$ for all x. However, any section over $[0,\varepsilon)$ takes the value $0 \in \mathscr{A}_x$ for $x \neq 0$, but can be arbitrary in $\mathscr{A}_0 \approx \mathbb{Z}$ for $x = 0$. The restriction $\mathscr{A}|(0,1]$ is constant. Thus $\mathscr{A}$ is a sheaf of rings but not a sheaf of rings *with unit.* (In the notation of 2.6 and Section 5, $\mathscr{A} \approx \mathbb{Z}_{\{0\}} \oplus \mathbb{Z}_{(0,1]}$.) $\diamond$

1.14. Example. A sheaf can also be described as being generated by a "presheaf" defined only on a *basis* of open sets. For example, on the circle $\mathbb{S}^1$, consider the basis $\mathscr{B}$ consisting of open arcs U of $\mathbb{S}^1$. For $U \in \mathscr{B}$ and for $x, y \in U$ we write $x > y$ if y is taken into x through U by a counterclockwise rotation. Fix a point $x_0 \in \mathbb{S}^1$. For $U \in \mathscr{B}$ let $A(U) = \mathbb{Z}$ and for $U, V \in \mathscr{B}$ with $V \subset U$ let $r_{V,U} = 1$ if $x_0 \in V$ or if $x_0 \notin U$ (i.e., if x_0 is in both U and V or in neither). If $x_0 \in U - V$ then let $r_{V,U} = 1$ if $x_0 > y$ for all $y \in V$, and $r_{V,U} = n$ (multiplication by the integer n) if $x_0 < y$ for all $y \in V$. This generates an interesting sheaf $\mathscr{A}_n$ on $\mathbb{S}^1$. It can be described directly (and more easily) as the quotient space $(-1,1) \times \mathbb{Z}$ modulo the identification $\langle t, k\rangle \sim \langle t-1, nk\rangle$ for $0 < t < 1$, and with the projection $[\langle t,k\rangle] \mapsto [t]$ to $\mathbb{S}^1 = (-1,1)/\{t \sim (t-1)\}$. Note, in particular, the cases $n = 0, -1$. The sheaf $\mathscr{A}_n$ is Hausdorff for $n \neq 0$ but not for $n = 0$. $\diamond$

2 Homomorphisms, subsheaves, and quotient sheaves

In this section we fix the base space X. A *homomorphism of presheaves* $h : A \to B$ is a collection of homomorphisms $h_U : A(U) \to B(U)$ commuting with restrictions. That is, h is a natural transformation of functors.

A *homomorphism of sheaves* $h : \mathscr{A} \to \mathscr{B}$ is a map such that $h(\mathscr{A}_x) \subset \mathscr{B}_x$ for all $x \in X$ and the restriction $h_x : \mathscr{A}_x \to \mathscr{B}_x$ of h to stalks is a homomorphism for all x.

A homomorphism of sheaves induces a homomorphism of the presheaves of sections in the obvious way. Conversely, let $h : A \to B$ be a homomorphism of presheaves (not necessarily satisfying (S1) and (S2)). For each

$x \in X$, h induces a homomorphism $h_x : \mathscr{A}_x = \varinjlim_{x \in U} A(U) \to \varinjlim_{x \in U} B(U) = \mathscr{B}_x$ and therefore a function $h : \mathscr{A} \to \mathscr{B}$. If $s \in A(U)$ then h maps the section $\theta(s) \in \mathscr{A}(U)$ onto the *section* $\theta(h(s)) \in \mathscr{B}(U)$. Thus h is continuous (since the projections to U are local homeomorphisms and take this function to the identity map).

The group of all homomorphisms $\mathscr{A} \to \mathscr{B}$ is denoted by $\operatorname{Hom}(\mathscr{A}, \mathscr{B})$.

2.1. Definition. *A "subsheaf" $\mathscr{A}$ of a sheaf $\mathscr{B}$ is an open subspace of $\mathscr{B}$ such that $\mathscr{A}_x = \mathscr{A} \cap \mathscr{B}_x$ is a subgroup of $\mathscr{B}_x$ for all $x \in X$. (That is, $\mathscr{A}$ is a subspace of $\mathscr{B}$ that is a sheaf on X with the induced algebraic structure.)*

If $h : \mathscr{A} \to \mathscr{B}$ is a homomorphism of sheaves, then

$$\operatorname{Ker} h = \{\alpha \in \mathscr{A} \mid h(\alpha) = 0\}$$

is a subsheaf of $\mathscr{A}$ and $\operatorname{Im} h$ is a subsheaf of $\mathscr{B}$. We define exact sequences of sheaves as usual; that is, the sequence $\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$ of sheaves is exact if $\operatorname{Im} f = \operatorname{Ker} g$. Note that such a sequence of sheaves is exact $\Leftrightarrow$ each $\mathscr{A}_x \to \mathscr{B}_x \to \mathscr{C}_x$ is exact.[4] Since $\varinjlim$ is an exact functor, it follows that the functor $A \mapsto \mathscr{S}\!heaf(A)$, from presheaves to sheaves, is exact.

Let $h : A \xrightarrow{f} B \xrightarrow{g} C$ be homomorphisms of presheaves. The induced sequence $\mathscr{A} \xrightarrow{f'} \mathscr{B} \xrightarrow{g'} \mathscr{C}$ of generated sheaves will be exact if and only if $\theta \circ g \circ f = 0$ and the following condition holds: For each open $U \subset X$, $x \in U$, and $s \in B(U)$ such that $g(s) = 0$, there exists a neighborhood $V \subset U$ of x such that $s|V = f(t)$ for some $t \in A(V)$. This is an elementary fact resulting from properties of direct limits and from the fact that $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is exact $\Leftrightarrow \mathscr{A}_x \to \mathscr{B}_x \to \mathscr{C}_x$ is exact for all $x \in X$. It will be used repeatedly. Note that the condition $\theta \circ g \circ f = 0$ is equivalent to the statement that for each $s \in A(U)$ and $x \in U$, there is a neighborhood $V \subset U$ of x such that $g(f(s|V)) = 0$, i.e., that $(g \circ f)(s)$ is "locally zero."

2.2. Proposition. *If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is an exact sequence of sheaves, then the induced sequence*

$$\boxed{0 \to \mathscr{A}'(Y) \to \mathscr{A}(Y) \to \mathscr{A}''(Y)}$$

is exact for all $Y \subset X$.

Proof. Since the restriction of this sequence to Y is still exact, it suffices to prove the statement in the case $Y = X$. The fact that the sequence of sections over X has order two and the exactness at $\mathscr{A}'(X)$ are obvious

[4]Caution: an exact sequence of sheaves is not necessarily an exact sequence of *presheaves*. See Proposition 2.2 and Example 2.3.

Categorical readers might check that these definitions give notions equivalent to those based on the fact that sheaves and presheaves form abelian categories.

(look at stalks). We can assume that $\mathscr{A}'$ is a subspace of $\mathscr{A}$. Then a section $s \in \mathscr{A}(X)$ going to $0 \in \mathscr{A}''(X)$ must take values in the subspace $\mathscr{A}'$, as is seen by looking at stalks. But this just means that it comes from a section in $\mathscr{A}'(X)$. □

2.3. Example. This example shows that $\mathscr{A}(Y) \to \mathscr{A}''(Y)$ need not be onto even if $\mathscr{A} \to \mathscr{A}''$ is onto. On the unit interval $\mathbb{I}$ let $\mathscr{A}$ be the sheaf $\mathbb{I}\times\mathbb{Z}_2$ and $\mathscr{A}''$ the sheaf with stalks $\mathbb{Z}_2$ at $\{0\}$ and $\{1\}$ and zero otherwise. (There is only one possible topology in $\mathscr{A}''$.) The canonical map $\mathscr{A} \to \mathscr{A}''$ is onto (with kernel being the subsheaf $\mathscr{A}' = (0,1)\times\mathbb{Z}_2 \cup [0,1]\times\{0\} \subset \mathbb{I}\times\mathbb{Z}_2$), but $\mathscr{A}(\mathbb{I}) \approx \mathbb{Z}_2$ while $\mathscr{A}''(\mathbb{I}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so that $\mathscr{A}(\mathbb{I}) \to \mathscr{A}''(\mathbb{I})$ is not surjective. Also see Example 2.5 and Exercises 13, 14, and 15.

2.4. Definition. *Let $\mathscr{A}$ be a subsheaf of a sheaf $\mathscr{B}$. The "quotient sheaf" $\mathscr{B}/\mathscr{A}$ is defined to be*

$$\boxed{\mathscr{B}/\mathscr{A} = \mathscr{S}\!\mathit{heaf}(U \mapsto \mathscr{B}(U)/\mathscr{A}(U)).}$$

The exact sequence of presheaves

$$0 \to \mathscr{A}(U) \to \mathscr{B}(U) \to \mathscr{B}(U)/\mathscr{A}(U) \to 0 \tag{1}$$

induces a sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A} \to 0$. On the stalks at x this is the direct limit of the sequences (1) for U ranging over the open neighborhoods of x. This sequence of stalks is exact since direct limits preserve exactness. Therefore, $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A} \to 0$ is exact.[5]

Suppose that $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ is an exact sequence of sheaves. We may regard $\mathscr{A}$ as a subsheaf of $\mathscr{B}$. The exact sequence

$$0 \to \mathscr{A}(U) \to \mathscr{B}(U) \to \mathscr{C}(U)$$

provides a monomorphism $\mathscr{B}(U)/\mathscr{A}(U) \to \mathscr{C}(U)$ of presheaves and hence a homomorphism of sheaves $\mathscr{B}/\mathscr{A} \to \mathscr{C}$, and the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{A} & \to & \mathscr{B} & \to & \mathscr{B}/\mathscr{A} & \to & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \to & \mathscr{A} & \to & \mathscr{B} & \to & \mathscr{C} & \to & 0 \end{array}$$

commutes. It follows, by looking at stalks, that $\mathscr{B}/\mathscr{A} \to \mathscr{C}$ is an isomorphism.

2.5. Example. Consider the sheaf $\mathscr{C}$ of germs of continuous real-valued functions on $X = \mathbb{R}^2 - \{0\}$. Let Z be the subsheaf of germs of locally constant functions with values the integer multiples of 2π. Then Z can be regarded as a subsheaf of $\mathscr{C}$. (Note that Z is a constant sheaf.) The polar angle θ is *locally* defined (ambiguously) as a section of $\mathscr{C}$, but it

[5]Note, however, that $(\mathscr{B}/\mathscr{A})(U) \neq \mathscr{B}(U)/\mathscr{A}(U)$ in general.

is not a global section. It does define (unambiguously) a section of the quotient sheaf $\mathscr{C}/Z$. This gives another example of an exact sequence $0 \to Z \to \mathscr{C} \to \mathscr{C}/Z \to 0$ of sheaves for which the sequence of sections is not right exact. Note that $\mathscr{C}/Z$ can be interpreted as the sheaf of germs of continuous functions on X with values in the circle group $\mathbb{S}^1$. Note also that $Z(X)$ is the group of constant functions on X with values in $2\pi\mathbb{Z}$ and hence is isomorphic to $\mathbb{Z}$; $\mathscr{C}(X)$ is the group of continuous real valued functions $X \to \mathbb{R}$; and $(\mathscr{C}/Z)(X)$ is the group of continuous functions $X \to \mathbb{S}^1$. The sequence

$$0 \to Z(X) \to \mathscr{C}(X) \xrightarrow{j} (\mathscr{C}/Z)(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

is exact by covering space theory, and so $\operatorname{Coker} j \approx \mathbb{Z}$.

2.6. Let A be a locally closed subspace of X and let $\mathscr{B}$ be a sheaf on A. It is easily seen (since A is locally closed) that there is a unique topology on the point set $\mathscr{B} \cup (X \times \{0\})$ such that $\mathscr{B}$ is a subspace and the projection onto X is a local homeomorphism (we identify $A \times \{0\}$ with the zero section of $\mathscr{B}$). With this topology and the canonical algebraic structure, $\mathscr{B} \cup (X \times \{0\})$ is a sheaf on X denoted by

$$\boxed{\mathscr{B}^X = \mathscr{B} \cup (X \times \{0\}).}$$

Thus $\mathscr{B}^X$ is the unique sheaf on X inducing $\mathscr{B}$ on A and 0 on $X - A$. Clearly, $\mathscr{B} \mapsto \mathscr{B}^X$ is an exact functor. The sheaf $\mathscr{B}^X$ is called the *extension of $\mathscr{B}$ by zero.*

Now let $\mathscr{A}$ be a sheaf on X and let $A \subset X$ be locally closed. We define

$$\boxed{\mathscr{A}_A = (\mathscr{A}|A)^X.}$$

For $U \subset X$ open, $\mathscr{A}_U$ is the subsheaf $\pi^{-1}(U) \cup (X \times \{0\})$ of $\mathscr{A}$, while for $F \subset X$ closed $\mathscr{A}_F$ is the quotient sheaf $\mathscr{A}_F = \mathscr{A}/\mathscr{A}_{X-F}$. If $A = U \cap F$, then $\mathscr{A}_A = (\mathscr{A}_U)_F = (\mathscr{A}_F)_U$.[6]

In this notation, the sheaf $\mathscr{A}'$ of 2.3 is $\mathscr{A}_{(0,1)}$, and $\mathscr{A}'' \approx \mathscr{A}_{\{0,1\}}$.

2.7. Example. Let U_i be the open disk of radius $1 - 2^{-i}$ in $X = \mathbb{D}^n$, the unit disk in $\mathbb{R}^n$. Put $A_1 = U_1$ and $A_i = U_i - U_{i-1}$ for $i > 1$. Note that for $i > 1$, $A_i \approx \mathbb{S}^{n-1} \times (0,1]$ is locally closed in X. Using the notation of 2.6, put

$$\begin{aligned} \mathscr{L}_1 &= \mathbb{Z}_{U_1} \\ \mathscr{L}_2 &= \mathscr{L}_1 \cup 2\mathbb{Z}_{U_2} \\ \mathscr{L}_3 &= \mathscr{L}_2 \cup 4\mathbb{Z}_{U_3} \\ &\cdots \end{aligned}$$

[6] Note that *any* locally closed subspace is the intersection of an open subspace with a closed subspace; see [19].

Let $\mathscr{L} = \bigcup \mathscr{L}_i \subset \mathbb{Z}$. Then the stalks of $\mathscr{L}$ are 0 on $\partial\mathbb{D}^n$ and are $2^i\mathbb{Z}$ on A_i. This is an example of a fairly complicated subsheaf of the constant sheaf $\mathbb{Z}$ on $\mathbb{D}^n$ even in the case $n = 1$. It is a counterexample to [40, Remark II-2.9.3]. ◇

3 Direct and inverse images

Let $f : X \to Y$ be a map and let $\mathscr{A}$ be a sheaf on X. The presheaf $U \mapsto \mathscr{A}(f^{-1}(U))$ on Y clearly satisfies (S1) and (S2) and hence is a sheaf. This sheaf on Y is denoted by $f\mathscr{A}$ and is called the *direct image* of $\mathscr{A}$.[7] Thus we have

$$\boxed{f\mathscr{A}(U) = \mathscr{A}(f^{-1}(U)).} \tag{2}$$

By 2.2 it is clear that $\mathscr{A} \mapsto f\mathscr{A}$ is a left exact covariant functor. The direct image is not generally right exact, and in fact, the theory of sheaves is largely concerned with the right derived functors of the direct image functor.

For the map $\varepsilon : X \to \star$, where $\star$ is the one point space, the direct image $\varepsilon\mathscr{A}$ is just the group $\Gamma(\mathscr{A}) = \mathscr{A}(X)$ (regarded as a sheaf on $\star$). Consequently, the direct image functor $\mathscr{A} \mapsto f\mathscr{A}$ is a generalization of the global section functor Γ.

Now let $\mathscr{B}$ be a sheaf on Y. The *inverse image* $f^*\mathscr{B}$ of $\mathscr{B}$ is the sheaf on X defined by

$$\boxed{f^*\mathscr{B} = \{\langle x, b\rangle \in X \times \mathscr{B} \mid f(x) = \pi(b)\},}$$

where $\pi : \mathscr{B} \to Y$ is the canonical projection. The projection $f^*\mathscr{B} \to Y$ is given by $\langle x, b\rangle \mapsto x$. To check that $f^*\mathscr{B}$ is indeed a sheaf, we note that if $U \subset Y$ is an open neighborhood of $f(x)$ and $s : U \to \mathscr{B}$ is a section of $\mathscr{B}$ with $s(f(x)) = b$, then the neighborhood $(f^{-1}(U) \times s(U)) \cap f^*\mathscr{B}$ of $\langle x, b\rangle \in f^*\mathscr{B}$ is precisely $\{\langle x', sf(x')\rangle \mid x' \in f^{-1}(U)\}$ and hence maps homeomorphically onto $f^{-1}(U)$. The group structure on $(f^*\mathscr{B})_x$ is defined so that the one-to-one correspondence

$$f_x^* : \mathscr{B}_{f(x)} \xrightarrow{\approx} (f^*\mathscr{B})_x, \tag{3}$$

defined by $f_x^*(b) = \langle x, b\rangle$, is an isomorphism. It is easy to check that the group operations are continuous.

We have already remarked that if $s : U \to \mathscr{B}$ is a section, then $x \mapsto \langle x, s(f(x))\rangle = f_x^*(s(f(x)))$ is a section of $f^*\mathscr{B}$ over $f^{-1}(U)$. Thus we have the canonical homomorphism

$$f_U^* : \mathscr{B}(U) \to (f^*\mathscr{B})(f^{-1}(U)) \tag{4}$$

defined by $f_U^*(s)(x) = f_x^*(s(f(x)))$.

[7] For a generalization of the direct image see IV-3.

From (3) it follows that f^* is an exact functor.

Note that for an inclusion $i : X \hookrightarrow Y$ and a sheaf $\mathscr{B}$ on Y, we have $\mathscr{B}|X \approx i^*\mathscr{B}$, as the reader is asked to detail in Exercise 1.

3.1. Example. Consider the constant sheaf $\mathbb{Z}$ on $X = [0,1]$ and its restriction $\mathscr{L} = \mathbb{Z}|(0,1)$. Let $i : (0,1) \hookrightarrow X$ be the inclusion. Then $i\mathscr{L} \approx \mathbb{Z}$, because for U a small open interval about 0 or 1, we have that $i\mathscr{L}(U) = \mathscr{L}(U \cap (0,1)) \approx \mathbb{Z}$. Also, $\mathscr{L}^X = \mathbb{Z}_{(0,1)}$. Therefore, $i\mathscr{L} \not\approx \mathscr{L}^X$ in general.

However, for an inclusion $i : F \hookrightarrow X$ of a *closed* subspace and for any sheaf $\mathscr{L}$ on F, it is true that $i\mathscr{L} \approx \mathscr{L}^X$, as the reader can verify. (This is essentially Exercise 2.) ◇

3.2. Example. Consider the constant sheaf $\mathscr{A}$ with stalks $\mathbb{Z}$ on $\mathbb{R} - \{0\}$ and let $i : \mathbb{R} - \{0\} \hookrightarrow \mathbb{R}$. Then $i\mathscr{A}$ has stalk $\mathbb{Z} \oplus \mathbb{Z}$ at 0 and stalk $\mathbb{Z}$ elsewhere, because, for example, $i\mathscr{A}(-\varepsilon, \varepsilon) = \mathscr{A}((-\varepsilon, 0) \cup (0, \varepsilon)) \approx \mathbb{Z} \oplus \mathbb{Z}$. A local section over a connected neighborhood of 0 taking value $(n, m) \in (i\mathscr{A})_0$ at 0 is $n \in (i\mathscr{A})_x$ for $x < 0$ and is $m \in (i\mathscr{A})_x$ for $x > 0$. ◇

3.3. Example. Consider the constant sheaf $\mathscr{A}$ with stalks $\mathbb{Z}$ on $X = \mathbb{S}^1$; let $Y = [-1, 1]$ and let $\pi : X \to Y$ be the projection. Then $\pi\mathscr{A}$ has stalks $\mathbb{Z}$ at -1 and at 1 but has stalks $(\pi\mathscr{A})_x \approx \mathbb{Z} \oplus \mathbb{Z}$ for $-1 < x < 1$. The reader should try to understand the topology connecting these stalks. For example, is it true that $\pi\mathscr{A} \approx \mathbb{Z} \oplus \mathbb{Z}_{(-1,1)}$, as defined in Section 5? Is there a sheaf on Y that is "locally isomorphic" to $\pi\mathscr{A}$ but not isomorphic to it? ◇

3.4. Example. Let Z be the constant sheaf with stalks $\mathbb{Z}$ on $X = \mathbb{S}^1$ and let Z^t denote the "twisted" sheaf with stalks $\mathbb{Z}$ on X (i.e., $Z^t = [0,1] \times \mathbb{Z}$ modulo the identifications $(0 \times n) \sim (1 \times -n)$). Let $f : X \to X$ be the covering map of degree 2. Then fZ is the sheaf on X with stalks $\mathbb{Z} \oplus \mathbb{Z}$ twisted by the exchange of basis elements in the stalks. Also, $f^*Z^t \approx Z$ since it is the locally constant sheaf with stalks $\mathbb{Z}$ on X twisted twice, which is no twist at all. Note that fZ has both Z and Z^t as subsheaves. The corresponding quotient sheaves are $(fZ)/Z \approx Z^t$ and $(fZ)/Z^t \approx Z$. However, $fZ \not\approx Z \oplus Z^t$ (defined in Section 5). ◇

3.5. Example. Let X and Z be as in Example 3.4 but let $f : X \to X$ be the covering map of degree 3. Then fZ is the locally constant sheaf with stalks $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ twisted by the cyclic permutation of factors. Thus fZ has the constant sheaf Z as a subsheaf (the "diagonal") with quotient sheaf the locally constant sheaf with stalks $\mathbb{Z} \oplus \mathbb{Z}$ twisted by the, essentially unique, nontrivial automorphism of period 3. ◇

3.6. Example. Let $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ be the canonical double covering. Then $\pi\mathbb{Z}$ is a twisted sheaf with stalks $\mathbb{Z} \oplus \mathbb{Z}$ on projective space, analogous to the sheaf fZ of Example 3.4. It contains the constant sheaf $\mathbb{Z}$ as a subsheaf with quotient sheaf Z^t, a twisted integer sheaf. If n is even, so that $\mathbb{RP}^n$ is nonorientable, Z^t is just the orientation sheaf $\mathscr{O}_n$ of Example 1.11. ◇

4 Cohomomorphisms

Throughout this section we let $f : X \to Y$ be a given map.

4.1. Definition. *If A and B are presheaves on X and Y respectively, then an "f-cohomomorphism" $k : B \rightsquigarrow A$ is a collection of homomorphisms $k_U : B(U) \to A(f^{-1}(U))$, for U open in Y, compatible with restrictions.*

4.2. Definition. *If $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X and Y respectively, then an "f-cohomomorphism" $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ is a collection of homomorphisms $k_x : \mathscr{B}_{f(x)} \to \mathscr{A}_x$ for each $x \in X$ such that for any section $s \in \mathscr{B}(U)$ the function $x \mapsto k_x(s(f(x)))$ is a section of $\mathscr{A}$ over $f^{-1}(U)$ (i.e., this function is continuous).*[8]

An f-cohomomorphism of sheaves induces an f-cohomomorphism of presheaves by putting $k_U(s)(x) = k_x(s(f(x)))$ where $U \subset Y$ is open and $s \in \mathscr{B}(U)$. Conversely, an f-cohomomorphism of presheaves $k : B \rightsquigarrow A$ induces, for $x \in X$, a homomorphism

$$k_x : \mathscr{B}_{f(x)} = \varinjlim B(U) \to \varinjlim A(f^{-1}U) \to \mathscr{A}_x$$

[where U ranges over neighborhoods of $f(x)$]. Then for $\theta : B(U) \to \mathscr{B}(U)$ the canonical homomorphism and for $s \in B(U)$, we have

$$\theta(k_U(s))(x) = k_x(s(f(x))),$$

so that $\{k_x\}$ is an f-cohomomorphism of sheaves $\mathscr{B} \rightsquigarrow \mathscr{A}$ (generated by B and A).

For any sheaf $\mathscr{B}$ on Y, the collection $f^* = \{f_x^*\}$ of (3) defines an f-cohomomorphism

$$f^* : \mathscr{B} \rightsquigarrow f^*\mathscr{B}.$$

If $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ is any f-cohomomorphism, let $h_x : (f^*\mathscr{B})_x \to \mathscr{A}_x$ be defined by $h_x = k_x \circ (f_x^*)^{-1}$. Together, the homomorphisms h_x define a function $h : f^*\mathscr{B} \to \mathscr{A}$. For $s \in \mathscr{B}(U)$, the equation

$$h(f_U^*(s)(x)) = h(f_x^*(s(f(x)))) = k_x(s(f(x))),$$

together with the fact that the $f_U^*(s)$ form a basis for the topology of $f^*\mathscr{B}$, implies that h is continuous. Thus any f-cohomomorphism k admits a unique factorization

$$k : \mathscr{B} \overset{f^*}{\rightsquigarrow} f^*\mathscr{B} \xrightarrow{h} \mathscr{A},$$

h being a homomorphism.

[8] Note that an f-cohomomorphism $\mathscr{B} \rightsquigarrow \mathscr{A}$ is not generally a function, since it is multiply valued unless f is one-to-one, and it is not defined everywhere unless f is onto. Of course, cohomomorphisms are the morphisms in the category of all sheaves on all spaces.

Similarly, for any sheaf $\mathscr{A}$ on X, the definition (2) provides an f-cohomomorphism $f : f\mathscr{A} \rightsquigarrow \mathscr{A}$. Since $f_U : f\mathscr{A}(U) \to \mathscr{A}(f^{-1}(U))$ is an isomorphism, it is clear that any f-cohomomorphism k admits a unique factorization

$$k : \mathscr{B} \xrightarrow{j} f\mathscr{A} \overset{f}{\rightsquigarrow} \mathscr{A}$$

(i.e., $k_U = f_U j_U$), where j is a homomorphism.

Thus to each f-cohomomorphism k there correspond unique homomorphisms $h : f^*\mathscr{B} \to \mathscr{A}$ and $j : \mathscr{B} \to f\mathscr{A}$. This correspondence is additive and natural in $\mathscr{A}$ and $\mathscr{B}$. Therefore, denoting the group of all f-cohomomorphisms from $\mathscr{B}$ to $\mathscr{A}$ by $f\text{-cohom}(\mathscr{B}, \mathscr{A})$, we have produced the following natural isomorphisms of functors:

$$\operatorname{Hom}(f^*\mathscr{B}, \mathscr{A}) \approx f\text{-}\operatorname{cohom}(\mathscr{B}, \mathscr{A}) \approx \operatorname{Hom}(\mathscr{B}, f\mathscr{A}).$$

Leaving out the middle term, we shall let φ denote this natural isomorphism

$$\boxed{\varphi : \operatorname{Hom}(f^*\mathscr{B}, \mathscr{A}) \xrightarrow{\approx} \operatorname{Hom}(\mathscr{B}, f\mathscr{A})} \tag{5}$$

of functors.[9]

Taking $\mathscr{A} = f^*\mathscr{B}$, we obtain the homomorphism

$$\boxed{\beta = \varphi(1) : \mathscr{B} \to f f^*\mathscr{B},} \tag{6}$$

and taking $\mathscr{B} = f\mathscr{A}$, we obtain the homomorphism

$$\boxed{\alpha = \varphi^{-1}(1) : f^* f\mathscr{A} \to \mathscr{A}.} \tag{7}$$

If $h : f^*\mathscr{B} \to \mathscr{A}$ is any homomorphism, then the naturality of φ implies that the diagram

$$\begin{array}{ccc} \operatorname{Hom}(f^*\mathscr{B}, f^*\mathscr{B}) & \xrightarrow{\varphi} & \operatorname{Hom}(\mathscr{B}, f f^*\mathscr{B}) \\ {\scriptstyle \operatorname{Hom}(f^*\mathscr{B}, h)} \downarrow & & \downarrow {\scriptstyle \operatorname{Hom}(\mathscr{B}, f(h))} \\ \operatorname{Hom}(f^*\mathscr{B}, \mathscr{A}) & \xrightarrow{\varphi} & \operatorname{Hom}(\mathscr{B}, f\mathscr{A}) \end{array}$$

commutes. That is,

$$\varphi(h) = f(h) \circ \varphi(1) = f(h) \circ \beta, \tag{8}$$

which means that $\varphi(h)$ is the composition

$$\mathscr{B} \xrightarrow{\beta} f f^*\mathscr{B} \xrightarrow{f(h)} f\mathscr{A}.$$

[9]The existence of such a natural isomorphism means that f^* and f are "adjoint functors."

Similarly, if $j : \mathscr{B} \to f\mathscr{A}$ is any homomorphism, then the diagram

$$\begin{array}{ccc} \mathrm{Hom}(f^*f\mathscr{A}, \mathscr{A}) & \xrightarrow{\varphi} & \mathrm{Hom}(f\mathscr{A}, f\mathscr{A}) \\ {\scriptstyle \mathrm{Hom}(f^*(j),\mathscr{A})}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(j,f\mathscr{A})} \\ \mathrm{Hom}(f^*\mathscr{B}, \mathscr{A}) & \xrightarrow{\varphi} & \mathrm{Hom}(\mathscr{B}, f\mathscr{A}) \end{array}$$

commutes, whence

$$\varphi^{-1}(j) = \varphi^{-1}(1) \circ f^*(j) = \alpha \circ f^*(j) \tag{9}$$

is the composition

$$f^*\mathscr{B} \xrightarrow{f^*(j)} f^*f\mathscr{A} \xrightarrow{\alpha} \mathscr{A}.$$

In particular, applying (9) to $j = \beta : \mathscr{B} \to ff^*\mathscr{B}$, we obtain that $\alpha \circ f^*(\beta) = \varphi^{-1}(\beta) = 1$. That is, the composition

$$f^*\mathscr{B} \xrightarrow{f^*(\beta)} f^*ff^*\mathscr{B} \xrightarrow{\alpha} f^*\mathscr{B}$$

is the identity. Thus $f^*(\beta)$ is a monomorphism, and since $(f^*\mathscr{B})_x = \mathscr{B}_{f(x)}$, it follows that

$$\beta : \mathscr{B} \to ff^*\mathscr{B}$$

is a *monomorphism* provided that $f : X \to Y$ is *surjective.*

In the next chapter we shall apply this to the special case in which $f : X_d \to X$ is the identity, where X_d denotes X with the discrete topology. In this case

$$(ff^*\mathscr{B})(U) = \prod_{x \in U} \mathscr{B}_x$$

is the group of "serrations" of $\mathscr{B}$ over U, where a *serration* is a possibly discontinuous cross section of $\mathscr{B}|U$. Then $\beta : \mathscr{B}(U) \to (ff^*\mathscr{B})(U)$ is just the inclusion of the group of (continuous) sections in that of serrations.

4.3. We conclude this section with a remark on cohomomorphisms in quotient sheaves. Let $\mathscr{A}'$ be a subsheaf of a sheaf $\mathscr{A}$ on X and $\mathscr{B}'$ a subsheaf of $\mathscr{B}$ on Y. Let $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ be an f-cohomomorphism that takes $\mathscr{B}'$ into $\mathscr{A}'$. Then k clearly induces an f-cohomomorphism

$$\mathscr{B}(U)/\mathscr{B}'(U) \rightsquigarrow \mathscr{A}(f^{-1}(U))/\mathscr{A}'(f^{-1}(U))$$

of presheaves, which, in turn, induces an f-cohomomorphism $\mathscr{B}/\mathscr{B}' \rightsquigarrow \mathscr{A}/\mathscr{A}'$ of the generated sheaves.

5 Algebraic constructions

In this section we shall consider covariant functors $F(G_1, G_2, \ldots)$ of several variables from the category of abelian groups to itself. (More generally, one may consider covariant functors from the category of "diagrams of abelian

groups of a given shape" to the category of abelian groups.) For general illustrative purposes we shall take the case of a functor of two variables.

We may also consider F as a functor from the category of presheaves on X to itself in the canonical way (since F is covariant). That is, we let

$$F(A,B)(U) = F(A(U),B(U)),$$

for presheaves A and B on X. The sheaf generated by the presheaf $F(A,B)$ will be denoted by $\mathcal{F}(A,B) = \mathcal{S}\!\mathit{heaf}(F(A,B))$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are sheaves on X then $\mathcal{F}(\mathcal{A},\mathcal{B}) = \mathcal{S}\!\mathit{heaf}(U \mapsto F(\mathcal{A},\mathcal{B})(U) = F(\mathcal{A}(U),\mathcal{B}(U)))$.[10]

Now suppose that the functor F commutes with direct limits. That is, suppose that the canonical map $\varinjlim F(G_\alpha,H_\alpha) \to F(\varinjlim G_\alpha, \varinjlim H_\alpha)$ is an isomorphism for direct systems $\{G_\alpha\}$ and $\{H_\alpha\}$ of abelian groups. Let $\mathcal{A}$ and $\mathcal{B}$ denote the sheaves generated by the presheaves A and B respectively. Then for U ranging over the neighborhoods of $x \in X$, we have $\varinjlim F(A,B)(U) = \varinjlim F(A(U),B(U)) \approx F(\varinjlim A(U), \varinjlim B(U)) = F(\mathcal{A}_x,\mathcal{B}_x)$ so that we have the natural isomorphism

$$\mathcal{F}(A,B)_x \approx F(\mathcal{A}_x,\mathcal{B}_x) \tag{10}$$

when F commutes with direct limits.

More generally, consider the natural maps $A(U) \to \mathcal{A}(U)$ and $B(U) \to \mathcal{B}(U)$. These give rise to a homomorphism $F(A,B) \to F(\mathcal{A},\mathcal{B})$ of presheaves and hence to a homomorphism

$$\mathcal{F}(A,B) \to \mathcal{F}(\mathcal{A},\mathcal{B}) \tag{11}$$

of the generated sheaves. If U ranges over the neighborhoods of $x \in X$, then the diagram

$$\begin{array}{ccc} \varinjlim F(A(U),B(U)) & \longrightarrow & \varinjlim F(\mathcal{A}(U),\mathcal{B}(U)) \\ \downarrow & & \downarrow \\ F(\varinjlim A(U), \varinjlim B(U)) & \longrightarrow & F(\varinjlim \mathcal{A}(U), \varinjlim \mathcal{B}(U)) \end{array}$$

commutes. The bottom homomorphism is an isomorphism by definition of $\mathcal{A}$ and $\mathcal{B}$. The top homomorphism is the restriction of (11) to the stalks at x. The vertical maps are isomorphisms when F commutes with direct limits. Thus we see that (11) is an *isomorphism* of sheaves provided that F commutes with direct limits. That is, in this case,

$$\boxed{\mathcal{S}\!\mathit{heaf}(U \mapsto F(\mathcal{A}(U),\mathcal{B}(U))) \approx \mathcal{S}\!\mathit{heaf}(U \mapsto F(A(U),B(U)))}$$

naturally.

[10]Our notation in some of the specific examples to follow will differ from the notation we are using in the general discussion.

We shall now discuss several explicit cases, starting with the tensor product. If $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X, we let

$$\boxed{\mathscr{A}\otimes\mathscr{B} = \mathscr{Sheaf}(U \mapsto \mathscr{A}(U)\otimes\mathscr{B}(U)).}$$

Since $\otimes$ commutes with direct limits, we have the natural isomorphism

$$\boxed{(\mathscr{A}\otimes\mathscr{B})_x \approx \mathscr{A}_x\otimes\mathscr{B}_x}$$

by (10). Since $\otimes$ is right exact for abelian groups, it will also be right exact for sheaves since exactness is a stalkwise property.

The following terminology will be useful:

5.1. Definition. *An exact sequence* $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ *of sheaves on* X *is said to be "pointwise split" if* $0 \to \mathscr{A}'_x \to \mathscr{A}_x \to \mathscr{A}''_x \to 0$ *splits for each* $x \in X$.

This condition clearly implies that $0 \to \mathscr{A}'\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}''\otimes\mathscr{B} \to 0$ is exact for every sheaf $\mathscr{B}$ on X.

Our second example concerns the torsion product. We shall use $G * H$ to denote $\operatorname{Tor}(G, H)$. For sheaves $\mathscr{A}$ and $\mathscr{B}$ on X we let

$$\boxed{\mathscr{A} * \mathscr{B} = \mathscr{Sheaf}(U \mapsto \mathscr{A}(U) * \mathscr{B}(U)).}$$

We have that

$$\boxed{(\mathscr{A} * \mathscr{B})_x \approx \mathscr{A}_x * \mathscr{B}_x}$$

since the torsion product $*$ commutes with direct limits.

Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be an exact sequence of sheaves. Then for each open set $U \subset X$, we have the exact sequence

$$0 \to \mathscr{A}'(U) * \mathscr{B}(U) \to \mathscr{A}(U) * \mathscr{B}(U) \to (\mathscr{A}(U)/\mathscr{A}'(U)) * \mathscr{B}(U)$$

$$\to \mathscr{A}'(U)\otimes\mathscr{B}(U) \to \mathscr{A}(U)\otimes\mathscr{B}(U) \to (\mathscr{A}(U)/\mathscr{A}'(U))\otimes\mathscr{B}(U) \to 0$$

of presheaves on X, where $\mathscr{B}$ is any sheaf. Now $\mathscr{A}''$ is canonically isomorphic to $\mathscr{Sheaf}(U \mapsto \mathscr{A}(U)/\mathscr{A}'(U))$. Thus this sequence of presheaves generates the exact sequence

$$\boxed{0 \to \mathscr{A}' * \mathscr{B} \to \mathscr{A} * \mathscr{B} \to \mathscr{A}'' * \mathscr{B} \to \mathscr{A}'\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}''\otimes\mathscr{B} \to 0} \quad (12)$$

of sheaves on X.[11]

Before passing on to other examples of our general considerations, we shall introduce some further notation concerned with tensor and torsion

[11] It should be noted that this is a special case of a general fact. Namely, if $\{F_n\}$ is an exact connected sequence of functors of abelian groups (as above), then the induced sequence of functors $\{\mathscr{F}_n\}$ on the category of sheaves to itself is also exact and connected.

products. If X and Y are spaces and $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are the projections, then for sheaves $\mathscr{A}$ on X and $\mathscr{B}$ on Y we define the *total tensor product* $\mathscr{A}\widehat{\otimes}\mathscr{B}$ to be the sheaf

$$\boxed{\mathscr{A}\widehat{\otimes}\mathscr{B} = (\pi_X^*\mathscr{A}) \otimes (\pi_Y^*\mathscr{B})}$$

on $X \times Y$. Similarly, the *total torsion product* is defined to be

$$\boxed{\mathscr{A}\widehat{*}\mathscr{B} = (\pi_X^*\mathscr{A}) * (\pi_Y^*\mathscr{B}).}$$

Clearly, we have natural isomorphisms

$$(\mathscr{A}\widehat{\otimes}\mathscr{B})_{\langle x,y\rangle} \approx \mathscr{A}_x \otimes \mathscr{B}_y,$$

$$(\mathscr{A}\widehat{*}\mathscr{B})_{\langle x,y\rangle} \approx \mathscr{A}_x * \mathscr{B}_y.$$

Another special case of our general discussion is provided by the direct sum functor. Thus, if $\{\mathscr{A}_\alpha\}$ is a family of sheaves on X, we let

$$\boxed{\bigoplus \mathscr{A}_\alpha = \mathscr{S}\!heaf(U \mapsto \bigoplus(\mathscr{A}_\alpha(U))).}$$

Since direct sums commute with direct limits, we have that

$$\boxed{\left(\bigoplus_\alpha \mathscr{A}_\alpha\right)_x \approx \bigoplus_\alpha (\mathscr{A}_\alpha)_x.}$$

In the case of the direct product, we note that the presheaf $U \mapsto \prod(\mathscr{A}_\alpha(U))$ satisfies (S1) and (S2) on page 6 and therefore is a sheaf. It is denoted by $\prod \mathscr{A}_\alpha$. However, direct products do not generally commute with direct limits, and in fact, $(\prod \mathscr{A}_\alpha)_x \not\approx \prod(\mathscr{A}_\alpha)_x$ in general. [For example let $\mathscr{A}_i = \mathbb{Z}_{[0,1/i)} \subset \mathbb{Z}$ for $i \geq 1$ on $X = [0,1]$. Then for $U = [0,1/n)$, we have that $\mathscr{A}_i(U) = 0$ for $i > n$, and so $\prod_{i=1}^\infty(\mathscr{A}_i(U)) = \mathbb{Z}^n$, whence $(\prod \mathscr{A}_i)_{\{0\}} \approx \varinjlim \mathbb{Z}^n = \mathbb{Z}^\infty$, the countable direct sum of copies of $\mathbb{Z}$. However, $\prod(\mathscr{A}_i)_{\{0\}} \approx \prod_{i=1}^\infty \mathbb{Z}$, which is uncountable. For another example, let $\mathscr{B}_n = \mathbb{Z}_{\{1/n\}}$. Then $\prod(\mathscr{B}_n)_{\{0\}} = 0$; but $(\prod \mathscr{B}_n)_{\{0\}} \neq 0$ since the sections $s_n \in \mathscr{B}_n(X)$ that are 1 at $1/n$ give a section $s = \prod s_n$ of the product that is not zero in any neighborhood of 0 and hence has nonzero germ at 0.]

For a finite number of variables (or, generally, for locally finite families), direct sums and direct products of sheaves coincide. For two variables (for example) $\mathscr{A}$ and $\mathscr{B}$, the direct sum is denoted by $\mathscr{A} \oplus \mathscr{B}$. (Note that $\mathscr{A}\triangle\mathscr{B}$ is the underlying topological space of $\mathscr{A} \oplus \mathscr{B}$.) The notation $\mathscr{A} \times \mathscr{B}$ is reserved for the *cartesian* product of $\mathscr{A}$ and $\mathscr{B}$, which with coordinatewise addition is a sheaf on $X \times Y$ when $\mathscr{A}$ is a sheaf on X and $\mathscr{B}$ is one on Y. Note that $\mathscr{A} \times \mathscr{B} = (\pi_X^*\mathscr{A}) \oplus (\pi_Y^*\mathscr{B})$.

Note that for $X = Y$, we have that $\mathscr{A} \oplus \mathscr{B} = (\mathscr{A} \times \mathscr{B})|\Delta$, where Δ is the diagonal of $X \times X$, identified with X, and similarly that $\mathscr{A} \otimes \mathscr{B} = (\mathscr{A}\widehat{\otimes}\mathscr{B})|\Delta$ and $\mathscr{A} * \mathscr{B} = (\mathscr{A}\widehat{*}\mathscr{B})|\Delta$.

Our next example is given by a functor on a category of "diagrams." Let A be a directed set. Consider *direct* systems $\{G_\alpha; \pi_{\alpha,\beta}\}$ (where $\pi_{\alpha,\beta} : G_\beta \to G_\alpha$ is defined for $\alpha > \beta$ in A and satisfies $\pi_{\alpha,\beta}\pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$) of abelian groups based on the directed set A. Let F be the (covariant) functor that assigns to each such direct system $\{G_\alpha; \pi_{\alpha,\beta}\}$ its direct limit

$$F(\{G_\alpha; \pi_{\alpha,\beta}\}) = \varinjlim G_\alpha.$$

Now let $\{\mathscr{A}_\alpha; \pi_{\alpha,\beta}\}$ be a direct system of sheaves based on A. Then we define

$$\boxed{\varinjlim \mathscr{A}_\alpha = \mathscr{Sheaf}\Big(U \mapsto \varinjlim_\alpha \mathscr{A}_\alpha(U)\Big).}$$

There are the compatible maps $\mathscr{A}_\alpha(U) \to \varinjlim_\alpha \mathscr{A}_\alpha(U)$ that induce canonical homomorphisms $\pi_\beta : \mathscr{A}_\beta \to \varinjlim_\alpha \mathscr{A}_\alpha$ such that $\pi_\beta = \pi_\alpha \circ \pi_{\alpha,\beta}$ whenever $\alpha > \beta$. Since direct limits commute with one another, we have that

$$\boxed{(\varinjlim \mathscr{A}_\alpha)_x \approx \varinjlim (\mathscr{A}_\alpha)_x.}$$

Now suppose that $\mathscr{A}$ is another sheaf on X and that we have a family of homomorphisms $h_\alpha : \mathscr{A}_\alpha \to \mathscr{A}$ that are compatible in the sense that $h_\beta = h_\alpha \circ \pi_{\alpha,\beta}$ whenever $\alpha > \beta$. These induce compatible maps $\mathscr{A}_\alpha(U) \to \mathscr{A}(U)$ for all open U and hence a homomorphism $\varinjlim(\mathscr{A}_\alpha(U)) \to \mathscr{A}(U)$ of presheaves. In turn this induces a homomorphism

$$h : \varinjlim \mathscr{A}_\alpha \to \mathscr{A}$$

of the generated sheaves such that $h \circ \pi_\alpha = h_\alpha$ for all α. That is, the direct limit of sheaves satisfies the "universal property" of direct limits.

In particular, if $\{\mathscr{A}_\alpha\}$ and $\{\mathscr{B}_\alpha\}$ are direct systems based on the same directed set, then the homomorphisms $\mathscr{A}_\alpha \to \varinjlim \mathscr{A}_\alpha$ and $\mathscr{B}_\alpha \to \varinjlim \mathscr{B}_\alpha$ induce compatible homomorphisms $\mathscr{A}_\alpha \otimes \mathscr{B}_\alpha \to \varinjlim \mathscr{A}_\alpha \otimes \varinjlim \mathscr{B}_\alpha$ and hence a homomorphism $\varinjlim(\mathscr{A}_\alpha \otimes \mathscr{B}_\alpha) \to \varinjlim \mathscr{A}_\alpha \otimes \varinjlim \mathscr{B}_\alpha$. On stalks this is an isomorphism since tensor products and direct limits commute. Thus it follows that this is an isomorphism

$$\boxed{\lambda : \varinjlim(\mathscr{A}_\alpha \otimes \mathscr{B}_\alpha) \xrightarrow{\approx} \varinjlim \mathscr{A}_\alpha \otimes \varinjlim \mathscr{B}_\alpha.} \tag{13}$$

The functor $\operatorname{Hom}(G, H)$ on abelian groups is covariant in only one of its variables, so that the general discussion does not apply. However, note that every homomorphism $\mathscr{A} \to \mathscr{B}$ of sheaves induces a homomorphism

$$\mathscr{A}|U \to \mathscr{B}|U.$$

Thus we see that the functor

$$U \mapsto \operatorname{Hom}(\mathscr{A}|U, \mathscr{B}|U) \tag{14}$$

defines a presheaf on X. We define

$$\boxed{\mathscr{H}om(\mathscr{A},\mathscr{B}) = \mathscr{S}heaf(U \mapsto \operatorname{Hom}(\mathscr{A}|U,\mathscr{B}|U)).}$$

It is clear that the presheaf (14) satisfies (S1) and (S2), so that

$$\boxed{\mathscr{H}om(\mathscr{A},\mathscr{B})(U) \approx \operatorname{Hom}(\mathscr{A}|U,\mathscr{B}|U).}$$

It is important to note that the last equation does not apply in general to sections over nonopen subspaces, and in particular that

$$\mathscr{H}om(\mathscr{A},\mathscr{B})_x \not\approx \operatorname{Hom}(\mathscr{A}_x,\mathscr{B}_x)$$

in general. For example, let $\mathscr{B}$ be the constant sheaf with stalks $\mathbb{Z}$ on $X = [0,1]$ and let $\mathscr{A} = \mathscr{B}_{\{0\}}$, which has stalk $\mathbb{Z}$ over $\{0\}$ and stalks 0 elsewhere. Then $\mathscr{H}om(\mathscr{A},\mathscr{B})(U) \approx \operatorname{Hom}(\mathscr{A}|U,\mathscr{B}|U) = 0$ for the open sets of the form $U = [0,\varepsilon)$, and hence $\mathscr{H}om(\mathscr{A},\mathscr{B})_{\{0\}} = 0$, whereas $\mathscr{A}_{\{0\}} = \mathbb{Z} = \mathscr{B}_{\{0\}}$, whence $\operatorname{Hom}(\mathscr{A}_{\{0\}},\mathscr{B}_{\{0\}}) \approx \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \approx \mathbb{Z}$.

If $\mathscr{R}$ is a sheaf of rings on X and if $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{R}$-modules, then one can define, in a similar manner, the sheaves

$$\mathscr{A} \otimes_{\mathscr{R}} \mathscr{B}, \qquad \mathscr{T}or_n^{\mathscr{R}}(\mathscr{A},\mathscr{B}), \qquad \mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B}).$$

6 Supports

A *paracompact space* is a Hausdorff space with the property that every open covering has an open, locally finite, refinement. The following facts are well known (see [34], [53] and [19]):

(1) *Every paracompact space is normal.*

(2) *A metric space is paracompact.*

(3) *A closed subspace of a paracompact space is paracompact.*

(4) *If $\{U_\alpha\}$ is a locally finite open cover of a normal space X, then there is an open cover $\{V_\alpha\}$ of X such that $\overline{V}_\alpha \subset U_\alpha$.*

(5) *A locally compact space is paracompact $\Leftrightarrow$ it is a disjoint union of open, σ-compact subspaces.*

A space is called *hereditarily paracompact* if every open subspace is paracompact. It is easily seen that this implies that every subspace is paracompact. Of course, metric spaces are hereditarily paracompact.

6.1. Definition. *A "family of supports" on X is a family Φ of closed subsets of X such that:*

(1) *a closed subset of a member of Φ is a member of Φ;*

(2) *Φ is closed under finite unions.*

Φ is said to be a "paracompactifying" family of supports if in addition:

(3) *each element of Φ is paracompact;*

(4) *each element of Φ has a (closed) neighborhood which is in Φ.*

We define the *extent* $E(\Phi)$ of a family of supports to be the union of the members of Φ. Note that $E(\Phi)$ is open when Φ is paracompactifying.

The family of all compact subsets of X is denoted by c. It is paracompactifying if X is locally compact. We use 0 to denote the family of supports whose only member is the empty set $\varnothing$. It is customary to "denote" the family of all closed subsets of X by the absence of a symbol, and we shall also use *cld* to denote this family.

Recall that for $s \in \mathscr{A}(X)$, $|s| = \{x \in X \mid s(x) \neq 0\}$ denotes the support of the section s. Now if A is a presheaf on X and $s \in A(X)$, we put $|s| = |\theta(s)|$, where $\theta : A(X) \to \mathscr{A}(X)$ is the canonical map, $\mathscr{A}$ being the sheaf generated by A.

Note that for $s \in A(X)$, $x \notin |s| \Leftrightarrow (s|U = 0$ for some neighborhood U of $x)$.

If $\mathscr{A}$ is a sheaf on X, we put

$$\boxed{\Gamma_\Phi(\mathscr{A}) = \{s \in \mathscr{A}(X) \mid |s| \in \Phi\}.}$$

Then $\Gamma_\Phi(\mathscr{A})$ is a subgroup of $\mathscr{A}(X)$, and for an exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$, the sequence

$$\boxed{0 \to \Gamma_\Phi(\mathscr{A}') \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}'')}$$

is exact.

For a presheaf A on X we put $A_\Phi(X) = \{s \in A(X) \mid |s| \in \Phi\}$.[12]

6.2. Theorem. *Let A be a presheaf on X that is conjunctive for coverings of X and let $\mathscr{A}$ be the sheaf generated by A. Then for any paracompactifying family Φ of supports on X, the sequence*

$$\boxed{0 \to A_0(X) \to A_\Phi(X) \xrightarrow{\theta} \Gamma_\Phi(\mathscr{A}) \to 0}$$

is exact.

Proof. The only nontrivial part is that θ is surjective. Let $s \in \Gamma_\Phi(\mathscr{A})$ and let U be an open neighborhood of $|s|$ with $\overline{U}$ paracompact. By covering $\overline{U}$ and then restricting to U, we can find a covering $\{U_\alpha\}$ of U that is locally finite in X and such that there exist $s_\alpha \in A(U_\alpha)$ with $\theta(s_\alpha) = s|U_\alpha$. Similarly, we can find a covering $\{V_\alpha\}$ of U with $U \cap \overline{V}_\alpha \subset U_\alpha$. Add $X - |s|$ to the collection $\{V_\alpha\}$, giving a locally finite covering of X, and use the zero section for the corresponding s_α.

[12]The Γ notation will be used only for sheaves and not for presheaves.

For $x \in X$ let $I(x) = \{\alpha \mid x \in \overline{V}_\alpha\}$, a finite set. For each $x \in X$ there is a neighborhood $W(x)$ such that $y \in W(x) \Rightarrow I(y) \subset I(x)$ and such that $W(x) \subset V_\alpha$ for each $\alpha \in I(x)$.

If $\alpha \in I(x)$, then $\theta(s_\alpha)(x) = s(x)$. Since $I(x)$ is finite, we may further assume that $W(x)$ is so small that $s_\alpha|W(x)$ is independent of $\alpha \in I(x)$ [since $\mathscr{A}_x = \varinjlim A(N)$, N ranging over the neighborhoods of x].

Let $s_x \in A(W(x))$ be the common value of $s_\alpha|W(x)$ for $\alpha \in I(x)$.

Suppose that $x, y \in X$ and $z \in W(x) \cap W(y)$. Let $\alpha \in I(z) \subset I(x) \cap I(y)$. Then $s_x = s_\alpha|W(y)$, so that $s_x|W(x) \cap W(y) = s_y|W(x) \cap W(y)$. Since A is conjunctive for coverings of X, there is a $t \in A(X)$ such that $t|W(x) = s_x$ for all $x \in X$. Clearly $\theta(t) = s$, and by definition, $|t| = |s| \in \Phi$. □

Note that $A_0(U) = 0$ for all open $U \subset X \Leftrightarrow A$ is a monopresheaf.

6.3. Definition. *If $A \subset X$ and Φ is a family of supports on X, then $\Phi \cap A$ denotes the family $\{K \cap A \mid K \in \Phi\}$ of supports on A, and $\Phi|A$ denotes the family $\{K \mid K \subset A$ and $K \in \Phi\}$ of supports on A or on X.*[13]

If X, Y are spaces with support families Φ and Ψ respectively, then $\Phi \times \Psi$ denotes the family on $X \times Y$ of all closed subsets of sets of the form $K \times L$ with $K \in \Phi$ and $L \in \Psi$.

If $f : X \to Y$ and Ψ is a family on Y, then $f^{-1}\Psi$ denotes the family on X of all closed subsets of sets of the form $f^{-1}K$ for $K \in \Psi$.

6.4. Example. For the purposes of this example, let us use the subscript Y on the family of supports *cld* or c to indicate the space to which these symbols apply. (In other places we let the context determine this.) Let $X = \mathbb{R}$ and $A = (0,1)$. Then $cld_X \cap A = cld_A$ and $cld_X|A = c_A$. Also, $c_X \cap A = cld_A$ and $c_X|A = c_A$.

If instead, $X = (0,1]$, then $cld_X \cap A = cld_A$ and $c_X|A = c_A$, while $cld_X|A$ is the family of closed subsets of A bounded away from 1. Also, $c_X \cap A$ is the family of closed subsets of A bounded away from 0.

If $X = \mathbb{R} = Y$, then the family $c_X \times c_Y = c_{X\times Y}$, while $c_X \times cld_Y$ is the family of all closed subsets of $X \times Y$ whose projection to X is bounded (but the projection need not be closed). This is the same as the family $\pi_X^{-1}c_X$. Also, $cld_X \times cld_Y = cld_{X\times Y} = \pi_X^{-1}cld_X$.

For any map $f : X \to Y$, the family $f^{-1}c_Y$ can be thought of as the family of (closed) "basewise compact" sets. In IV-5 we shall define what can be thought of as the family of "fiberwise compact" sets. ◇

6.5. Proposition. *If Φ is a paracompactifying family of supports on X and if $Y \subset X$ is locally closed, then $\Phi|Y$ is a paracompactifying family of supports on Y.*

Proof. For $Y = U \cap F$ with U open and F closed, we have that $\Phi|Y = (\Phi|U)|(U \cap F)$, so that it suffices to consider the two cases Y open and Y closed. These cases are obvious. □

[13] Note that $\Phi|F = \Phi \cap F$ for F closed.

6.6. Proposition. *Let $A \subset X$ be locally closed, let Φ be a family of supports on X, and let $\mathscr{B}$ be a sheaf on A. Then the restriction of sections $\Gamma(\mathscr{B}^X) \to \Gamma(\mathscr{B}^X|A) = \Gamma(\mathscr{B})$ induces an isomorphism*

$$\boxed{\Gamma_\Phi(\mathscr{B}^X) \xrightarrow{\approx} \Gamma_{\Phi|A}(\mathscr{B}).}$$

Similarly, for a sheaf $\mathscr{A}$ on X, the restriction of sections induces an isomorphism

$$\boxed{\Gamma_\Phi(\mathscr{A}_A) \xrightarrow{\approx} \Gamma_{\Phi|A}(\mathscr{A}|A).}$$

Proof. A section $s \in \Gamma_\Phi(\mathscr{B}^X)$ must have support in A since $\mathscr{B}^X$ vanishes outside of A. Thus $|s| \in \Phi|A$. Moreover, $s|A$ can be zero only if s is zero. Now suppose that $t \in \Gamma_{\Phi|A}(\mathscr{B})$, and let $s : X \to \mathscr{B}^X$ be the extension of t by zero. It suffices to show that s is continuous. Since s coincides with the zero section on the open set $X - |t|$, it suffices to restrict our attention to the neighborhood of any point $x \in |t|$. Let $v \in \mathscr{B}^X(U)$ be a section of $\mathscr{B}^X$ with $v(x) = t(x) = s(x)$. We may assume, by changing the open neighborhood U of x, that $v|U \cap A = t|U \cap A$. But v must vanish on $U - A$, so that $v = s|U$. Hence s is continuous on U, and this completes the proof of the first statement. The second statement is immediate from the identity $(\mathscr{A}|A)^X = \mathscr{A}_A$. □

6.7. In this book the Γ notation will be used only for the group of *global* sections. Thus the group of sections over $A \subset X$ of a sheaf $\mathscr{A}$ on X is denoted by $\Gamma(\mathscr{A}|A)$. In the literature, but not here, it is often denoted by $\Gamma(A, \mathscr{A})$. Of course, for the case of a support family Φ on X, there are at least two variations: $\Gamma_{\Phi\cap A}(\mathscr{A}|A)$ and $\Gamma_{\Phi|A}(\mathscr{A}|A)$.

7 Classical cohomology theories

As examples of the use of Theorem 6.2 and also for future reference we will briefly describe the "classical" singular, Alexander-Spanier, de Rham, and Čech cohomology theories.

Alexander-Spanier cohomology

Let G be a fixed abelian group. For $U \subset X$ open let $A^p(U; G)$ be the group of all functions $f : U^{p+1} \to G$ under pointwise addition. Then the functor $U \mapsto A^p(U; G)$ is a conjunctive presheaf on X. [For if $f_\alpha : U_\alpha^{p+1} \to G$ are functions such that f_α and f_β agree on $U_\alpha^{p+1} \cap U_\beta^{p+1}$, then define $f : U^{p+1} \to G$, where $U = \bigcup U_\alpha$, by $f(x) = f_\alpha(x)$ if $x \in U_\alpha^{p+1}$ and $f(x)$ arbitrary if $x \notin U_\alpha^{p+1}$ for any α.] Let $\mathscr{A}^p(X; G) = \mathscr{S}\!heaf(U \mapsto A^p(U; G))$.

The differential (or "coboundary") $d : A^p(U;G) \to A^{p+1}(U;G)$ is defined by

$$df(x_0,\dots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i f(x_0,\dots,\widehat{x}_i,\dots,x_{p+1}),$$

where $f : U^{p+1} \to G$.

Now d is a homomorphism of presheaves and $d^2 = 0$. Thus d induces a differential

$$d : \mathscr{A}^p(X;G) \to \mathscr{A}^{p+1}(X;G)$$

with $d^2 = 0$.

The classical definition of Alexander-Spanier cohomology with supports in the family Φ is

$$\boxed{{}_{AS}H^p_\Phi(X;G) = H^p(A^*_\Phi(X;G)/A^*_0(X;G)).}$$

Note that $A^p_0(X;G)$ is the set of all functions $X^{p+1} \to G$ that vanish in some neighborhood of the diagonal. Thus two functions $f,g : X^{p+1} \to G$ represent the same element of $A^p(X;G)/A^p_0(X;G) \Leftrightarrow$ they coincide in some neighborhood of the diagonal.[14]

Thus Theorem 6.2 implies that if Φ is a *paracompactifying* family of supports, then there is a natural isomorphism

$$\boxed{{}_{AS}H^p_\Phi(X;G) \approx H^p(\Gamma_\Phi(\mathscr{A}^*(X;G))).} \tag{15}$$

There is a "cup product" $\cup : A^p(U;G_1)\otimes A^q(U;G_2) \to A^{p+q}(U;G_1\otimes G_2)$ given by the Alexander-Whitney formula

$$(f\cup g)(x_0,\dots,x_{p+q}) = f(x_0,\dots,x_p)\otimes g(x_p,\dots,x_{p+q}),$$

with $d(f\cup g) = df\cup g + (-1)^p f\cup dg$ and $|f\cup g| \subset |f|\cap|g|$. This induces products

$$\cup : \mathscr{A}^p(X;G_1)\otimes\mathscr{A}^q(X;G_2) \to \mathscr{A}^{p+q}(X;G_1\otimes G_2),$$

$$\Gamma_\Phi(\mathscr{A}^p(X;G_1))\otimes\Gamma_\Psi(\mathscr{A}^q(X;G_2)) \to \Gamma_{\Phi\cap\Psi}(\mathscr{A}^{p+q}(X;G_1\otimes G_2)),$$

and

$$H^p(\Gamma_\Phi(\mathscr{A}^q(X;G_1)))\otimes H^q(\Gamma_\Psi(\mathscr{A}^q(X;G_2))) \to H^{p+q}(\Gamma_{\Phi\cap\Psi}(\mathscr{A}^*(X;G_1\otimes G_2)));$$

i.e.,

$${}_{AS}H^p_\Phi(X;G_1)\otimes{}_{AS}H^q_\Psi(X;G_2) \to {}_{AS}H^{p+q}_{\Phi\cap\Psi}(X;G_1\otimes G_2).$$

In particular, for a base ring L with unit and an L-module G, $\mathscr{A}^0(X;L)$ is a sheaf of rings with unit, and each $\mathscr{A}^n(X;G)$ is an $\mathscr{A}^0(X;L)$-module.

[14]Note that it is the taking of the quotient by the elements of empty support that brings the topology of X into the cohomology groups, since $A^*(X;G)$ itself is totally independent of the topology.

Singular cohomology

Let $\mathscr{A}$ be a locally constant sheaf on X. (Classically $\mathscr{A}$ is called a "bundle of coefficients.") For $U \subset X$, let $S^p(U;\mathscr{A})$ be the group of singular p-cochains of U with values in $\mathscr{A}$. That is, an element $f \in S^p(U;\mathscr{A})$ is a function that assigns to each singular p-simplex $\sigma : \Delta_p \to U$ of U, a cross section $f(\sigma) \in \Gamma(\sigma^*(\mathscr{A}))$, where Δ_p denotes the standard p-simplex.

Since $\mathscr{A}$ is locally constant and Δ_p is simply connected, $\sigma^*(\mathscr{A})$ is a constant sheaf on Δ_p (as $\sigma^*(\mathscr{A})$ is just the induced bundle on Δ_p). It follows that we can define the coboundary operator

$$d : S^p(U;\mathscr{A}) \to S^{p+1}(U;\mathscr{A})$$

by $df(\tau) = f(\partial\tau) \in \Gamma(\tau^*(\mathscr{A}))$.

Let $\mathscr{S}^p(X;\mathscr{A}) = \mathscr{S}\!heaf(U \mapsto S^p(U;\mathscr{A}))$ with the induced differential. The presheaf $S^p(\bullet\,;\mathscr{A})$ is conjunctive since if $\{U_\alpha\}$ is a collection of open sets with union U and if $f(\sigma)$ is defined whenever σ is a singular simplex in some U_α with value that is independent of the particular index α, then just define $f(\sigma) = 0$ (or anything) if $\sigma \not\subset U_\alpha$ for any α, and this extends f to be an element of $S^p(U;\mathscr{A})$.

The classical definition of singular cohomology (with the local coefficients $\mathscr{A}$ and supports in Φ) is

$$\boxed{{}_{\Delta}H^p_\Phi(X;\mathscr{A}) = H^p(S^*_\Phi(X;\mathscr{A})).}$$

However, it is a well-known consequence of the operation of subdivision that

$$H^p(S^*_0(X;\mathscr{A})) = 0 \quad \text{for all } p.$$

[We indicate the proof: Let $\mathfrak{U} = \{U_\alpha\}$ be a covering of X by open sets and let $S^p(\mathfrak{U};\mathscr{A})$ be the group of singular cochains based on $\mathfrak{U}$-small singular simplices. Then a subdivision argument shows that the surjection

$$j_{\mathfrak{U}} : S^*(X;\mathscr{A}) \twoheadrightarrow S^*(\mathfrak{U};\mathscr{A})$$

induces a cohomology isomorphism. Therefore, if we let $K^*_{\mathfrak{U}} = \operatorname{Ker} j_{\mathfrak{U}}$, then $H^*(K^*_{\mathfrak{U}}) = 0$ by the long exact cohomology sequence induced by the short exact cochain sequence $0 \to K^*_{\mathfrak{U}} \to S^*(X;\mathscr{A}) \to S^*(\mathfrak{U};\mathscr{A}) \to 0$. However, $S^*_0(X;\mathscr{A}) = \bigcup K^*_{\mathfrak{U}} = \varinjlim K^*_{\mathfrak{U}}$, so that

$$H^*(S^*_0(X;\mathscr{A})) = H^*(\varinjlim K^*_{\mathfrak{U}}) \approx \varinjlim H^*(K^*_{\mathfrak{U}}) = 0.]$$

Therefore, if Φ is paracompactifying, then the exact sequence

$$0 \to S^*_0 \to S^*_\Phi \to \Gamma_\Phi(\mathscr{S}^*) \to 0$$

of 6.2 yields the isomorphism

$$\boxed{{}_{\Delta}H^p_\Phi(X;\mathscr{A}) \approx H^p(\Gamma_\Phi(\mathscr{S}^*(X;\mathscr{A}))).} \tag{16}$$

As with the case of Alexander-Spanier cohomology, the singular cup product makes $\mathscr{S}^0(X;L)$ into a sheaf of rings and each $\mathscr{S}^n(X;\mathscr{A})$ into an $\mathscr{S}^0(X;L)$-module, where $\mathscr{A}$ is a locally constant sheaf of L-modules.

Remark: If X is a differentiable manifold and we let $S^*(U;\mathscr{A})$ be the complex of singular cochains based on C^∞ singular simplices, a similar discussion applies.

de Rham cohomology

Let X be a differentiable manifold and let $\Omega^p(U)$ be the group of differential p-forms on U with $d:\Omega^p(U)\to\Omega^{p+1}(U)$ being the exterior derivative.[15]

The de Rham cohomology group of X is defined to be

$$\boxed{{}_\Omega H^p_\Phi(X) = H^p(\Omega^*_\Phi(X)).}$$

However, the presheaf $U\mapsto\Omega^p(U)$ is a conjunctive monopresheaf and hence is a sheaf. Thus, trivially, we have

$$\boxed{{}_\Omega H^p_\Phi(X) \approx H^p(\Gamma_\Phi(\Omega^*))} \tag{17}$$

for any family Φ of supports.

Čech cohomology

Let $\mathfrak{U}=\{U_\alpha;\alpha\in I\}$ be an open covering of a space X indexed by a set I and let G be a presheaf on X. Then an n-cochain c of $\mathfrak{U}$ is a function defined on ordered $(n+1)$-tuples $(\alpha_0,\dots,\alpha_n)$ of members of I such that $U_{\alpha_0,\dots,\alpha_n}=U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}\neq\varnothing$ with value

$$c(\alpha_0,\dots,\alpha_n)\in G(U_{\alpha_0,\dots,\alpha_n}).$$

These form a group denoted by $\check{C}^n(\mathfrak{U};G)$. An open set V of X is covered by $\mathfrak{U}\cap V=\{U_\alpha\cap V;\alpha\in I\}$. Thus we have the cochain group $\check{C}^n(\mathfrak{U}\cap V;G)$, and the assignment $V\mapsto\check{C}^n(\mathfrak{U}\cap V;G)$ gives a presheaf on X, and hence a sheaf

$$\check{\mathscr{C}}^n(\mathfrak{U};G)=\mathscr{S}\!heaf(V\mapsto\check{C}^n(\mathfrak{U}\cap V;G)).$$

Thus it makes sense to speak of the support $|c|$ of a cochain, i.e., $|c|=|\theta(c)|$, where $\theta:\check{C}^n(\mathfrak{U};G)\to\Gamma(\check{\mathscr{C}}^n(\mathfrak{U};G))$. This defines the cochain group $\check{C}^n_\Phi(\mathfrak{U};G)$ for a family Φ of supports on X. The coboundary operator $d:\check{C}^n_\Phi(\mathfrak{U};G)\to\check{C}^{n+1}_\Phi(\mathfrak{U};G)$ is defined by

$$dc(\alpha_0,\dots,\alpha_{n+1})=\sum_{i=0}^{n+1}(-1)^i c(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_{n+1})|U_{\alpha_0,\dots,\alpha_{n+1}}.$$

[15] See, for example, [19, Chapters II and V].

It is easy to see that $d^2 = 0$ and so there are the cohomology groups

$$\check{H}^n_\Phi(\mathfrak{U}; G) = H^n(\check{C}^*_\Phi(\mathfrak{U}; G)).$$

A *refinement* of $\mathfrak{U}$ is another open covering $\mathfrak{V} = \{V_\beta; \beta \in J\}$ together with a function (called a *refinement projection*) $\varphi : J \to I$ such that $V_\beta \subset U_{\varphi(\beta)}$ for all $\beta \in J$. This yields a chain map $\varphi^* : \check{C}^*_\Phi(\mathfrak{U}; G) \to \check{C}^*_\Phi(\mathfrak{V}; G)$ by

$$\varphi^*(c)(\beta_0, \ldots, \beta_n) = c(\varphi(\beta_0), \ldots, \varphi(\beta_n))|V_{\beta_0,\ldots,\beta_n}.$$

If $\psi : J \to I$ is another refinement projection, then the functions $D : \check{C}^{n+1}_\Phi(\mathfrak{U}; G) \to \check{C}^n_\Phi(\mathfrak{V}; G)$ given by

$$Dc(\beta_0, \ldots, \beta_n) = \sum_{i=0}^{n} (-1)^i c(\varphi(\beta_0), \ldots, \varphi(\beta_i), \psi(\beta_i), \ldots, \psi(\beta_n))|V_{\beta_0,\ldots,\beta_n}$$

provide a chain homotopy between φ^* and ψ^*. Therefore, there is a homomorphism

$$j^n_{\mathfrak{V},\mathfrak{U}} : \check{H}^n_\Phi(\mathfrak{U}; G) \to \check{H}^n_\Phi(\mathfrak{V}; G)$$

induced by φ^* but *independent* of the particular refinement projection φ used to define it. Thus we can define the Čech cohomology group as

$$\boxed{\check{H}^n_\Phi(X; G) = \varinjlim_{\mathfrak{U}} \check{H}^n_\Phi(\mathfrak{U}; G).}$$

Since it does not affect the direct limit to restrict the coverings to a cofinal set of coverings, it is legitimate to restrict attention to coverings $\mathfrak{U} = \{U_x; x \in X\}$ such that $x \in U_x$ for all x. In this case there is a *canonical* refinement projection, the identity map $X \to X$, for a refinement $\mathfrak{V} = \{V_x; x \in X\}$, $V_x \subset U_x$. Thus there is a canonical chain map

$$\check{C}^*_\Phi(\mathfrak{U}; G) \to \check{C}^*_\Phi(\mathfrak{V}; G),$$

and so it is legitimate to pass to the limit and define the Čech cochain group

$$\boxed{\check{C}^*_\Phi(X; G) = \varinjlim_{\mathfrak{U}} \check{C}^*_\Phi(\mathfrak{U}; G).}$$

Since the direct limit functor is exact, it commutes with cohomology, i.e., there is a canonical isomorphism

$$\boxed{\check{H}^*_\Phi(X; G) \approx H^*(\check{C}^*_\Phi(X; G)),}$$

which we shall regard as equality.

We shall study this further in Chapter III. For the present, let us restrict attention to the case in which G is an abelian group regarded as a constant presheaf. We wish to define a natural homomorphism from the Alexander-Spanier groups to the Čech groups. If $f : X^{n+1} \to G$ is an Alexander-Spanier cochain and $\mathfrak{U} = \{U_x; x \in X\}$ is a covering of X, then f induces

an element $f_{\mathfrak{U}} \in \check{C}^n(\mathfrak{U};G)$ by putting $f_{\mathfrak{U}}(x_0,\dots,x_n) = f(x_0,\dots,x_n)$ when $U_{x_0,\dots,x_n} \neq \varnothing$. Consequently, f induces $f_\infty = \varinjlim f_{\mathfrak{U}} \in \check{C}^*(X;G)$. We claim that $|f_\infty| = |f|$. Indeed,

$$\begin{aligned}
x \notin |f_\infty| \ &\Leftrightarrow\ \exists \mathfrak{U}, V,\ x \in V \text{ and } f_{\mathfrak{U}}|V = 0 \text{ in } \check{C}(\mathfrak{U}\cap V;G)\\
&\Leftrightarrow\ \exists \mathfrak{U}, V,\ x \in V \ni x_0,\dots,x_n \in V \ \Rightarrow\ f_{\mathfrak{U}}(x_0,\dots,x_n) = 0\\
&\Leftrightarrow\ \exists W,\ x \in W \ni x_0,\dots,x_n \in W \ \Rightarrow\ f(x_0,\dots,x_n) = 0\\
&\Leftrightarrow\ \exists W,\ x \in W \ni f|W^{n+1} = 0\\
&\Leftrightarrow\ x \notin |f|.
\end{aligned}$$

Also, $|f_\infty| = \varnothing \Leftrightarrow f_\infty = 0$. Now, given $g \in \check{C}^*(X;G)$, g comes from some $g_{\mathfrak{U}} \in \check{C}^n(\mathfrak{U};G)$, and it is clear that $g_{\mathfrak{U}}$ extends arbitrarily to an Alexander-Spanier cochain g. It follows that $f \mapsto f_\infty$ induces an *isomorphism*

$$A^n_\Phi(X;G)/A^n_0(X;G) \xrightarrow{\approx} \check{C}^n_\Phi(X;G),$$

whence

$$\boxed{{}_{AS}H^n_\Phi(X;G) \approx \check{H}^n_\Phi(X;G)} \qquad (18)$$

for all spaces X and families Φ of supports on X.

Now, the Čech cohomology groups are not altered by restriction to any cofinal system of coverings. Therefore, if X is compact, we can restrict the coverings used to finite coverings. Similarly, if X is paracompact, we can restrict attention to locally finite coverings.

Finally, if the covering dimension[16] $\operatorname{covdim} X = n < \infty$ then we can restrict attention to locally finite coverings $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ such that $U_{\alpha_0,\dots,\alpha_{n+1}} = \varnothing$ for distinct α_i. Now, $\check{C}^*(\mathfrak{U};G)$ is the ordered simplicial cochain complex $C^*(N(\mathfrak{U});G)$ of an n-dimensional abstract simplicial complex, namely the *nerve* $N(\mathfrak{U})$ of $\mathfrak{U}$.[17] For $c \in \check{C}^p(\mathfrak{U};G)$ we have that $|c| = \bigcup\{\overline{U}_{\alpha_0,\dots,\alpha_p} \mid c(\alpha_0,\dots,\alpha_p) \neq 0\}$, since $\mathfrak{U}$ is locally finite. Thus

$$\begin{aligned}
\check{C}^*_\Phi(\mathfrak{U};G) &= \{c \in \check{C}^*(\mathfrak{U};G) \mid \exists K \in \Phi \ni c(\alpha_0,\dots,\alpha_p) = 0 \text{ if } \overline{U}_{\alpha_0,\dots,\alpha_p} \not\subset K\}\\
&= \varinjlim_{K\in\Phi}\{c \in \check{C}^*(\mathfrak{U};G) \mid c(\alpha_0,\dots,\alpha_p) = 0 \text{ if } \overline{U}_{\alpha_0,\dots,\alpha_p} \not\subset K\}\\
&= \varinjlim_{K\in\Phi} C^*(N(\mathfrak{U}), N_K(\mathfrak{U});G),
\end{aligned}$$

where $N_K(\mathfrak{U}) = \{\{\alpha_0,\dots,\alpha_p\} \in N(\mathfrak{U}) \mid \overline{U}_{\alpha_0,\dots,\alpha_p} \not\subset K\}$, which is a subcomplex of $N(\mathfrak{U})$. But $C^*(N(\mathfrak{U}), N_K(\mathfrak{U});G)$ is chain equivalent to the corresponding *oriented* simplicial cochain complex that vanishes above degree n. Therefore $\check{H}^p_\Phi(\mathfrak{U};G) = 0$ for $p > n$, whence $\check{H}^p_\Phi(X;G) = 0$ for $p > n$. Consequently,

$$\boxed{{}_{AS}H^p_\Phi(X;G) = 0 \text{ for } p > \operatorname{covdim} X.} \qquad (19)$$

[16]The covering dimension of X is the least integer n (or ∞) such that every covering of X has a refinement for which no point is contained in more than $n+1$ distinct members of the covering.

[17]This has the members of I as vertices and the subsets $\{\alpha_0,\dots,\alpha_p\} \subset I$, where $U_{\alpha_0,\dots,\alpha_p} \neq \varnothing$, as the p-simplices.

Singular homology

Even though the definition of singular cohomology requires a *locally constant* sheaf as coefficients,[18] one can define singular *homology* with coefficients in an arbitrary sheaf $\mathscr{A}$. To do this, define the group of singular n-chains by

$$\boxed{S_n(X;\mathscr{A}) = \bigoplus_{\sigma} \Gamma(\sigma^*\mathscr{A}),}$$

where the sum ranges over all singular simplices $\sigma : \Delta_n \to X$ of X. If $F_i : \Delta_{n-1} \to \Delta_n$ is the ith face map, then we have the induced homomorphism

$$\eta_i : \Gamma(\sigma^*\mathscr{A}) \to \Gamma(F_i^*\sigma^*\mathscr{A}) = \Gamma((\sigma \circ F_i)^*\mathscr{A})$$

of Section 4, and so the boundary operator

$$\partial : S_n(X;\mathscr{A}) \to S_{n-1}(X;\mathscr{A})$$

can be defined by

$$\partial s = \sum_{i=0}^{n} (-1)^i \eta_i(s)$$

for $s \in \Gamma(\sigma^*\mathscr{A})$.

When $\mathscr{A}$ is locally constant, then this, and the case of cohomology, is equivalent to Steenrod's definition of (co)homology with "local coefficients"; see [75] for the definition of the latter.

The functor $U \mapsto S_n(U;\mathscr{A})$ is *covariant*, and so it is not a presheaf. Thus it has a different nature than do the cohomology theories. See, however, Exercise 12 for a different description of singular homology that has a closer relationship to the cohomology theories.

Exercises

1. Ⓢ If $\mathscr{A}$ is a sheaf on X and $i : B \hookrightarrow X$, show that $i^*\mathscr{A} \approx \mathscr{A}|B$.
2. Ⓢ If $\mathscr{B}$ is a sheaf on B and $i : B \hookrightarrow X$, show that $(i\mathscr{B})|B \approx \mathscr{B}$.
3. Let $\{B_\alpha, \pi_{\alpha,\beta}\}$ be a direct system of *presheaves* [that is, for $U \subset X$, $\{B_\alpha(U), \pi_{\alpha,\beta}(U)\}$ is a direct system of groups such that the $\pi_{\alpha,\beta}$'s commute with restrictions]. Let $B = \varinjlim B_\alpha$ denote the *presheaf* $U \mapsto \varinjlim B_\alpha(U)$. Let $\mathscr{B}_\alpha = \mathscr{S}\!heaf(B_\alpha)$ and $\mathscr{B} = \mathscr{S}\!heaf(B)$. Show that $\mathscr{B}$ and $\varinjlim \mathscr{B}_\alpha$ are canonically isomorphic.
4. Ⓢ A sheaf $\mathscr{P}$ on X is called *projective* if the following commutative diagram, with exact row, can always be completed as indicated:

$$\begin{array}{ccccc} \mathscr{B} & \longrightarrow & \mathscr{C} & \longrightarrow & 0 \\ & \nwarrow & \uparrow & & \\ & & \mathscr{P}. & & \end{array}$$

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[18]This will be generalized by another method in Chapter III.

Show that the constant sheaf $\mathbb{Z}$ on the unit interval is not the quotient of a projective sheaf. (Thus there are not "sufficiently many projectives" in the category of sheaves.)

More generally, show that on a locally connected Hausdorff space without isolated points the only projective sheaf is 0.

5. Show that the tensor product of two sheaves satisfies the universal property of tensor products. That is, if $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are sheaves on X and if $f : \mathscr{A}\triangle\mathscr{B} \to \mathscr{C}$ is a map that commutes with the projections onto X and is bilinear on each stalk, then there exists a unique homomorphism $h : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ such that $f = hk$, where $k : \mathscr{A}\triangle\mathscr{B} \to \mathscr{A} \otimes \mathscr{B}$ takes $(a,b) \in \mathscr{A}_x \times \mathscr{B}_x = (\mathscr{A}\triangle\mathscr{B})_x$ into $a \otimes b \in \mathscr{A}_x \otimes \mathscr{B}_x = (\mathscr{A} \otimes \mathscr{B})_x$.

   Treat the direct sum in a similar manner.

6. Show that the functor $\operatorname{Hom}(\bullet, \bullet)$ of sheaves is left exact.

7. Let $f : X \to Y$ and let $\mathscr{R}$ be a sheaf of rings on Y. Show that the natural equivalence (5) of Section 4 restricts to a natural equivalence

$$\varphi : \operatorname{Hom}_{f^*\mathscr{R}}(f^*\mathscr{B}, \mathscr{A}) \xrightarrow{\approx} \operatorname{Hom}_{\mathscr{R}}(\mathscr{B}, f\mathscr{A}),$$

   where $\mathscr{B}$ is an $\mathscr{R}$-module and $\mathscr{A}$ is an $f^*\mathscr{R}$-module. [The $\mathscr{R}$-module structure of $f\mathscr{A}$ is given by the composition

$$\mathscr{R}(U) \otimes (f\mathscr{A})(U) \to (f^*\mathscr{R})(f^{-1}U) \otimes \mathscr{A}(f^{-1}U) \to \mathscr{A}(f^{-1}U).]$$

8. Ⓢ Let $f : X \to Y$ and let $\mathscr{A}$ be a sheaf on X. Show that $\Gamma_\Phi(f\mathscr{A}) = \Gamma_{f^{-1}\Phi}(\mathscr{A})$, under the defining equality $(f\mathscr{A})(Y) = \mathscr{A}(X)$, for any family Φ of supports on Y. [Also, see IV-3.]

9. Ⓢ Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be an exact sequence of sheaves on a locally connected Hausdorff space X. Suppose that $\mathscr{A}'$ and $\mathscr{A}''$ are locally constant and that the stalks of $\mathscr{A}''$ are finitely generated (over some constant base ring). Show that $\mathscr{A}$ is also locally constant. [*Hint*: For $x \in X$ find a neighborhood U such that $\mathscr{A}(U) \to \mathscr{A}''(U)$ is surjective and such that $\mathscr{A}'(U) \to \mathscr{A}'_y$ and $\mathscr{A}''(U) \to \mathscr{A}''_y$ are isomorphisms for every $y \in U$.]

10. Ⓢ Show by example that Exercise 9 does not hold without the condition that the stalks of $\mathscr{A}''$ are finitely generated.

11. Ⓢ If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is an exact sequence of *constant* sheaves on X, show that the sequence

$$0 \to \Gamma_\Phi(\mathscr{A}') \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}'') \to 0$$

   is exact for any family Φ of supports on X.

12. Ⓢ Let $\Delta_*(X, A)$ [respectively, $\Delta_*^c(X, A)$] be the chain complex of locally finite (respectively, finite) singular chains of X modulo those chains in A. Show that the homomorphism of generated sheaves induced by the obvious homomorphism

$$\Delta_*^c(X, X - U) \hookrightarrow \Delta_*(X, X - U)$$

   of presheaves is an isomorphism. Denote this generated sheaf by $\varDelta_*$. Show that the presheaf $U \mapsto \Delta_*(X, X - U)$ (which generates $\varDelta_*$) is a monopresheaf and that it is conjunctive for *coverings* of X. Deduce that

$$\theta : \Delta_*(X) \to \Gamma(\varDelta_*)$$

is an isomorphism when X is paracompact. [Note, however, that $U \mapsto \Delta_*(X, X-U)$ is not fully conjunctive and hence is not itself a sheaf.] Also show that θ induces an isomorphism

$$\Delta_*^c(X) \xrightarrow{\approx} \Gamma_c(\Delta_*).$$

(This provides another approach to the definition of singular homology with coefficients in a sheaf, by putting $\Delta_*^c(X;\mathscr{A}) = \Gamma_c(\Delta_* \otimes \mathscr{A})$.)

13. Let X be the complex line (real 2-dimensional) and let C denote the constant sheaf of complex numbers. Let $\mathscr{A}$ be the sheaf of germs of complex analytic functions on X. Show that

$$0 \to C \xrightarrow{i} \mathscr{A} \xrightarrow{d} \mathscr{A} \to 0$$

is exact, where i is the canonical inclusion and d is differentiation. For $U \subset X$ open show that $d : \mathscr{A}(U) \to \mathscr{A}(U)$ need not be surjective. For which open sets U is it surjective?

14. Ⓢ Let X be the unit circle in the plane. Let $\mathbb{R}$ denote the constant sheaf of real numbers; $\mathscr{D}$ the sheaf of germs of continuously differentiable real-valued functions on X; and $\mathscr{C}$ the sheaf of germs of continuous real-valued functions on X. Show that $0 \to \mathbb{R} \xrightarrow{i} \mathscr{D} \xrightarrow{d} \mathscr{C} \to 0$ is exact, where d is differentiation. Show that $\operatorname{Coker}\{d : \mathscr{D}(X) \to \mathscr{C}(X)\}$ is isomorphic to the group of real numbers.

15. Ⓢ Let X be the real line. Let $\mathscr{F}$ be the sheaf of germs of all integer-valued functions on X and let $i : \mathbb{Z} \hookrightarrow \mathscr{F}$ be the canonical inclusion. Let $\mathscr{G}$ be the quotient sheaf of $\mathscr{F}$ by $\mathbb{Z}$. Show that $\mathscr{F}(X) \to \mathscr{G}(X)$ is surjective, while $\operatorname{Coker}\{\Gamma_c(\mathscr{F}) \to \Gamma_c(\mathscr{G})\} \approx \mathbb{Z}$.

16. Show that there are natural isomorphisms

$$\operatorname{Hom}\left(\bigoplus_\lambda \mathscr{A}_\lambda, \mathscr{B}\right) \approx \prod_\lambda \operatorname{Hom}(\mathscr{A}_\lambda, \mathscr{B})$$

and

$$\operatorname{Hom}\left(\mathscr{A}, \prod_\lambda \mathscr{B}_\lambda\right) \approx \prod_\lambda \operatorname{Hom}(\mathscr{A}, \mathscr{B}_\lambda).$$

17. Prove or disprove that there is the following natural isomorphism of functors of sheaves $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ on X:

$$\mathscr{H}om(\mathscr{A}, \mathscr{H}om(\mathscr{B}, \mathscr{C})) \approx \mathscr{H}om(\mathscr{A} \otimes \mathscr{B}, \mathscr{C}).$$

18. Ⓢ If $f : A \to X$ is a map with $f(A)$ dense in X and $\mathscr{M}$ is a subsheaf of a constant sheaf on X, then show that the canonical map $\beta : \mathscr{M} \to ff^*\mathscr{M}$ of Section 4 is a monomorphism. Also, give an example showing that this is false for arbitrary sheaves $\mathscr{M}$ on X.

19. Ⓢ For a given point x in the Hausdorff space X, let x also denote the family $\{\{x\}, \varnothing\}$ of supports on X. Show that ${}_{AS}H^n_x(X; G) = 0$ for all $n > 0$. (Compare II-18.)

Chapter II

Sheaf Cohomology

In this chapter we shall define the sheaf-theoretic cohomology theory and shall develop many of its basic properties.

The cohomology groups of a space with coefficients in a sheaf are defined in Section 2 using the canonical resolution of a sheaf due to Godement. In Section 3 it is shown that the category of sheaves contains "enough injectives," and it follows from the results of Sections 4 and 5 that the sheaf cohomology groups are just the right derived functors of the left exact functor Γ that assigns to a sheaf its group of sections.

A sheaf $\mathscr{A}$ is said to be *acyclic* if the higher cohomology groups with coefficients in $\mathscr{A}$ are zero. Such sheaves provide a means of "computing" cohomology in particular situations. In Sections 5 and 9 some important classes of acyclic sheaves are defined and investigated.

In Section 6 we prove a theorem concerning the existence and uniqueness of extensions of a natural transformation of functors (of several variables) to natural transformations of "connected systems" of functors. This result is applied in Section 7 to define, and to give axioms for, the cup product in sheaf cohomology theory. These sections are central to our treatment of many of the fundamental consequences of sheaf theory.

The cohomology homomorphism induced by a map is defined in Section 8. The relationship between the cohomology of a subspace and that of its neighborhoods is investigated in Section 10, and the important notion of "tautness" of a subspace is introduced there.

In Section 11 we prove the Vietoris mapping theorem and use it to prove that sheaf-theoretic cohomology, with constant coefficients, satisfies the invariance under homotopy property for general topological spaces.

Relative cohomology theory is introduced into sheaf theory in Section 12, and its properties, such as invariance under excision, are developed. In Section 13 we derive some exact sequences of the Mayer-Vietoris type.

Sections 14, 15, and 17 are concerned, almost exclusively, with locally compact spaces. In Section 14 we prove the "continuity" property, both for spaces and for coefficient sheaves. This property is an important feature of sheaf-theoretic cohomology that is not satisfied for such theories as singular cohomology. A general Künneth formula is derived in Section 15. Section 17 treats local connectivity in higher degrees. This section really has nothing to do with sheaf theory, but the results of this section are used repeatedly in later parts of the book.

In Section 16 we study the concept of cohomological dimension, which

has important applications to other parts of the book. Section 18 contains definitions of "local cohomology groups" and of cohomology groups of the "ideal boundary."

If G is a finite group acting on a space X and if $\pi : X \to X/G$ is the "orbit map," then π induces, as does any map, a homomorphism from the cohomology of X/G to that of X. In Section 19 the "transfer map," which takes the cohomology of X into that of X/G, is defined. When G is cyclic of prime order we also obtain the exact sequences of P. A. Smith relating the cohomology of the fixed point set of G to that of X (a general Hausdorff space on which G acts).

In Sections 20 and 21 we define the Steenrod cohomology operations (the squares and pth powers) in sheaf cohomology and derive several of their properties. This material is not used elsewhere in the book.

All of the sections of this chapter, except for Sections 18 through 21, are used repeatedly in other parts of the book.

Most of Chapter III can be read after Section 9 of the present chapter.

1 Differential sheaves and resolutions

1.1. Definition. *A "graded sheaf" $\mathscr{L}^*$ is a sequence $\{\mathscr{L}^p\}$ of sheaves, p ranging over the integers. A "differential sheaf" is a graded sheaf together with homomorphisms $d : \mathscr{L}^p \to \mathscr{L}^{p+1}$ such that $d^2 : \mathscr{L}^p \to \mathscr{L}^{p+2}$ is zero for all p. A "resolution" of a sheaf $\mathscr{A}$ is a differential sheaf $\mathscr{L}^*$ with $\mathscr{L}^p = 0$ for $p < 0$ together with an "augmentation" homomorphism $\varepsilon : \mathscr{A} \to \mathscr{L}^0$ such that the sequence $0 \to \mathscr{A} \xrightarrow{\varepsilon} \mathscr{L}^0 \xrightarrow{d} \mathscr{L}^1 \xrightarrow{d} \mathscr{L}^2 \to \cdots$ is exact.*

Similarly, one can define graded and differential presheaves.

Since exact sequences and direct limits commute, it follows that the functor $\mathscr{S}\!heaf$, assigning to a presheaf its associated sheaf, is an exact functor. Thus if $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of *presheaves* of order two [i.e., $g(U) \circ f(U) = 0$ for all U] and if $\mathscr{A} \xrightarrow{f'} \mathscr{B} \xrightarrow{g'} \mathscr{C}$ is the induced sequence of generated sheaves, then $\operatorname{Im} f'$ and $\operatorname{Ker} g'$ are generated respectively by the presheaves $\operatorname{Im} f$ and $\operatorname{Ker} g$. Similarly, the sheaf $\operatorname{Ker} g'/\operatorname{Im} f'$ is (naturally isomorphic to) the sheaf generated by the presheaf $\operatorname{Ker} g/\operatorname{Im} f : U \mapsto \operatorname{Ker} g(U)/\operatorname{Im} f(U)$.

If $\mathscr{L}^*$ is a differential sheaf then we define its homology sheaf (or "derived sheaf") to be the graded sheaf $\mathscr{H}^*(\mathscr{L}^*)$, where as usual,

$$\mathscr{H}^p(\mathscr{L}^*) = \operatorname{Ker}(d : \mathscr{L}^p \to \mathscr{L}^{p+1})/\operatorname{Im}(d : \mathscr{L}^{p-1} \to \mathscr{L}^p).$$

The preceding remarks show that $\mathscr{H}^p(\mathscr{L}^*) = \mathscr{S}\!heaf(U \mapsto H^p(\mathscr{L}^*(U)))$, and in general, if $\mathscr{L}^*$ is generated by the differential *presheaf* L^*, then $\mathscr{H}^p(\mathscr{L}^*) = \mathscr{S}\!heaf(U \mapsto H^p(L^*(U)))$.

Note that in general, $\mathscr{H}^p(\mathscr{L}^*)(U) \not\approx H^p(\mathscr{L}^*(U))$. [For example, if we let $\mathscr{L}^0 = \mathscr{L}^1$ be the "twisted" sheaf with stalks $\mathbb{Z}$ on $X = \mathbb{S}^1$ and let

$\mathscr{L}^2 = \mathbb{Z}_2$, the constant sheaf, then $0 \to \mathscr{L}^0 \xrightarrow{2} \mathscr{L}^1 \to \mathscr{L}^2 \to 0$ is exact, so that $\mathscr{H}^p(\mathscr{L}^*)(X) = 0$ for all p. However, $\mathscr{L}^0(X) = 0 = \mathscr{L}^1(X)$ and $\mathscr{L}^2(X) \approx \mathbb{Z}_2$, so that $H^2(\mathscr{L}^*(X)) \approx \mathbb{Z}_2$.]

1.2. Example. In singular cohomology let G be the coefficient group (that is, the constant sheaf with stalk G; this is no loss of generality since we are interested here in local matters).

We have the differential presheaf

$$0 \to G \to S^0(U;G) \to S^1(U;G) \to \cdots, \tag{1}$$

where $G \to S^0(U;G)$ is the usual augmentation. Here we regard G as the constant presheaf $U \mapsto G(U) = G$. This generates the differential sheaf

$$0 \to G \to \mathscr{S}^0(X;G) \to \mathscr{S}^1(X;G) \to \cdots \tag{2}$$

When is this exact? That is, when is $\mathscr{S}^*(X;G)$ a *resolution* of G? Clearly this sequence is exact at G since (1) is. The homology of (1) is just the *reduced* singular cohomology group ${}_\Delta\tilde{H}^*(U;G)$. Thus (2) is exact $\Leftrightarrow$ the sheaf ${}_\Delta\tilde{\mathscr{H}}^*(X;G) = \mathscr{Sheaf}(U \mapsto {}_\Delta\tilde{H}^*(U;G))$ is trivial. This is the case $\Leftrightarrow$

$$\varinjlim {}_\Delta\tilde{H}^*(U;G) = 0 \tag{3}$$

for all $x \in X$, where U ranges over the neighborhoods of x. In the terminology of Spanier, (3) is the condition that the point x be "taut" with respect to singular cohomology with coefficients in G. Note that this condition is implied by the condition $HLC^\infty_\mathbb{Z}$. [A space X is said to be HLC^n_L (homologically locally connected) if for each $x \in X$ and neighborhood U of x, there is a neighborhood $V \subset U$ of x, depending on p, such that the homomorphism ${}_\Delta\tilde{H}_p(V;L) \to {}_\Delta\tilde{H}_p(U;L)$ is trivial for $p \leq n$. Obviously any locally contractible space, and hence any manifold or CW-complex, is HLC.] An example of a space that does not satisfy this condition is the union X of circles of radius $1/n$ all tangent to the x-axis at the origin. It is clear that this is not HLC^1, and at least in the case of rational coefficients, the sheaf ${}_\Delta\mathscr{H}^1(X;\mathbb{Q}) \neq 0$ for this space. ◇

1.3. Example. The Alexander-Spanier presheaf $A^*(\bullet\,;G)$ provides a differential presheaf

$$0 \to G \to A^0(U;G) \to A^1(U;G) \to \cdots \tag{4}$$

and hence a differential sheaf. However, in this case (4) is already exact. [For if $f : U^{p+1} \to G$ and $df = 0$, define $g : U^p \to G$ by $g(x_0,\dots,x_{p-1}) = f(x,x_0,\dots,x_{p-1})$, where x is an arbitrary element of U. Then

$$\begin{aligned} dg(x_0,\dots,x_p) &= \textstyle\sum(-1)^i g(x_0,\dots,\widehat{x}_i,\dots,x_p) \\ &= \textstyle\sum(-1)^i f(x,x_0,\dots,\widehat{x}_i,\dots,x_p) \\ &= f(x_0,\dots,x_p) - df(x,x_0,\dots,\widehat{x}_i,\dots,x_p) \\ &= f(x_0,\dots,x_p), \end{aligned}$$

so that $f = dg$.]

Thus the Alexander-Spanier sheaf $\mathscr{A}^*(X; G)$ is always a resolution of G. ◇

1.4. Example. The de Rham sheaf Ω^* on any differentiable manifold is a differential sheaf and has an augmentation $\mathbb{R} \to \Omega^0$ defined by taking a real number r into the constant function on X with value r. Moreover, $0 \to \mathbb{R} \to \Omega^0 \to \Omega^1 \to \cdots$ is an exact sequence of sheaves, as follows from the Poincaré Lemma, which states that every closed differential form on euclidean space is exact; see [19, V-9.2]. Therefore, Ω^* is a resolution of $\mathbb{R}$. ◇

2 The canonical resolution and sheaf cohomology

For any sheaf $\mathscr{A}$ on X and open set $U \subset X$ we let $C^0(U; \mathscr{A})$ be the collection of all functions (not necessarily continuous) $f : U \dashrightarrow \mathscr{A}$ such that $\pi \circ f$ is the identity on U, $\pi : \mathscr{A} \to X$ being the canonical projection. Such possibly discontinuous sections are called *serrations*, a terminology introduced by Bourgin [10]. That is,

$$\boxed{C^0(U; \mathscr{A}) = \prod_{x \in U} \mathscr{A}_x.}$$

Under pointwise operations, this is a group, and the functor $U \mapsto C^0(U; \mathscr{A})$ is a conjunctive monopresheaf on X. Hence this presheaf is a sheaf, which will be denoted by $\mathscr{C}^0(X; \mathscr{A})$. Note that if X_d denotes the point set of X with the discrete topology and if $f : X_d \to X$ is the canonical map, then $\mathscr{C}^0(X; \mathscr{A}) \approx f f^* \mathscr{A}$, as already mentioned in I-4.

Inclusion of the collection of sections of $\mathscr{A}$ in the collection of all serrations gives an inclusion $\mathscr{A}(U) \hookrightarrow C^0(U; \mathscr{A}) = \mathscr{C}^0(X; \mathscr{A})(U)$ and hence provides a natural monomorphism

$$\boxed{\varepsilon : \mathscr{A} \rightarrowtail \mathscr{C}^0(X; \mathscr{A}).}$$

(For $f : X_d \to X$ as above, this inclusion coincides with the monomorphism $\beta : \mathscr{A} \rightarrowtail f f^* \mathscr{A}$ of (6) on page 15.)

If Φ is a family of supports on X, we put

$$C^0_\Phi(X; \mathscr{A}) = \Gamma_\Phi(\mathscr{C}^0(X; \mathscr{A})).$$

Then for any exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ of sheaves, the corresponding sequence of *presheaves*

$$0 \to C^0(\bullet; \mathscr{A}') \to C^0(\bullet; \mathscr{A}) \to C^0(\bullet; \mathscr{A}'') \to 0$$

is obviously exact. Moreover, for any family Φ of supports, the sequence

$$0 \to C^0_\Phi(\bullet\,;\mathscr{A}') \to C^0_\Phi(\bullet\,;\mathscr{A}) \to C^0_\Phi(\bullet\,;\mathscr{A}'') \to 0$$

is exact. [To see that the last map is onto, we recall that $f \in C^0_\Phi(X;\mathscr{A}'')$ can be regarded as a serration $\hat{f} : X \dashrightarrow \mathscr{A}''$. Then $|f|$ is the *closure* of $\{x \mid \hat{f}(x) = 0\}$. Clearly $\hat{f}$ is the image of a serration $\hat{g}$ of $\mathscr{A}$ that vanishes wherever $\hat{f}$ vanishes. Thus $|g| = |f| \in \Phi$.]

Let $\mathscr{Z}^1(X;\mathscr{A}) = \operatorname{Coker}\{\varepsilon : \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A})\}$, so that the sequence

$$0 \to \mathscr{A} \xrightarrow{\varepsilon} \mathscr{C}^0(X;\mathscr{A}) \xrightarrow{\partial} \mathscr{Z}^1(X;\mathscr{A}) \to 0$$

is exact. We also define, inductively,

$$\mathscr{C}^n(X;\mathscr{A}) = \mathscr{C}^0(X;\mathscr{Z}^n(X;\mathscr{A}))$$

$$\mathscr{Z}^{n+1}(X;\mathscr{A}) = \mathscr{Z}^1(X;\mathscr{Z}^n(X;\mathscr{A}))$$

so that

$$0 \to \mathscr{Z}^n(X;\mathscr{A}) \xrightarrow{\varepsilon} \mathscr{C}^n(X;\mathscr{A}) \xrightarrow{\partial} \mathscr{Z}^{n+1}(X;\mathscr{A}) \to 0$$

is exact. Let $d = \varepsilon \circ \partial$ be the composition

$$\mathscr{C}^n(X;\mathscr{A}) \xrightarrow{\partial} \mathscr{Z}^{n+1}(X;\mathscr{A}) \xrightarrow{\varepsilon} \mathscr{C}^{n+1}(X;\mathscr{A}).$$

Then the sequence

$$\boxed{0 \to \mathscr{A} \xrightarrow{\varepsilon} \mathscr{C}^0(X;\mathscr{A}) \xrightarrow{d} \mathscr{C}^1(X;\mathscr{A}) \xrightarrow{d} \mathscr{C}^2(X;\mathscr{A}) \xrightarrow{d} \cdots}$$

is exact. That is, $\mathscr{C}^*(X;\mathscr{A})$ is a resolution of $\mathscr{A}$. It is called the *canonical resolution* of $\mathscr{A}$ and is due to Godement [40].

This resolution satisfies the stronger property of being naturally "pointwise homotopically trivial." In fact, for $x \in U \subset X$ consider the homomorphism $C^0(U;\mathscr{A}) \to \mathscr{A}_x$ that assigns to a serration $U \dashrightarrow \mathscr{A}$ its value at x. Passing to the limit over neighborhoods of x, this induces a homomorphism $\eta_x : \mathscr{C}^0(X;\mathscr{A})_x \to \mathscr{A}_x$. Clearly $\eta_x \circ \varepsilon : \mathscr{A}_x \to \mathscr{A}_x$ is the identity. Thus, defining $\nu_x : \mathscr{Z}^1(X;\mathscr{A})_x \to \mathscr{C}^0(X;\mathscr{A})_x$ by $\nu_x \circ \partial = 1 - \varepsilon \circ \eta_x$ (which is unambiguous), we obtain the pointwise splitting

$$\mathscr{A}_x \underset{\eta_x}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}^0(X;\mathscr{A}) \underset{\nu_x}{\overset{\partial}{\rightleftarrows}} \mathscr{Z}^1(X;\mathscr{A})_x.$$

Replacing $\mathscr{A}$ by $\mathscr{Z}^n(X;\mathscr{A})$ we obtain, generally, the splittings

$$\mathscr{Z}^n(X;\mathscr{A})_x \underset{\eta_x}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}^n(X;\mathscr{A}) \underset{\nu_x}{\overset{\partial}{\rightleftarrows}} \mathscr{Z}^{n+1}(X;\mathscr{A})_x.$$

[Note that as a consequence of these splittings, all these sheaves are torsion free (i.e., have torsion free stalks) when $\mathscr{A}$ is torsion free.] Let $D_x = \nu_x \circ \eta_x$: $\mathscr{C}^n(X;\mathscr{A})_x \to \mathscr{C}^{n-1}(X;\mathscr{A})_x$, for $n > 0$. Then on $\mathscr{C}^n(X;\mathscr{A})_x$ for $n > 0$ we have $dD_x + D_x d = \varepsilon\partial\nu_x\eta_x + \nu_x\eta_x\varepsilon\partial = \varepsilon\eta_x + \nu_x\partial = 1$, while on $\mathscr{C}^0(X;\mathscr{A})_x$ we have $D_x d = \nu_x\eta_x\varepsilon\partial = \nu_x\partial = 1 - \varepsilon\eta_x$. These three equations,

$$\begin{cases} dD_x + D_x d = 1, & \text{in positive degrees,} \\ D_x d = 1 - \varepsilon\eta_x, & \text{in degree zero,} \\ \eta_x\varepsilon = 1, & \text{on } \mathscr{A}, \end{cases} \tag{5}$$

show that $\mathscr{C}^*(X;\mathscr{A})_x$ is a homotopically trivial resolution of $\mathscr{A}_x$, and this is what we mean when we say that $\mathscr{C}^*(X;\mathscr{A})$ is *pointwise homotopically trivial.* Moreover, the D_x and η_x are natural in $\mathscr{A}$.

2.1. Lemma. *If $\mathscr{A}^*$ is a pointwise homotopically trivial resolution of a sheaf $\mathscr{A}$ on X (e.g., $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$), then $\mathscr{A}^* \otimes \mathscr{B}$ is a resolution of $\mathscr{A} \otimes \mathscr{B}$ for any sheaf $\mathscr{B}$ on X.*

Proof. The hypothesis means that there are the homomorphisms D_x : $\mathscr{A}^n_x \to \mathscr{A}^{n-1}_x$ for $n > 0$ and $\eta_x : \mathscr{A}^0_x \to \mathscr{A}_x$ satisfying (5). Tensoring with $\mathscr{B}$ preserves these equations and the result follows immediately. □

Now suppose that $\mathscr{R}$ is a sheaf of rings and that $\mathscr{A}$ is an $\mathscr{R}$-module. Then $\mathscr{C}^0(X;\mathscr{A})$ is a $\mathscr{C}^0(X;\mathscr{R})$-module and, a fortiori, it is an $\mathscr{R}$-module. Also note that $\varepsilon : \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A})$ is an $\mathscr{R}$-module homomorphism. Thus $\mathscr{Z}^1(X;\mathscr{A})$ is an $\mathscr{R}$-module. By induction, each $\mathscr{C}^n(X;\mathscr{A})$ and $\mathscr{Z}^n(X;\mathscr{A})$ is an $\mathscr{R}$-module, and d is an $\mathscr{R}$-module homomorphism.

Since $\mathscr{Z}^n(X;\mathscr{A})$ is an $\mathscr{R}$-module, when $\mathscr{A}$ is an $\mathscr{R}$-module, it follows that $\mathscr{C}^n(X;\mathscr{A}) = \mathscr{C}^0(X;\mathscr{Z}^n(X;\mathscr{A}))$ is a $\mathscr{C}^0(X;\mathscr{R})$-module. We remark, however, that d is *not* a $\mathscr{C}^0(X;\mathscr{R})$-module homomorphism (for if it were, then it would turn out that the cohomology theory we are going to develop would all be trivial).

Since $\mathscr{C}^0(X;\mathscr{A})$ is an exact functor of $\mathscr{A}$, so is $\mathscr{Z}^1(X;\mathscr{A})$. By induction it follows that $\mathscr{C}^n(X;\mathscr{A})$ and $\mathscr{Z}^n(X;\mathscr{A})$ are all exact functors of $\mathscr{A}$.

For a family Φ of supports on X we put

$$\boxed{C^n_\Phi(X;\mathscr{A}) = \Gamma_\Phi(\mathscr{C}^n(X;\mathscr{A})) = C^0_\Phi(X;\mathscr{Z}^n(X;\mathscr{A})).}$$

Since $C^0_\Phi(X;\bullet)$ and $\mathscr{Z}^n(X;\bullet)$ are exact functors, it follows that

$$\boxed{C^n_\Phi(X;\bullet) \text{ is an exact functor.}}$$

2.2. Definition. *For a family Φ of supports on X and for a sheaf $\mathscr{A}$ on X we define*

$$\boxed{H^n_\Phi(X;\mathscr{A}) = H^n(C^*_\Phi(X;\mathscr{A})).}$$

Since $0 \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{C}^0(X;\mathscr{A})) \to \Gamma_\Phi(\mathscr{C}^1(X;\mathscr{A}))$ is exact, we obtain the natural isomorphism

$$\boxed{\Gamma_\Phi(\mathscr{A}) \xrightarrow{\approx} H^0_\Phi(X;\mathscr{A})}$$

of functors of $\mathscr{A}$.

From a short exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ of sheaves on X we obtain a short exact sequence

$$0 \to C^*_\Phi(X;\mathscr{A}') \to C^*_\Phi(X;\mathscr{A}) \to C^*_\Phi(X;\mathscr{A}'') \to 0$$

of chain complexes and hence an induced long exact sequence

$$\boxed{\cdots \to H^p_\Phi(X;\mathscr{A}') \to H^p_\Phi(X;\mathscr{A}) \to H^p_\Phi(X;\mathscr{A}'') \xrightarrow{\delta} H^{p+1}_\Phi(X;\mathscr{A}') \to \cdots}$$

2.3. If $\mathscr{A}$ is a sheaf on X and $A \subset X$, and if Ψ is a family of supports on A, then we shall often use the abbreviation

$$\boxed{H^p_\Psi(A;\mathscr{A}) = H^p_\Psi(A;\mathscr{A}|A).}$$

2.4. We shall now describe another type of canonical resolution, also due to Godement, which is of value in certain situations. Most of the details will be left to the reader since this resolution will play a minor role in this book. Let $\mathscr{F}^p(X;\mathscr{A})$ be defined inductively by $\mathscr{F}^0(X;\mathscr{A}) = \mathscr{C}^0(X;\mathscr{A})$ and $\mathscr{F}^p(X;\mathscr{A}) = \mathscr{C}^0(X;\mathscr{F}^{p-1}(X;\mathscr{A}))$. Define $F^p_\Phi(X;\mathscr{A}) = \Gamma_\Phi(\mathscr{F}^p(X;\mathscr{A}))$ for any family Φ of supports on X. Then both $\mathscr{F}^p(X;\mathscr{A})$ and $F^p(X;\mathscr{A})$ are exact functors of $\mathscr{A}$. We shall define a differential $\delta : \mathscr{F}^p(X;\mathscr{A}) \to \mathscr{F}^{p+1}(X;\mathscr{A})$ that makes $\mathscr{F}^*(X;\mathscr{A})$ into a resolution of $\mathscr{A}$. To do this we first give a description of $F^p(X;\mathscr{A}) = \Gamma(\mathscr{F}^p(X;\mathscr{A}))$ that is analogous to the definition of Alexander-Spanier cochains.

Denote by $M^p(X;\mathscr{A})$ the set of all functions defined on $(p+1)$-tuples $(x_0, x_1, \ldots, x_p)$ of points in X such that $f(x_0,\ldots,x_p) \in \mathscr{A}_{x_p}$. We shall define an epimorphism

$$\psi_p : M^p(X;\mathscr{A}) \twoheadrightarrow F^p(X;\mathscr{A})$$

by induction on p. Let ψ_0 be the identity. If ψ_{p-1} has been defined, let $f \in M^p(X;\mathscr{A})$, and for each $x_0 \in X$, let $f_{x_0} \in M^{p-1}(X;\mathscr{A})$ be defined by

$$f_{x_0}(x_1,\ldots,x_p) = f(x_0,x_1,\ldots,x_p).$$

The assignment

$$x_0 \mapsto \psi_{p-1}(f_{x_0})(x_0) \in \mathscr{F}^{p-1}(X;\mathscr{A})_{x_0}$$

is a serration of $\mathscr{F}^{p-1}(X;\mathscr{A})$ and hence defines an element of

$$C^0(X;\mathscr{F}^{p-1}(X;\mathscr{A})) = F^p(X;\mathscr{A}).$$

We let $\psi_p(f)$ be this element. Clearly, ψ_p is surjective.

By an induction on p it is easy to check that $\operatorname{Ker}\psi_p$ consists of all elements $f \in M^p(X;\mathscr{A})$ such that for each $(q+1)$-tuple $(x_0,\dots,x_q)$ there is a neighborhood $U(x_0,\dots,x_q)$ of x_q such that if

$$\begin{aligned} &x_1 \in U(x_0),\\ &x_2 \in U(x_0,x_1),\\ &\quad\cdots\\ &x_p \in U(x_0,\dots,x_{p-1}),\end{aligned}$$

then $f(x_0,\dots,x_p)=0$.

We now define the differential δ. If $t \in \mathscr{A}_x$, let $S(t)$ be any serration of $\mathscr{A}$ that is continuous in some neighborhood of x and with $S(t)(x)=t$. [Note that S induces the canonical inclusion $\varepsilon : \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A}) = \mathscr{F}^0(X;\mathscr{A})$.] For $f \in M^p(X;\mathscr{A})$ let $\delta f \in M^{p+1}(X;\mathscr{A})$ be defined by

$$\delta f(x_0,\dots,x_{p+1}) = \sum_{i=0}^{p}(-1)^i f(x_0,\dots,\widehat{x_i},\dots,x_{p+1}) + (-1)^{p+1}S(f(x_0,\dots,x_p))(x_{p+1}).$$

The reader may check that $\delta(\operatorname{Ker}\psi_p) \subset \operatorname{Ker}\psi_{p+1}$ and that $\delta\delta f \in \operatorname{Ker}\psi_{p+2}$ for all $f \in M^p(X;\mathscr{A})$. Thus, we may define a differential δ on $F^p(X;\mathscr{A})$ by $\psi_{p+1}\delta = \delta\psi_p$. Note that on $F^p(X;\mathscr{A})$, δ does not depend on the particular choice function S. On $F^p(X;\mathscr{A})$ we have that $\delta\delta = 0$. These definitions are natural with respect to inclusions of open sets $U \subset X$, and thus we obtain a differential δ on $\mathscr{F}^p(X;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}(U \mapsto F^p(U;\mathscr{A}))$.

Consider the homomorphism $E_x : M^p(X;\mathscr{A}) \to M^{p-1}(X;\mathscr{A})$, for $p > 0$, given by $E_x(f) = f_x$. We claim that this induces a homomorphism $D_x : \mathscr{F}^p(X;\mathscr{A})_x \to \mathscr{F}^{p-1}(X;\mathscr{A})_x$. To see this, let $\theta_x^p : M^p(X;\mathscr{A}) \to \mathscr{F}^p(X;\mathscr{A})_x$ be the composition of $\psi_p : M^p(X;\mathscr{A}) \to F^p(X;\mathscr{A})$ with the restriction $F^p(X;\mathscr{A}) = \Gamma(\mathscr{F}^p(X;\mathscr{A})) \to \mathscr{F}^p(X;\mathscr{A})_x$. If $f \in \operatorname{Ker}\theta_x^p$ then there exists a neighborhood U of x such that $f|U \in M^p(U;\mathscr{A})$ is in $\operatorname{Ker}\psi_p$. Thus there is a sequence of neighborhoods $U(x_0,\dots,x_q)$ in U as above such that if each $x_q \in U(x_0,\dots,x_{q-1})$, then $f(x_0,\dots,x_p)=0$. Specializing to $x_0 = x$ we can cut U down so that $U = U(x)$. Put $V(x_1,\dots,x_q) = U(x,x_1\dots,x_q)$. Then if $x_2 \in V(x_1)$, $x_3 \in V(x_1,x_2)$, etc., we have that $f_x(x_1,x_2,\dots,x_p) = f(x,x_1,\dots,x_p) = 0$, whence $E_x(f) = f_x \in \operatorname{Ker}\theta_x^{p-1}$, as claimed.

Now, it is easy to compute that $E_x\delta + \delta E_x = 1$, whence $D_x\delta + \delta D_x = 1$. We also have $\eta_x : \mathscr{F}^0(X;\mathscr{A}) = \mathscr{C}^0(X;\mathscr{A}) \to \mathscr{A}$, as before, and the reader may check that $D_x\delta = 1 - \varepsilon\eta_x$. Thus, we have produced a pointwise splitting of $\mathscr{F}^*(X;\mathscr{A})$. In particular, this implies that $\mathscr{F}^*(X;\mathscr{A})$ is a resolution of $\mathscr{A}$.

One advantage of this resolution is that it has a semisimplicial structure. The description above in terms of the Alexander-Spanier type of cochains has an analogue for $\mathscr{C}^*(X;\mathscr{A})$. For these facts we refer the reader to [40].

3 Injective sheaves

Let $\mathscr{R}$ be a sheaf of rings with unit on X. All sheaves on X in this section will be $\mathscr{R}$-modules; homomorphisms will be $\mathscr{R}$-module homomorphisms and so on. An $\mathscr{R}$-module $\mathscr{I}$ on X is said to be *injective* (with respect to $\mathscr{R}$) if, for any subsheaf $\mathscr{A}$ of a sheaf $\mathscr{B}$ on X and for any homomorphism $h : \mathscr{A} \to \mathscr{I}$ (of $\mathscr{R}$-modules) there exists an extension $\mathscr{B} \to \mathscr{I}$ of h. That is, $\mathscr{I}$ is injective if the contravariant functor

$$\mathrm{Hom}_{\mathscr{R}}(\bullet, \mathscr{I})$$

is right exact and hence exact (see I-Exercise 6).

In this section we shall show that there are "enough" injective sheaves in the category of $\mathscr{R}$-modules.[1]

First, we need the following preliminary result:

3.1. Theorem. *Let $f : W \to X$ be a map and let $\mathscr{L}$ be an $f^*\mathscr{R}$-injective sheaf on W. Then $f\mathscr{L}$ is an $\mathscr{R}$-injective sheaf on X.*

Proof. We must show that the functor $\mathrm{Hom}_{\mathscr{R}}(\bullet, f\mathscr{L})$ is exact. But by I-Exercise 7, it is naturally equivalent to the functor $\mathrm{Hom}_{f^*\mathscr{R}}(f^*(\bullet), \mathscr{L})$, which is exact since f^* is exact and $\mathscr{L}$ is $f^*\mathscr{R}$-injective. □

We shall now apply this result to the case in which $W = X_d$, the discrete space with the same underlying point set as X. Here $\mathscr{R}$ and $f^*\mathscr{R}$ have the same stalks, and it is clear that a sheaf $\mathscr{L}$ on X_d is injective $\Leftrightarrow$ each stalk $\mathscr{L}_x$ is an injective $\mathscr{R}_x$-module. Thus 3.1 immediately yields the result that if $I(x)$ is an injective $\mathscr{R}_x$-module for each $x \in X$, then the sheaf $\mathscr{I}$ on X defined by

$$\mathscr{I}(U) = \prod_{x \in U} I(x), \tag{6}$$

with the obvious restriction maps, is an injective sheaf on X, since it is just the direct image of the sheaf on X_d whose stalk at x is $I(x)$.

3.2. Theorem. *Any sheaf $\mathscr{A}$ on X is a subsheaf of some injective sheaf.*

Proof. With the previous notation, let $I(x)$ be some injective $\mathscr{R}_x$-module containing $\mathscr{A}_x$ as a submodule. Then the inclusion

$$\prod_{x \in U} \mathscr{A}_x \hookrightarrow \prod_{x \in U} I(x)$$

provides a monomorphism $\mathscr{C}^0(X; \mathscr{A}) \rightarrowtail \mathscr{I}$. Composing this with the canonical monomorphism $\mathscr{A} \rightarrowtail \mathscr{C}^0(X; \mathscr{A})$ yields the desired monomorphism $\mathscr{A} \rightarrowtail \mathscr{I}$. □

[1] We shall make use of this fact only in the case in which $\mathscr{R}$ is a constant sheaf.

3.3. Proposition. *Let L be an integral domain and suppose that $\mathscr{R}$ is the constant sheaf with stalks L. If $\mathscr{I}$ is an L-injective sheaf on X, then $\Gamma_\Phi(\mathscr{I})$ is divisible (with respect to L) for any family Φ of supports on X.*[2]

Proof. Let Q be the field of quotients of L and let $s \in \mathscr{I}(X)$ with $K = |s| \in \Phi$. Then s defines a homomorphism g of the constant sheaf L into $\mathscr{I}$ by $g(\lambda_x) = \lambda_x s(x)$. Since this kills the subsheaf L_{X-K}, it factors to give a homomorphism $h : L_K \to \mathscr{I}$ such that s is the image of the unity section 1 of L via $L \to L_K \xrightarrow{h} \mathscr{I}$. Since $\mathscr{I}$ is injective, h can be extended to $h' : Q_K \to \mathscr{I}$. Now if $k \in L$, then $1/k \in Q$ gives a section of Q_K that is carried by h' to a section $t \in \mathscr{I}(X)$ with $|t| = K$ and $kt = s$. □

3.4. Proposition. *If $\mathscr{I}$ is an injective sheaf, then so is $\mathscr{I}|U$ for any open set $U \subset X$.*

Proof. Let $0 \to \mathscr{A} \to \mathscr{B}$ be an exact sequence of sheaves on U and let $\mathscr{A} \to \mathscr{I}|U$ be any homomorphism. The exact commutative diagram[3]

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{A}^X & \longrightarrow & \mathscr{B}^X \\ & & \downarrow & & \vdots \\ & & (\mathscr{I}|U)^X & \longrightarrow & \mathscr{I} \end{array}$$

can be completed as indicated since $\mathscr{I}$ is injective, yielding the required homomorphism $\mathscr{B} = \mathscr{B}^X|U \to \mathscr{I}|U$. □

Because of 3.2, standard methods of homological algebra can be applied to the theory of sheaves. For example, every sheaf has an injective resolution.[4] Moreover, suppose that $\mathscr{I}^*$ is a differential sheaf with each $\mathscr{I}^p$ injective and that $\mathscr{B} \to \operatorname{Ker}(\mathscr{I}^0 \to \mathscr{I}^1)$ is a given homomorphism. Then for any *resolution* $\mathscr{L}^*$ of a sheaf $\mathscr{A}$, any homomorphism $h : \mathscr{A} \to \mathscr{B}$ admits an extension to a chain map $\mathscr{L}^* \to \mathscr{I}^*$, that is, a commutative diagram

$$\begin{array}{ccccccc} \mathscr{A} & \to & \mathscr{L}^0 & \xrightarrow{d} & \mathscr{L}^1 & \xrightarrow{d} & \cdots \\ h\downarrow & & \downarrow & & \downarrow & & \\ \mathscr{B} & \to & \mathscr{I}^0 & \xrightarrow{d} & \mathscr{I}^1 & \xrightarrow{d} & \cdots \end{array}$$

Moreover, any two such chain maps $\mathscr{L}^* \to \mathscr{I}^*$ extending h are chain homotopic, that is, if $h', h'' : \mathscr{L}^* \to \mathscr{I}^*$ are chain maps extending h, then there is a sequence of homomorphisms

$$\eta : \mathscr{L}^p \to \mathscr{I}^{p-1} \quad \text{such that} \quad d\eta + \eta d = h' - h''.$$

[2] Also see Exercise 6.

[3] Recall that $(\mathscr{I}|U)^X = \mathscr{I}_U \subset \mathscr{I}$.

[4] For the benefit of readers with deficient background in homological algebra, we detail this and some other items we need about injective resolutions at the end of this section.

Any chain map $\mathscr{L}^* \to \mathscr{I}^*$ induces a chain map of complexes $\Gamma_\Phi(\mathscr{L}^*) \to \Gamma_\Phi(\mathscr{I}^*)$. Since a chain homotopy induces a chain homotopy, we see that there is a canonical homomorphism

$$h^* : H^*(\Gamma_\Phi(\mathscr{L}^*)) \to H^*(\Gamma_\Phi(\mathscr{I}^*))$$

induced by $h : \mathscr{A} \to \mathscr{B}$, independent of all choices.

Now let $\mathscr{I}^*$ be an injective *resolution* of $\mathscr{B}$. If $\mathscr{L}^*$ is also an injective resolution of $\mathscr{A}$ and if $\mathscr{A} = \mathscr{B}$, h being the identity, we see that h^* is an isomorphism, since a map in the other direction exists and both compositions must be the identity because of the uniqueness. Thus $H^*(\Gamma_\Phi(\mathscr{I}^*))$ depends only on $\mathscr{B}$ and not on the particular injective resolution.

The functor $\mathscr{B} \mapsto H^p(\Gamma_\Phi(\mathscr{I}^*))$ is called the pth *right derived functor* of the left exact functor $\mathscr{B} \mapsto \Gamma_\Phi(\mathscr{B})$. We shall show in Section 5 that this functor is precisely the functor

$$\mathscr{B} \mapsto H^p_\Phi(X; \mathscr{B}).$$

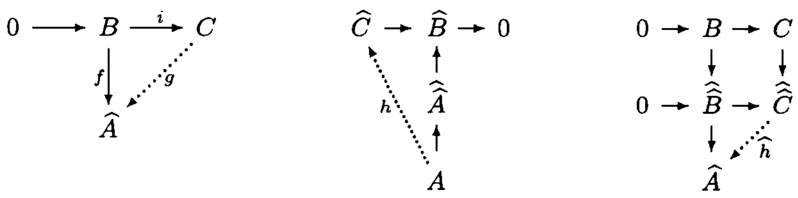
More generally, the functor $\mathscr{B} \mapsto H^p(\Gamma_\Phi(\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{I}^*)))$ is the pth derived functor of the left exact functor $\mathscr{B} \mapsto \Gamma_\Phi(\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B}))$ and is denoted by $\operatorname{Ext}^p_{\Phi,\mathscr{R}}(\mathscr{A}, \mathscr{B})$. Since $\mathscr{H}om_{\mathscr{R}}(\mathscr{R}, \mathscr{B}) \approx \mathscr{B}$ (naturally) we will have that

$$H^p_\Phi(X; \mathscr{B}) \approx \operatorname{Ext}^p_{\Phi,\mathscr{R}}(\mathscr{R}, \mathscr{B}).$$

3.5. Sometimes it is convenient to have a *canonical* injective resolution of a sheaf. To obtain this we need only choose, canonically, the injective module $I(x)$. We now show how to do this.

Let L be a ring with unit and let A be an L-module. Recall that an abelian group is injective as a $\mathbb{Z}$-module $\Leftrightarrow$ it is divisible.[5] We let T denote the group of rationals modulo the integers. Let $\widehat{A}$ denote the L-module $\operatorname{Hom}_{\mathbb{Z}}(A, T)$ and note that $A \mapsto \widehat{A}$ is an exact contravariant functor since T is injective as a $\mathbb{Z}$-module. Since T is injective, it is easily seen that the map $A \to \widehat{\widehat{A}}$, taking $a \mapsto a^*$ where $a^*(f) = f(a)$, is a monomorphism.

We claim that if A is projective then $\widehat{A}$ is injective (as L-modules). To see this, consult the following diagrams



in which we are to produce the map g. The map h exists since A is projective. This dualizes to give the third diagram, and we let g be the composition $C \rightarrowtail \widehat{\widehat{C}} \xrightarrow{\widehat{h}} \widehat{A}$. The composition $B \rightarrowtail \widehat{\widehat{B}} \to \widehat{A}$ is easily seen to be the

[5] See, for example, [19, V-6.2].

original map f. Thus, by commutativity of the third diagram, $f = g \circ i$, as desired.

Now, for an L-module B, let $F(B)$ be the free L-module on the *nonzero* elements of B. (More accurately, $F(B)$ should be considered as the free module on B as a basis modulo the cyclic submodule generated by $0 \in B$.) Then F is a covariant functor. Moreover, there is a natural surjection $F(B) \twoheadrightarrow B$ and hence a natural monomorphism $\widehat{B} \rightarrowtail \widehat{F(B)}$.

The composition $A \rightarrowtail \widehat{A} \rightarrowtail \widehat{F(\widehat{A})}$ is then a natural monomorphism into the injective L-module $I(A) = \widehat{F(\widehat{A})}$. Also, $A \mapsto I(A)$ is a covariant functor. Note, however, that $I(\bullet)$ is not an exact functor since F is not exact.

If $\mathscr{A}$ is a sheaf of $\mathscr{R}$-modules, where $\mathscr{R}$ is a sheaf of rings with unit, let $\mathscr{I}^0(X;\mathscr{A})$ denote the sheaf $\mathscr{I}$ of (6) on page 41, where $I(x)$ is taken to be $I(\mathscr{A}_x)$ as constructed above (for the ring $\mathscr{R}_x$). The proof of 3.2 gives a monomorphism $\mathscr{A} \rightarrowtail \mathscr{I}^0(X;\mathscr{A})$, which we shall regard as an inclusion. Define, inductively,

$$\begin{aligned}\mathscr{Z}^1(X;\mathscr{A}) &= \mathscr{I}^0(X;\mathscr{A})/\mathscr{A},\\ \mathscr{I}^n(X;\mathscr{A}) &= \mathscr{I}^0(X;\mathscr{Z}^n(X;\mathscr{A})),\\ \mathscr{Z}^n(X;\mathscr{A}) &= \mathscr{Z}^1(X;\mathscr{Z}^{n-1}(X;\mathscr{A})).\end{aligned}$$

Then we have the exact sequences

$$0 \to \mathscr{A} \to \mathscr{I}^0(X;\mathscr{A}) \to \mathscr{Z}^1(X;\mathscr{A}) \to 0$$

and

$$0 \to \mathscr{Z}^n(X;\mathscr{A}) \to \mathscr{I}^n(X;\mathscr{A}) \to \mathscr{Z}^{n+1}(X;\mathscr{A}) \to 0,$$

which concatenate to give the natural exact sequence

$$0 \to \mathscr{A} \to \mathscr{I}^0(X;\mathscr{A}) \to \mathscr{I}^1(X;\mathscr{A}) \to \mathscr{I}^2(X;\mathscr{A}) \to \cdots$$

That is, $\mathscr{I}^*(X;\mathscr{A})$ is an injective resolution of $\mathscr{A}$ and is a covariant functor of $\mathscr{A}$.[6] It will be called the *canonical injective resolution* of $\mathscr{A}$.

3.6. For the benefit of readers who have insufficient background concerning resolutions, we indicate here the derivations of some of the statements we have made about injective resolutions and others we shall need later. Readers already at home with this subject should skip to the next section. Although we discuss this subject in the language of sheaves, everything here can be done in general abelian categories.

The discussion in 3.5 already indicated how one can construct an injective resolution of a sheaf just given the fact that any sheaf can be embedded in an injective sheaf.

If $h : \mathscr{A} \to \mathscr{B}$ is any homomorphism of sheaves on X, and if $\varepsilon : \mathscr{A} \to \mathscr{L}^*$ is a resolution and $\eta : \mathscr{B} \to \mathscr{I}^*$ is a degree zero map of $\mathscr{B}$ into an injective

[6] Unfortunately, however, it is not an *exact* functor.

differential sheaf, let us show how to construct an extension $h^*: \mathscr{L}^* \to \mathscr{I}^*$ of h. Consider the diagram

$$\begin{array}{ccccc} 0 \to & \mathscr{A} & \xrightarrow{\varepsilon} & \mathscr{L}^0 \\ & {\scriptstyle h}\downarrow & & \vdots{\scriptstyle h^0} \\ & \mathscr{B} & \xrightarrow{\eta} & \mathscr{I}^0 \end{array}$$

in which the top row is exact. The map h^0 extending $\eta \circ h$ exists since $\mathscr{I}^0$ is injective. This makes the diagram commute. Next, the induced diagram

$$\begin{array}{cccc} 0 \to & \mathscr{L}^0/\operatorname{Im}\varepsilon & \xrightarrow{\varepsilon'} & \mathscr{L}^1 \\ & \downarrow & & \vdots{\scriptstyle h^1} \\ & \mathscr{I}^0/\operatorname{Im}\eta & \xrightarrow{\eta'} & \mathscr{I}^1 \end{array}$$

can similarly be completed. The inductive step for the completion of the argument is now clear.

Now suppose that $h^*, k^* : \mathscr{L}^* \to \mathscr{I}^*$ are two such extensions of the same map $h: \mathscr{A} \to \mathscr{B}$. We wish to construct a chain homotopy g^* between them. That is, we want homomorphisms $g^n : \mathscr{L}^n \to \mathscr{I}^{n-1}$ with $g^0 = 0$ and $d^{n-1}g^n + g^{n+1}d^n = h^n - k^n$, where d^* stands for the differentials in both $\mathscr{L}^*$ and $\mathscr{I}^*$. We have the commutative diagram

$$\begin{array}{cccccccccc} 0 \to & \mathscr{A} & \xrightarrow{\varepsilon} & \mathscr{L}^0 & \xrightarrow{d^0} & \mathscr{L}^1 & \xrightarrow{d^1} & \mathscr{L}^2 & \xrightarrow{d^2} & \cdots \\ & {\scriptstyle h-k=0}\downarrow & & {\scriptstyle h^0-k^0}\downarrow & & {\scriptstyle h^1-k^1}\downarrow & & {\scriptstyle h^2-k^2}\downarrow & & \\ & \mathscr{B} & \xrightarrow{\eta} & \mathscr{I}^0 & \xrightarrow{d^0} & \mathscr{I}^1 & \xrightarrow{d^1} & \mathscr{I}^2 & \xrightarrow{d^2} & \cdots \end{array}$$

which has $(h^0 - k^0) \circ \varepsilon = \eta \circ 0 = 0$. Thus $h^0 - k^0$ factors through $\mathscr{L}^0/\mathscr{A}$, and d^0 induces a monomorphism $\mu : \mathscr{L}^0/\mathscr{A} \rightarrowtail \mathscr{L}^1$. Since $\mathscr{I}^0$ is injective, μ extends to give a homomorphism $g^1 : \mathscr{L}^1 \to \mathscr{I}^0$ such that $g^1 \circ d^0 = h^0 - k^0$. Since $g^0 = 0$ by definition, this is the desired equation $d^{-1}g^0 + g^1 d^0 = h^0 - k^0$. Now,

$$(h^1 - k^1 - d^0 g^1)d^0 = (h^1 - k^1)d^0 - d^0(g^1 d^0) = (h^1 - k^1)d^0 - d^0(h^0 - k^0) = 0$$

by the commutativity of the diagram. Thus $(h^1 - k^1 - d^0 g^1) : \mathscr{L}^1 \to \mathscr{I}^1$ factors through $\mathscr{L}^1/\operatorname{Im} d^0 = \mathscr{L}^1/\operatorname{Ker} d^1$. Therefore, since $\mathscr{I}^1$ is injective, there exists a map $g^2 : \mathscr{L}^2 \to \mathscr{I}^1$ such that $d^1 g^2 = h^1 - k^1 - d^0 g^1$, which is exactly the equation we are after. The inductive step is now clear.

Finally, given an exact sequence $0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{j} \mathscr{C} \to 0$ of sheaves, we wish to construct injective resolutions $\mathscr{A}^*$, $\mathscr{B}^*$, and $\mathscr{C}^*$ of them and an exact sequence $0 \to \mathscr{A}^* \to \mathscr{B}^* \to \mathscr{C}^* \to 0$ extending the original sequence. First, let $\varepsilon : \mathscr{C} \to \mathscr{C}^*$ be an injective resolution and let $\kappa : \mathscr{B} \rightarrowtail \mathscr{A}^0$ be a monomorphism into an injective sheaf. Put $\mathscr{B}^0 = \mathscr{A}^0 \oplus \mathscr{C}^0$ and let $j^0 : \mathscr{B}^0 \to \mathscr{C}^0$ be the projection. Then $\eta = \langle \kappa, \varepsilon j \rangle : \mathscr{B} \to \mathscr{B}^0$ is a

monomorphism and $j^0 \circ \eta = \varepsilon \circ j$. Next, by looking at the quotient of $\mathscr{B}^0$ by $\operatorname{Im}\eta$ there is a map $\kappa^0 : \mathscr{B}^0 \to \mathscr{A}^1$ with $\operatorname{Ker}\kappa^0 = \operatorname{Im}\eta$, where $\mathscr{A}^1$ is some injective sheaf. Then define $\mathscr{B}^1 = \mathscr{A}^1 \oplus \mathscr{C}^1$, with $j^1 : \mathscr{B}^1 \to \mathscr{C}^1$ being the projection and with $d^0 = \langle \kappa^0, d^0 j^0 \rangle : \mathscr{B}^0 = \mathscr{A}^0 \oplus \mathscr{C}^0 \to \mathscr{A}^1 \oplus \mathscr{C}^1 = \mathscr{B}^1$. Then $j^1 d^0 = d^0 j^0$. The inductive step should now be clear, and it gives a resolution $\mathscr{B}^* = \mathscr{A}^* \oplus \mathscr{C}^*$ of $\mathscr{B}^*$ (but the differential is not the direct sum of that on $\mathscr{C}^*$ and the one to be put on $\mathscr{A}^*$) and a map $j^* : \mathscr{B}^* \to \mathscr{C}^*$ of resolutions extending j. Also, j^* is the projection and thus has kernel $\mathscr{A}^*$, which was chosen to consist of injective sheaves (and hence $\mathscr{B}^*$ also consists of injective sheaves). By a simple diagram chase, $\mathscr{A}^*$ inherits a differential and is a resolution of $\mathscr{A}$, as follows by looking at the long exact homology sequence (of derived sheaves) associated with the short exact sequence $0 \to \mathscr{A}^* \to \mathscr{B}^* \to \mathscr{C}^* \to 0$ of differential sheaves.

4 Acyclic sheaves

Let Φ be a family of supports on X and $\mathscr{A}$ a sheaf on X. The sheaf $\mathscr{A}$ is said to be *Φ-acyclic* if

$$H^p_\Phi(X;\mathscr{A}) = 0, \quad \text{for all } p > 0.$$

We shall see in Section 5 that injective sheaves are Φ-acyclic for all Φ.

Let $\mathscr{L}^*$ be a resolution of the sheaf $\mathscr{A}$. We shall describe a natural homomorphism

$$\gamma : H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^p_\Phi(X;\mathscr{A}).$$

Let $\mathscr{Z}^p = \operatorname{Ker}(\mathscr{L}^p \to \mathscr{L}^{p+1}) = \operatorname{Im}(\mathscr{L}^{p-1} \to \mathscr{L}^p)$, where $\mathscr{Z}^0 = \mathscr{A}$. The exact sequence $0 \to \mathscr{Z}^{p-1} \to \mathscr{L}^{p-1} \to \mathscr{Z}^p \to 0$ induces the exact sequence

$$0 \to \Gamma_\Phi(\mathscr{Z}^{p-1}) \to \Gamma_\Phi(\mathscr{L}^{p-1}) \to \Gamma_\Phi(\mathscr{Z}^p) \to H^1_\Phi(X;\mathscr{Z}^{p-1}).$$

(Note that the next term is 0 if $\mathscr{L}^{p-1}$ is Φ-acyclic.) Thus we obtain the monomorphism

$$H^p(\Gamma_\Phi(\mathscr{L}^*)) = \frac{\Gamma_\Phi(\mathscr{Z}^p)}{\operatorname{Im}(\Gamma_\Phi(\mathscr{L}^{p-1}) \to \Gamma_\Phi(\mathscr{Z}^p))} \rightarrowtail H^1_\Phi(X;\mathscr{Z}^{p-1}) \tag{7}$$

[since $0 \to \Gamma_\Phi(\mathscr{Z}^p) \to \Gamma_\Phi(\mathscr{L}^p) \to \Gamma_\Phi(\mathscr{L}^{p+1})$ is exact], which is an isomorphism when $\mathscr{L}^{p-1}$ is Φ-acyclic. Moreover, the exact sequence $0 \to \mathscr{Z}^{p-r} \to \mathscr{L}^{p-r} \to \mathscr{Z}^{p-r+1} \to 0$ induces the connecting homomorphism

$$H^{r-1}_\Phi(X;\mathscr{Z}^{p-r+1}) \to H^r_\Phi(X;\mathscr{Z}^{p-r}), \tag{8}$$

and we let γ be the composition

$$H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^1_\Phi(X;\mathscr{Z}^{p-1}) \to H^2_\Phi(X;\mathscr{Z}^{p-2}) \to \cdots \to H^p_\Phi(X;\mathscr{Z}^0).$$

Now (7) and (8) are isomorphisms if each $\mathscr{L}^i$ is Φ-acyclic. Thus we have:

4.1. Theorem. *If $\mathscr{L}^*$ is a resolution of $\mathscr{A}$ by Φ-acyclic sheaves, then the natural map*

$$\gamma : H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^p_\Phi(X;\mathscr{A})$$

is an isomorphism for all p. □

Naturality means that if the diagram

$$\begin{array}{ccc} \mathscr{A} & \to & \mathscr{L}^* \\ \downarrow f & & \downarrow h \\ \mathscr{B} & \to & \mathscr{M}^* \end{array}$$

commutes (h being a homomorphism of resolutions, i.e., a chain map) then

$$\begin{array}{ccc} H^p(\Gamma_\Phi(\mathscr{L}^*)) & \xrightarrow{\gamma} & H^p_\Phi(X;\mathscr{A}) \\ \big\downarrow h^* & & \big\downarrow f^* \\ H^p(\Gamma_\Phi(\mathscr{M}^*)) & \xrightarrow{\gamma} & H^p_\Phi(X;\mathscr{B}) \end{array}$$

also commutes. A similar statement holds regarding connecting homomorphisms, but we shall not need it.

4.2. Note that it follows from 4.1 that if $\mathscr{L}^* \to \mathscr{M}^*$ is a homomorphism of resolutions by Φ-acyclic sheaves of the same sheaf $\mathscr{A}$, then the induced map

$$H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^p(\Gamma_\Phi(\mathscr{M}^*))$$

is an isomorphism.

4.3. Corollary. *If $0 \to \mathscr{L}^0 \to \mathscr{L}^1 \to \mathscr{L}^2 \to \cdots$ is an exact sequence of Φ-acyclic sheaves, then the corresponding sequence*

$$0 \to \Gamma_\Phi(\mathscr{L}^0) \to \Gamma_\Phi(\mathscr{L}^1) \to \Gamma_\Phi(\mathscr{L}^2) \to \Gamma_\Phi(\mathscr{L}^3) \to \cdots$$

is exact.

Proof. We must show that $H^p(\Gamma_\Phi(\mathscr{L}^*)) = 0$ for all p. By 4.1 we have $H^p(\Gamma_\Phi(\mathscr{L}^*)) \approx H^p_\Phi(X;0)$ since $\mathscr{L}^*$ is a Φ-acyclic resolution of the zero sheaf 0. But $\mathscr{C}^0(X;0) = 0$, and it follows that $\mathscr{C}^n(X;0) = 0$ for all n, whence $C^n_\Phi(X;0) = 0$ and $H^n_\Phi(X;0) = 0$. □

5 Flabby sheaves

In this section we define and study an important class of Φ-acyclic sheaves containing the class of injective sheaves. In particular, we will show that injective sheaves are Φ-acyclic for all Φ.

5.1. Definition. *A sheaf $\mathscr{A}$ on X is "flabby" if $\mathscr{A}(X) \to \mathscr{A}(U)$ is surjective for every open set $U \subset X$.*

5.2. Proposition. *If $\mathscr{A}$ is flabby on X and Φ is any family of supports on X, then $\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi\cap U}(\mathscr{A}|U)$ is surjective for all open $U \subset X$.*

Proof. Let $s \in \Gamma_{\Phi\cap U}(\mathscr{A}|U)$. Then $|s| = K \cap U$ for some $K \in \Phi$. Then s extends by 0 to the open set $U \cup (X - K)$. Extend this arbitrarily to $t \in \Gamma(\mathscr{A})$ using that $\mathscr{A}$ is flabby. Then $|t| \subset K$ and so $t \in \Gamma_\Phi(\mathscr{A})$. $\square$

For any sheaf $\mathscr{A}$ on X, it follows directly from the definitions that $\mathscr{C}^0(X;\mathscr{A})$, and hence $\mathscr{C}^n(X;\mathscr{A})$ and $\mathscr{F}^n(X;\mathscr{A})$, are flabby.

5.3. Proposition. *Every injective module $\mathscr{I}$ over a sheaf $\mathscr{R}$ of rings with unit is flabby.*

Proof. Let $s \in \mathscr{I}(U)$. The map $\mathscr{R}|U \to \mathscr{I}|U$ defined by $\rho_x \mapsto \rho_x \cdot s(x)$, where $\rho_x \in \mathscr{R}_x$ for $x \in U$ clearly extends to a homomorphism $h : \mathscr{R}_U \to \mathscr{I}$. Since $\mathscr{I}$ is injective, h extends to a homomorphism $g : \mathscr{R} \to \mathscr{I}$. Thus the section $x \mapsto g(1_x)$ of $\mathscr{I}$ extends s to X. $\square$

5.4. Theorem. *Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be exact and suppose that $\mathscr{A}'$ is flabby. Then for any family Φ of supports on X,*

$$0 \to \Gamma_\Phi(\mathscr{A}') \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}'') \to 0$$

is exact. In particular, since $\mathscr{A}'|U$ is flabby for U open in X,

$$0 \to \mathscr{A}'(U) \to \mathscr{A}(U) \to \mathscr{A}''(U) \to 0$$

is exact. Moreover, $\mathscr{A}$ is flabby $\Leftrightarrow$ $\mathscr{A}''$ is flabby.

Proof. We may consider $\mathscr{A}'$ as a subsheaf of $\mathscr{A}$. Let $s \in \Gamma_\Phi(\mathscr{A}'')$. Let $\mathscr{C}$ be the collection of all pairs (U,t), U open in X, $t \in \mathscr{A}(U)$, such that t represents $s|U \in \mathscr{A}''(U)$. Order $\mathscr{C}$ by $(U,t) < (U',t')$ if $U \subset U'$ and $t'|U = t$. Then $\mathscr{C}$ is inductively ordered (i.e., a chain in $\mathscr{C}$ has an upper bound, its union) and hence has a maximal element, say (V,t). Suppose that $V \neq X$. Let $x \notin V$ and let W be a neighborhood of x such that $(W,t') \in \mathscr{C}$ for some $t' \in \mathscr{A}(W)$ (which clearly exists).

Now, $t|V \cap W - t'|V \cap W \in \mathscr{A}'(V \cap W)$ and hence extends to some $t'' \in \mathscr{A}'(W)$ since $\mathscr{A}'$ is flabby. Then t and $t' + t''$ agree on $V \cap W$, so that together they define an element of $\mathscr{A}(V \cup W)$ extending t and representing s on $V \cup W$. This contradicts the maximality of (V,t) and shows that $V = X$.

Now put $U = X - |s|$ and note that $t|U \in \mathscr{A}'(U)$. We can extend $t|U$ to some $t' \in \mathscr{A}'(X)$ since $\mathscr{A}'$ is flabby. Then $t - t'$ represents s on X and is zero on U. Therefore $|t - t'| = |s| \in \Phi$.

The last statement follows from the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{A}'(X) & \to & \mathscr{A}(X) & \to & \mathscr{A}''(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{A}'(U) & \to & \mathscr{A}(U) & \to & \mathscr{A}''(U) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

and a diagram chase. $\square$

5.5. Theorem. *A flabby sheaf is Φ-acyclic for any Φ.*

Proof. Since $\mathscr{C}^0(X;\mathscr{A})$ is always flabby, it follows from the last part of 5.4 that $\mathscr{Z}^1(X;\mathscr{A})$ is flabby when $\mathscr{A}$ is flabby. By induction, all $\mathscr{Z}^n(X;\mathscr{A})$ are flabby when $\mathscr{A}$ is flabby.

Thus if we apply the functor Γ_Φ to the exact sequences

$$0 \to \mathscr{Z}^n(X;\mathscr{A}) \to \mathscr{C}^n(X;\mathscr{A}) \to \mathscr{Z}^{n+1}(X;\mathscr{A}) \to 0$$

$[n = 0, 1, \dots$ and where $\mathscr{Z}^0(X;\mathscr{A}) = \mathscr{A}]$, we obtain exact sequences. It follows immediately that the concatenated sequence

$$0 \to \Gamma_\Phi(\mathscr{A}) \to C^0_\Phi(X;\mathscr{A}) \to C^1_\Phi(X;\mathscr{A}) \to \cdots$$

is exact, as was to be shown. □

5.6. Corollary. *The functor $\mathscr{A} \mapsto H^p_\Phi(X;\mathscr{A})$ is the pth right derived functor of the left exact functor Γ_Φ.*

Proof. This follows immediately from 4.1, 5.3 and 5.5. □

5.7. Proposition. *If $f : X \to Y$ and $\mathscr{A}$ is a flabby sheaf on X, then $f\mathscr{A}$ is flabby on Y.*

Proof. This is immediate from the definition of $f\mathscr{A}$ and of "flabby." □

5.8. Corollary. *If $F \subset X$ is closed and $\mathscr{B}$ is a flabby sheaf on F, then $\mathscr{B}^X$ is flabby on X.*

Proof. If $i : F \hookrightarrow X$ is the inclusion, then $\mathscr{B}^X = i\mathscr{B}$. □

5.9. Proposition. *If $\{\mathscr{L}_\lambda \mid \lambda \in \Lambda\}$ is a family of flabby sheaves on X, then $\prod_\lambda \mathscr{L}_\lambda$ is flabby.*

Proof. This follows from the definition $(\prod_\lambda \mathscr{L}_\lambda)(U) = \prod_\lambda(\mathscr{L}_\lambda(U))$. □

A family $\{\mathscr{L}_\lambda \mid \lambda \in \Lambda\}$ of sheaves on X is said to be *locally finite* if each point $x \in X$ has a neighborhood U such that $\mathscr{L}_\lambda|U = 0$ for all but a finite collection of indices λ.

5.10. Corollary. *If $\{\mathscr{L}_\lambda \mid \lambda \in \Lambda\}$ is a locally finite family of sheaves on X then*

$$\boxed{H^p(X;\textstyle\prod_\lambda \mathscr{L}_\lambda) \approx \prod_\lambda H^p(X;\mathscr{L}_\lambda).}$$

If Λ is finite then this holds for arbitrary support families.

Proof. If we let $\mathscr{L}_\lambda^* = \mathscr{C}^*(X;\mathscr{L}_\lambda)$, then $\{\mathscr{L}_\lambda^*\}$ is also locally finite, and it follows that $(\prod_\lambda \mathscr{L}_\lambda^*)_x = \prod_\lambda(\mathscr{L}_\lambda^*)_x$ since locally near x, $\prod_\lambda \mathscr{L}_\lambda^* = \bigoplus_\lambda \mathscr{L}_\lambda^*$. Since $\prod_\lambda$ is exact, it follows that $\prod_\lambda \mathscr{L}_\lambda^*$ is a resolution of $\prod_\lambda \mathscr{L}_\lambda$ for locally finite collections. It is flabby by 5.9. The result then follows from 4.1, the fact that $\prod_\lambda$ commutes with Γ, and the fact that $\prod_\lambda$ is exact. The last statement follows from the fact that $\Gamma_\Phi(\prod_\lambda \mathscr{L}_\lambda^*) = \prod_\lambda(\Gamma_\Phi(\mathscr{L}_\lambda^*))$ when Λ is finite, due to the fact that the union of a finite collection of members of Φ is a member of Φ. □

5.11. Example. This example shows that 5.9 does not hold for (infinite) direct sums instead of direct products. Let $\mathscr{L}_n = \mathbb{Z}_{\{1/n\}}$ on $X = \mathbb{R}$. Then $\mathscr{L}_n$ is flabby but $\mathscr{L} = \bigoplus_n \mathscr{L}_n \approx \mathbb{Z}_{\{1,1/2,\dots\}}$ is not. (The unique serration of $\mathscr{L}$ that is 1 at all the points $1/n$ is continuous over $(0,\infty)$ but not at 0 since it does not coincide with the zero section in some neighborhood of 0.) Note that for this example, $(\mathscr{L}_n)_{\{0\}} = 0$ for all n and $(\bigoplus_n \mathscr{L}_n)_{\{0\}} = 0$, but $(\prod_n \mathscr{L}_n)_{\{0\}} \neq 0$. ◇

5.12. Example. This example shows that 5.10 does not hold for general support families. Let $\mathscr{L}_n = \mathbb{Z}_{\{n\}}$ on $X = \mathbb{R}$. Then $\{\mathscr{L}_n\}$ is locally finite. Clearly $H_c^0(X;\prod_n \mathscr{L}_n) = \Gamma_c(\prod_n \mathscr{L}_n) \approx \bigoplus_n \mathbb{Z}$, which is countable, while $\prod_n(H_c^0(X;\mathscr{L}_n)) = \prod_n(\Gamma_c(\mathscr{L}_n)) \approx \prod_n \mathbb{Z}$ is uncountable. ◇

5.13. Proposition. *Let L be a principal ideal domain. If $\mathscr{A}$ is a flabby sheaf of L-modules on X and $\mathscr{M}$ is a locally constant sheaf of L-modules on X with finitely generated stalks and with $\mathscr{A} *_L \mathscr{M} = 0$, then $\mathscr{A} \otimes_L \mathscr{M}$ is flabby.*

Proof. Since flabbiness is a local property by Exercise 10, it suffices to treat the case in which $\mathscr{M}$ is constant. Since a direct product of flabby sheaves is flabby by 5.9, it suffices to consider the case of a cyclic L-module $\mathscr{M} = L_k = L/kL$ for which $\mathscr{A}$ has no k-torsion. Then the exact sequence $0 \to \mathscr{A} \otimes_L L \xrightarrow{k} \mathscr{A} \otimes_L L \to \mathscr{A} \otimes_L L_k \to 0$ shows that $\mathscr{A} \otimes_L L_k$ is flabby by 5.4 since $\mathscr{A} \otimes_L L \approx \mathscr{A}$ is flabby. □

5.14. Corollary. *Let L be a principal ideal domain. Let $\mathscr{A}$ be a sheaf of L-modules on X and let $\mathscr{M}$ be a locally constant sheaf of L-modules on X with finitely generated stalks such that $\mathscr{A} *_L \mathscr{M} = 0$. Then with $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$, there is a natural isomorphism*

$$\boxed{H_\Phi^*(X;\mathscr{A} \otimes_L \mathscr{M}) \approx H^*(\Gamma_\Phi(\mathscr{A}^* \otimes_L \mathscr{M}))}$$

for any family Φ of supports on X. If $\mathscr{M}$ is constant with stalks M, then also

$$\boxed{H_\Phi^*(X;\mathscr{A} \otimes_L \mathscr{M}) \approx H^*(\Gamma_\Phi(\mathscr{A}^*) \otimes_L M).}$$

Proof. Since $\mathscr{A}^*$ has no torsion that $\mathscr{A}$ does not have, $\mathscr{A}^* *_L \mathscr{M} = 0$, and so $\mathscr{A}^* \otimes_L \mathscr{M}$ is a flabby resolution of $\mathscr{A} \otimes_L \mathscr{M}$ by 5.13 and 2.1. Thus the first statement follows from 4.1. If M is a finitely generated L-module, then there is an exact sequence $0 \to R \to F \to M \to 0$ with R and F free and finitely generated. The sequence

$$0 \to \Gamma_\Phi(\mathscr{A}^* \otimes_L R) \to \Gamma_\Phi(\mathscr{A}^* \otimes_L F) \to \Gamma_\Phi(\mathscr{A}^* \otimes_L M) \to 0$$

is exact since $\mathscr{A}^* \otimes_L R$ is flabby by 5.13. Now $\Gamma_\Phi(\mathscr{A}^* \otimes_L F) \approx \Gamma_\Phi(\mathscr{A}^*) \otimes_L F$, and similarly for R, since it is just a finite direct sum of copies of $\Gamma_\Phi(\mathscr{A}^*)$. It follows that $\Gamma_\Phi(\mathscr{A}^* \otimes_L M) \approx \Gamma_\Phi(\mathscr{A}^*) \otimes_L M$, whence the second statement follows from the first. □

5.15. We conclude this section with an improved version of the map γ of Section 4. Let $\mathscr{A}$ be a sheaf on X, $\mathscr{L}^*$ a resolution of $\mathscr{A}$, and $\mathscr{I}^*$ an injective resolution of $\mathscr{A}$. Denote $\mathscr{C}^*(X;\mathscr{A})$ by $\mathscr{A}^*$. We can find homomorphisms of resolutions (unique up to homotopy)

$$\mathscr{L}^* \to \mathscr{I}^* \leftarrow \mathscr{A}^*.$$

By 4.2 together with 5.3 and 5.5 the induced map

$$H^*(\Gamma_\Phi(\mathscr{A}^*)) \to H^*(\Gamma_\Phi(\mathscr{I}^*))$$

is an isomorphism. Let

$$\rho : H^*(\Gamma_\Phi(\mathscr{L}^*)) \to H^*_\Phi(X;\mathscr{A})$$

be the composition $H^*(\Gamma_\Phi(\mathscr{L}^*)) \to H^*(\Gamma_\Phi(\mathscr{I}^*)) \approx H^*(\Gamma_\Phi(\mathscr{A}^*)) = H^*_\Phi(X;\mathscr{A})$. It is easy to see that ρ does not depend on the choices made. Also, $\mathscr{I}^*$ could be replaced by any Φ-acyclic resolution, provided the indicated maps from $\mathscr{L}^*$ and $\mathscr{A}^*$ exist.

Note that ρ is an isomorphism when $\mathscr{L}^*$ is Φ-acyclic. One can see without much difficulty that $\rho = (-1)^{p(p+1)/2}\gamma$, where γ is the isomorphism of Section 4, but this is immaterial and will not be proved here; see [24, V-7.1]. It is of more importance that ρ is easier to work with. For example, when $\mathscr{L}^* = \mathscr{C}^*(X;\mathscr{A}) = \mathscr{A}^*$, it is clear that ρ is the identity.

Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves on X; let $\mathscr{L}^*$, $\mathscr{N}^*$, and $\mathscr{M}^*$ be resolutions of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ respectively; and let $\mathscr{L}^* \to \mathscr{N}^* \to \mathscr{M}^*$ extend $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$. Assume further that $0 \to \Gamma_\Phi(\mathscr{L}^*) \to \Gamma_\Phi(\mathscr{N}^*) \to \Gamma_\Phi(\mathscr{M}^*) \to 0$ is exact, so that there is an induced long exact sequence in homology. Then it can be shown that ρ maps this induced sequence into the cohomology sequence of $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ so as to form a commutative ladder. (Use the fact from 3.6 that a short exact sequence can always be extended to a short exact sequence of injective resolutions and use Exercise 47.)

6 Connected sequences of functors

In this section we prove an elementary theorem on functors of sheaves that will be basic in following sections. It will be useful in defining and proving the uniqueness of various homomorphisms involving cohomology groups. A basic reference is [58, Chapter XII]. Also see [80, Chapter III]. All sheaves in this section are sheaves on a *fixed* base space X. All functors will be *additive covariant* functors from the category of sheaves to that of abelian groups.[7]

A class $\mathscr{E}$ of short exact sequences $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ will be said to be *admissible* if the following three conditions hold:

- $\mathscr{E}$ is closed under isomorphisms of exact sequences.
- Every *pointwise split*[8] short exact sequence is in $\mathscr{E}$.
- If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is in $\mathscr{E}$, then regarding $\mathscr{A}'$ as a subsheaf of $\mathscr{A}$, the sequence

$$0 \to \mathscr{A}' \to \mathscr{C}^0(X;\mathscr{A}) \to \mathscr{C}^0(X;\mathscr{A})/\mathscr{A}' \to 0$$

  is in $\mathscr{E}$.

Remark: The class of pointwise split short exact sequences is admissible because there is a splitting $\mathscr{A}_x \to \mathscr{A}'_x$ by definition and a splitting $\mathscr{C}^0(X;\mathscr{A})_x \to \mathscr{A}_x$ as shown in Section 2. Also, any "proper class" in the sense of Mac Lane [58, p. 367] is admissible provided it contains all pointwise split sequences.

An *$\mathscr{E}$-connected sequence* of (covariant) functors (F^*, δ) is a sequence of functors $F^n : \mathscr{A} \mapsto F^n(\mathscr{A})$, n an integer, together with natural transformations $\delta : F^n(\mathscr{A}'') \to F^{n+1}(\mathscr{A}')$ defined for all exact sequences $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ in $\mathscr{E}$ such that the induced long sequence

$$\cdots \to F^n(\mathscr{A}') \to F^n(\mathscr{A}) \to F^n(\mathscr{A}'') \xrightarrow{\delta} F^{n+1}(\mathscr{A}') \to \cdots \tag{9}$$

is of order two, and such that a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{A}' & \to & \mathscr{A} & \to & \mathscr{A}'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{B}' & \to & \mathscr{B} & \to & \mathscr{B}'' & \to & 0 \end{array}$$

(the rows being in $\mathscr{E}$) induces a commutative diagram

$$\begin{array}{ccc} F^n(\mathscr{A}'') & \xrightarrow{\delta} & F^{n+1}(\mathscr{A}') \\ \downarrow & & \downarrow \\ F^n(\mathscr{B}'') & \xrightarrow{\delta} & F^{n+1}(\mathscr{B}'). \end{array}$$

[7] The results of this section remain true if we replace the category of all sheaves by that of $\mathscr{R}$-modules, where $\mathscr{R}$ is a fixed sheaf of rings with unit.

[8] See I-5.1.

"Connected" will stand for $\mathscr{E}$-connected when $\mathscr{E}$ consists of *all* short exact sequences.

An *$\mathscr{E}$-connected system of multifunctors* is a system $(F^{*,\dots,*}, \delta_1, ..., \delta_n)$, where $F^{*,\dots,*}$ is a system of functors of n variables (where n is fixed) indexed by n-tuples of integers, such that if all indices $p_1, ..., p_n$ are fixed except for the ith, and all variables $\mathscr{A}_1, ..., \mathscr{A}_n$ are fixed except the ith, then together with δ_i, we obtain an $\mathscr{E}$-connected sequence of functors of one variable, and we also require that $\delta_i\delta_j = \delta_j\delta_i$ for all i, j when this is defined.

6.1. We shall call a connected sequence of functors (F^*, δ) *fundamental* if the following three conditions hold:

- $F^p(\mathscr{A}) = 0$ if $p < 0$.
- $F^p(\mathscr{A}) = 0$ if $p > 0$ and $\mathscr{A}$ has the form $\mathscr{C}^0(X; \mathscr{B})$ for some sheaf $\mathscr{B}$.
- For any short exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$, the induced sequence (9) is *exact.*

These conditions imply that F^p is the pth right derived functor of the left exact functor F^0.[9] The standard examples are, of course, $F^*(\mathscr{A}) = H^*_\Phi(X; \mathscr{A})$ for any given family Φ of supports on X.

6.2. Theorem. *Suppose that $(F_1^*, \delta_1), \dots, (F_n^*, \delta_n)$ are fundamental connected sequences of functors and that $(F^{*,\dots,*}, \delta_1', \dots, \delta_n')$ is an $\mathscr{E}$-connected system of multifunctors of n variables, with $\mathscr{E}$ admissible. Assume that we are given a natural transformation of functors*

$$T^{0,\dots,0} : F_1^0(\mathscr{A}_1) \otimes \cdots \otimes F_n^0(\mathscr{A}_n) \longrightarrow F^{0,\dots,0}(\mathscr{A}_1, \dots, \mathscr{A}_n).$$

Then:

(a) *There exists a unique extension of $T^{0,\dots,0}$ to natural transformations*

$$T^{p_1,\dots,p_n} : F_1^{p_1}(\mathscr{A}_1) \otimes \cdots \otimes F_n^{p_n}(\mathscr{A}_n) \longrightarrow F^{p_1,\dots,p_n}(\mathscr{A}_1, \dots, \mathscr{A}_n)$$

that are compatible with respect to δ_i and δ_i' for all i; (see below for the meaning of "compatible").

(b) *If for fixed indices p_j, $j \neq i$, and for fixed variables $\mathscr{A}_j$, $j \neq i$, the sequence of functors $\mathscr{A}_i \mapsto F^{p_1,\dots,p_n}(\mathscr{A}_1, \dots, \mathscr{A}_n)$ is $\mathscr{E}'$-connected, for some admissible $\mathscr{E}' \supset \mathscr{E}$, by a transformation δ_i'' extending δ_i', then δ_i and δ_i'' are compatible.*

(c) *If $n = 1$ and (F^*, δ_1') is also fundamental, then:*

  (i) *If $T^0 : F_1^0(\mathscr{A}) \to F^0(\mathscr{A})$ is surjective for all $\mathscr{A}$, then so are all the T^p.*

  (ii) *If T^0 is injective for all $\mathscr{A}$ and is an isomorphism for $\mathscr{A}$ flabby, then T^p is an isomorphism for all $\mathscr{A}$ and p.*

[9] For one can resolve any sheaf by injective sheaves of the form $\mathscr{C}^0(X; \bullet)$.

Proof. The "compatibility" of δ_i and δ_i' means that if

$$0 \to \mathscr{A}_i' \to \mathscr{A}_i \to \mathscr{A}_i'' \to 0$$

is in $\mathscr{E}$ and if $\alpha_j \in F_j^{p_j}(\mathscr{A}_j)$ for $j \neq i$, and $\alpha_i \in F_i^{p_i}(\mathscr{A}_j'')$, then

$$T^{p_1,\dots,p_i+1,\dots,p_n}(\alpha_1 \otimes \cdots \otimes \delta_i\alpha_i \otimes \cdots \otimes \alpha_n) = \delta_i' T^{p_1,\dots,p_n}(\alpha_1 \otimes \cdots \otimes \alpha_n).$$

Let us use the abbreviations $\mathscr{Z}_i^p = \mathscr{Z}^p(X;\mathscr{A}_i)$ and $\mathscr{C}_i^p = \mathscr{C}^p(X;\mathscr{A}_i)$, where $\mathscr{Z}_i^0 = \mathscr{A}_i$ and $\mathscr{C}_i^{-1} = 0$. Also, for brevity, let

$$T = T^{0,\dots,0} \quad \text{and} \quad F = F^{0,\dots,0}.$$

Applying F_i^* to the sequences

$$0 \to \mathscr{Z}_i^p \to \mathscr{C}_i^p \to \mathscr{Z}_i^{p+1} \to 0 \tag{10}$$

we obtain the sequence of homomorphisms

$$F_i^0(\mathscr{C}_i^{p-1}) \to F_i^0(\mathscr{Z}_i^p) \twoheadrightarrow F_i^1(\mathscr{Z}_i^{p-1}) \xrightarrow{\approx} F_i^2(\mathscr{Z}_i^{p-2}) \xrightarrow{\approx} \cdots \xrightarrow{\approx} F_i^p(\mathscr{A}_i),$$

which is exact at the second term.

Then the composition of the δ_i of the sequences (10) yields canonical surjections

$$\gamma_i : F_i^0(\mathscr{Z}_i^{p_i}) \twoheadrightarrow F_i^{p_i}(\mathscr{A}_i)$$

whose kernel is the image of $\lambda_i : F_i^0(\mathscr{C}_i^{p_i-1}) \to F_i^0(\mathscr{Z}_i^{p_i})$.

Similarly for $F^{p_1,\dots,p_n}$, using the fact that $\delta_i'\delta_j' = \delta_j'\delta_i'$, the composition of the δ_i''s of the sequences (10) in *any order* yields a canonical homomorphism:

$$\eta : F(\mathscr{Z}_1^{p_1},\dots,\mathscr{Z}_n^{p_n}) \to F^{p_1,\dots,p_n}(\mathscr{A}_1,\dots,\mathscr{A}_n).$$

Moreover, the kernel of this map *contains* the images of all of the induced maps $\rho_i =$

$$T(1,\dots,\gamma_i,\dots,1) : F(\mathscr{Z}_1^{p_1},\dots,\mathscr{C}_i^{p_i-1},\dots,\mathscr{Z}_n^{p_n}) \to F(\mathscr{Z}_1^{p_1},\dots,\mathscr{Z}_i^{p_i-1},\dots,\mathscr{Z}_n^{p_n})$$

(the $\mathscr{C}$ term in only the ith variable) so that $\eta\rho_i = 0$. Now, if $b_i \in \mathscr{Z}_i^{p_i}$ and $\gamma_j(b_j) = 0$ for some j, then $b_j = \lambda_j(c_j)$ for some $c_j \in F_j^0(\mathscr{C}_j^{p_j-1})$, so that

$$\begin{aligned}\eta(T(b_1 \otimes \cdots \otimes b_j \otimes \cdots \otimes b_n)) &= \eta(T(b_1 \otimes \cdots \otimes \lambda_j(c_j) \otimes \cdots \otimes b_n)) \\ &= \eta(\rho_j(T(b_1 \otimes \cdots \otimes b_j \otimes \cdots \otimes b_n))) \\ &= 0.\end{aligned}$$

Therefore the map $T : F_1^0(\mathscr{Z}_1^{p_1}) \otimes \cdots \otimes F_n^0(\mathscr{Z}_n^{p_n}) \to F(\mathscr{Z}_1^{p_1},\dots,\mathscr{Z}_n^{p_n})$ induces a unique compatible map

$$T^{p_1,\dots,p_n} : F_1^{p_1}(\mathscr{A}_1) \otimes \cdots \otimes F_n^{p_n}(\mathscr{A}_n) \to F^{p_1,\dots,p_n}(\mathscr{A}_1,\dots,\mathscr{A}_n)$$

given by $T^{p_1,\dots,p_n}(a_1 \otimes \cdots \otimes a_n) = \eta(T(\gamma_1^{-1}(a_1) \otimes \cdots \otimes \gamma_n^{-1}(a_n)))$.

It is clear from the definition of $T^{p_1,\dots,p_n}$ that the δ's are compatible on the class of short exact sequences of the form $0 \to \mathscr{A} \to \mathscr{C}^0(\mathscr{A}) \to \mathscr{Z}^1(\mathscr{A}) \to 0$. The sequences (10) are of this form. Thus to complete the proof of (a) we need only prove (b). But then it suffices to prove the case of *one* variable, for if we fix p_j, $\mathscr{A}_j$, and $\alpha_j \in F_j^{p_j}(\mathscr{A}_j)$ for $j \neq i$, we need only consider the functors F_i^* and $\mathscr{B} \mapsto F^{p_1,\dots,p_n}(\mathscr{A}_1, \dots, \mathscr{B}, \dots, \mathscr{A}_n)$ of one variable $\mathscr{B} = \mathscr{A}_i$ and the transformation $S^{p_i}(\beta) = T^{p_1,\dots,p_n}(\alpha_1, \dots, \beta, \dots, \alpha_n)$: $F_i^{p_i}(\mathscr{B}) \to F^{p_1,\dots,p_n}(\mathscr{A}_1, \dots, \mathscr{B}, \dots, \mathscr{A}_n)$.

For one variable, (b) is contained in (a). Therefore we need only complete the proof of (a) for the case of one variable. Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be in $\mathscr{E}$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{A}' & \to & \mathscr{A} & \xrightarrow{j} & \mathscr{A}'' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \to & \mathscr{A}' & \to & \mathscr{C}^0(\mathscr{A}) & \xrightarrow{k} & \mathscr{C}^0(\mathscr{A})/\mathscr{A}' & \to & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle g} & & \\
0 & \to & \mathscr{A}' & \to & \mathscr{C}^0(\mathscr{A}') & \xrightarrow{l} & \mathscr{Z}^1(\mathscr{A}') & \to & 0
\end{array} \tag{11}$$

all of whose rows are in $\mathscr{E}$. Applying F_1 (respectively F) we obtain a commutative diagram[10]

$$\begin{array}{ccccccc}
F_1^p(\mathscr{A}) & \longrightarrow & F_1^p(\mathscr{A}'') & \xrightarrow{\delta_1} & F_1^{p+1}(\mathscr{A}') & & \\
\downarrow & & \downarrow{\scriptstyle f^*} & & \| & & \\
F_1^p(\mathscr{C}^0(\mathscr{A})) & \xrightarrow{k^*} & F_1^p(\mathscr{C}^0(\mathscr{A})/\mathscr{A}') & \xrightarrow{\delta_1} & F_1^{p+1}(\mathscr{A}') & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle g^*} & & \| & & \\
F_1^p(\mathscr{C}^0(\mathscr{A}')) & \xrightarrow{l^*} & F_1^p(\mathscr{Z}^1(\mathscr{A}')) & \xrightarrow{\delta_1} & F_1^{p+1}(\mathscr{A}') & \longrightarrow & 0
\end{array}$$

and a map of this into the corresponding diagram for F. Denoting δ_1' by δ, we must show that if $\alpha \in F_1^p(\mathscr{A}'')$, then

$$\delta T^p(\alpha) = T^{p+1}(\delta_1 \alpha).$$

However,

$$\delta T^p(\alpha) = \delta f^* T^p(\alpha) = \delta T^p(f^*\alpha) \quad \text{and} \quad T^{p+1}(\delta_1 \alpha) = T^{p+1}(\delta_1 f^* \alpha).$$

Thus it suffices to show, for $\alpha' \in F_1^p(\mathscr{C}^0(\mathscr{A})/\mathscr{A}')$, that

$$\delta T^p(\alpha') = T^{p+1}(\delta_1 \alpha'). \tag{12}$$

However, note that $F_1^p(\mathscr{Z}^1(\mathscr{A}'))$ maps *onto* $F_1^p(\mathscr{C}^0(\mathscr{A})/\mathscr{A}')/(\operatorname{Im} k^*)$. (Also note that the kernel is $\operatorname{Im} l^*$.) Thus it suffices to prove (12) for $\alpha' = g^*(\alpha'')$

[10]Note that for $p > 0$ the bottom two groups in the first column are zero.

for some $\alpha'' \in F_1^p(\mathscr{J}^1(\mathscr{A}'))$. We compute[11]

$$\begin{aligned}\delta T^p(\alpha') = \delta T^p(g^*\alpha'') = \delta g^* T^p(\alpha'') = \delta T^p(\alpha'')\\ =_{\text{def}} T^{p+1}(\delta_1\alpha'') = T^{p+1}(\delta_1 g^*\alpha'') = T^{p+1}(\delta_1\alpha')\end{aligned}$$

as was to be shown; completing the proof of (a) and (b).

Part (c) is an elementary application of the five-lemma, which will be left to the reader. □

Remark: The conditions defining a fundamental connected sequence are more restrictive than is necessary, but they are sufficient for our purposes. See [58, Chapter XII] in this regard.

6.3. Example. Let $f : X \to Y$ and let Φ and Ψ be families of supports on X and Y respectively such that $f^{-1}\Psi \subset \Phi$. Then $F_1^*(\mathscr{B}) = H_\Psi^*(Y;\mathscr{B})$ is a fundamental connected sequence of functors of sheaves $\mathscr{B}$ on Y. Also, $F_2^*(\mathscr{B}) = H_\Phi^*(X; f^*\mathscr{B})$ is a connected system of functors of $\mathscr{B}$, since $\mathscr{B} \mapsto f^*\mathscr{B}$ is exact. We have the natural transformation

$$F_1^0(\mathscr{B}) = \Gamma_\Psi(\mathscr{B}) \to \Gamma_{f^{-1}\Psi}(f^*\mathscr{B}) \hookrightarrow \Gamma_\Phi(f^*\mathscr{B}) = F_2^0(\mathscr{B}).$$

Therefore, Theorem 6.2 gives us a natural transformation

$$\boxed{f^* : H_\Psi^*(Y;\mathscr{B}) \to H_\Phi^*(X; f^*\mathscr{B})}$$

of functors compatible with connecting homomorphisms. This means that an exact sequence $0 \to \mathscr{B}' \to \mathscr{B} \to \mathscr{B}'' \to 0$ of sheaves gives rise to a commutative ladder

$$\begin{array}{ccccccccc}\cdots\to & H_\Psi^p(Y;\mathscr{B}') & \to & H_\Psi^p(Y;\mathscr{B}) & \to & H_\Psi^p(Y;\mathscr{B}'') & \to & H_\Psi^{p+1}(Y;\mathscr{B}') & \to\cdots\\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\to & H_\Phi^p(X;f^*\mathscr{B}') & \to & H_\Phi^p(X;f^*\mathscr{B}) & \to & H_\Phi^p(X;f^*\mathscr{B}'') & \to & H_\Phi^{p+1}(X;f^*\mathscr{B}') & \to\cdots\end{array}$$

in cohomology. This will be generalized and discussed at more length in Section 8. ◇

7 Axioms for cohomology and the cup product

A *cohomology theory* (in the sense of Cartan) on a space X with supports in Φ is a *fundamental* connected sequence of functors $\overline{H}_\Phi^*(X;\bullet)$ (additive and covariant from the category of sheaves on X to that of abelian groups) together with a natural isomorphism of functors

$$T^0 : \Gamma_\Phi(\bullet) \xrightarrow{\approx} \overline{H}_\Phi^0(X;\bullet).$$

[11] In this computation g^* stands for both $F^p(g)$ and $F_1^p(g)$ and similarly for f^*.

By Theorem 6.2, any cohomology theory is naturally isomorphic to $H^*_\Phi(X;\bullet)$.

The following theorem similarly defines and gives axioms for the cup product in sheaf cohomology.

7.1. Theorem. *Let Φ, Ψ be families of supports on X. Then there exists a unique natural transformation of functors (on the category of sheaves on X to abelian groups)*

$$\cup : H^p_\Phi(X;\mathscr{A}) \otimes H^q_\Psi(X;\mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B}) \tag{13}$$

called the "cup product" and satisfying the following three properties:

(a) *For $p = q = 0$, $\cup : \Gamma_\Phi(\mathscr{A}) \otimes \Gamma_\Psi(\mathscr{B}) \to \Gamma_{\Phi\cap\Psi}(\mathscr{A}\otimes\mathscr{B})$ is the transformation induced by the canonical map $\mathscr{A}(X)\otimes\mathscr{B}(X) \to (\mathscr{A}\otimes\mathscr{B})(X)$ of* I-Section 5.

(b) *If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ and $0 \to \mathscr{A}'\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}''\otimes\mathscr{B} \to 0$ are exact, then for $\alpha \in H^p_\Phi(X;\mathscr{A}'')$ and $\beta \in H^q_\Psi(X;\mathscr{B})$ we have $\delta(\alpha\cup\beta) = \delta\alpha\cup\beta$.*

(c) *If $0 \to \mathscr{B}' \to \mathscr{B} \to \mathscr{B}'' \to 0$ and $0 \to \mathscr{A}\otimes\mathscr{B}' \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B}'' \to 0$ are exact, then for $\alpha \in H^p_\Phi(X;\mathscr{A})$ and $\beta \in H^q_\Psi(X;\mathscr{B}'')$ we have $\delta(\alpha\cup\beta) = (-1)^p\alpha\cup\delta\beta$.*

Proof. Let $F_1^p(\mathscr{A}) = H^p_\Phi(X;\mathscr{A})$ and $F_2^q(\mathscr{B}) = H^q_\Psi(X;\mathscr{B})$, and

$$F^{p,q}(\mathscr{A},\mathscr{B}) = H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B}).$$

Let $\mathscr{E}$ be the class of pointwise split short exact sequences.

If $E_1 : 0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is in $\mathscr{E}$, then $0 \to \mathscr{A}'\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}''\otimes\mathscr{B} \to 0$ is exact for all $\mathscr{B}$, and we take $\delta_1' : F^{p,q}(\mathscr{A}'',\mathscr{B}) \to F^{p+1,q}(\mathscr{A}',\mathscr{B})$ to be the connecting homomorphism δ on $H^{p+q}_{\Phi\cap\Psi}(X;\bullet)$ of this sequence.

Similarly, if $E_2 : 0 \to \mathscr{B}' \to \mathscr{B} \to \mathscr{B}'' \to 0$ is in $\mathscr{E}$, then $0 \to \mathscr{A}\otimes\mathscr{B}' \to \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}\otimes\mathscr{B}'' \to 0$ is exact, and $\delta_2' : F^{p,q}(\mathscr{A},\mathscr{B}'') \to F^{p,q+1}(\mathscr{A},\mathscr{B}')$ is taken to be $(-1)^p\delta$.

If E_1 and E_2 are both in $\mathscr{E}$, then applying $C^*_{\Phi\cap\Psi}$ to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{A}'\otimes\mathscr{B}' & \to & \mathscr{A}\otimes\mathscr{B}' & \to & \mathscr{A}''\otimes\mathscr{B}' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{A}'\otimes\mathscr{B} & \to & \mathscr{A}\otimes\mathscr{B} & \to & \mathscr{A}''\otimes\mathscr{B} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{A}'\otimes\mathscr{B}'' & \to & \mathscr{A}\otimes\mathscr{B}'' & \to & \mathscr{A}''\otimes\mathscr{B}'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we obtain a commutative diagram of chain complexes, and it is a well-known algebraic fact that the induced square

$$\begin{array}{ccc} H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}''\otimes\mathscr{B}'') & \xrightarrow{\delta} & H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}'\otimes\mathscr{B}'') \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}''\otimes\mathscr{B}') & \xrightarrow{\delta} & H^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}'\otimes\mathscr{B}') \end{array}$$

anticommutes. Thus, by definition, $\delta_1'\delta_2' = \delta_2'\delta_1'$.

The theorem now follows from 6.2. [To get the full (b), one needs to show that the class of sequences $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ satisfying the conditions of (b), for $\mathscr{B}$ given, is admissible; and similarly for (c). This is immediate because the only thing needing proof is that $\mathscr{A}'\otimes\mathscr{B} \to \mathscr{C}^0(X;\mathscr{A})\otimes\mathscr{B}$ is a monomorphism, and this is just the composition of the monomorphisms $\mathscr{A}'\otimes\mathscr{B} \rightarrowtail \mathscr{A}\otimes\mathscr{B}$ and $\mathscr{A}\otimes\mathscr{B} \rightarrowtail \mathscr{C}^0(X;\mathscr{A})\otimes\mathscr{B}$, the latter being a monomorphism because the canonical monomorphism $\mathscr{A} \rightarrowtail \mathscr{C}^0(X;\mathscr{A})$ splits pointwise.] □

Remark: For uniqueness, (b) and (c) can be restricted to sequences of the form

$$0 \to \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A}) \to \mathscr{Z}^1(X;\mathscr{A}) \to 0.$$

7.2. Corollary. *Let $\tau : \mathscr{B}\otimes\mathscr{A} \to \mathscr{A}\otimes\mathscr{B}$ be the canonical isomorphism. Then for $\alpha \in H^p_\Phi(X;\mathscr{A})$ and $\beta \in H^q_\Psi(X;\mathscr{B})$, we have that*

$$\boxed{\alpha \cup \beta = (-1)^{pq}\tau^*(\beta\cup\alpha).}$$

Proof. If $T : H^p_\Phi(X;\mathscr{A})\otimes H^q_\Psi(X;\mathscr{B}) \to H^q_\Psi(X;\mathscr{B})\otimes H^p_\Phi(X;\mathscr{A})$ is given by $T(\alpha\otimes\beta) = (-1)^{pq}\beta\otimes\alpha$, then $\tau^*\circ\cup\circ T$ is immediately seen to satisfy the hypotheses of 7.1, and so it must coincide with the cup product. Hence $\alpha\cup\beta = (\tau^*\circ\cup\circ T)(\alpha\cup\beta) = \tau^*((-1)^{pq}\beta\cup\alpha)$. □

Remark: Clearly either of the conditions (b) or (c) of 7.1 could be replaced by the commutativity formula of 7.2.

Remark: If we restrict attention to the category of $\mathscr{R}$-modules where $\mathscr{R}$ is a sheaf of rings, then in (13) $\mathscr{A}\otimes\mathscr{B}$ can be replaced by the tensor product over $\mathscr{R}$ and the other tensor product by the tensor product over $\Gamma(\mathscr{R})$. This is an easy consequence of the following result.

7.3. Proposition. *The cup product is associative.*

Proof. We are to show that the two ways of defining the map

$$H^p_\Phi(X;\mathscr{A})\otimes H^q_\Psi(X;\mathscr{B})\otimes H^r_\Theta(X;\mathscr{C}) \to H^{p+q+r}_{\Phi\cap\Psi\cap\Theta}(X;\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{C})$$

agree. Thus we apply Theorem 6.2 to the $\mathscr{E}$-connected system of trifunctors (with $\mathscr{E}$ being the class of pointwise split short exact sequences)

$$F^{p,q,r}(\mathscr{A},\mathscr{B},\mathscr{C}) = H^{p+q+r}_{\Phi\cap\Psi\cap\Theta}(X;\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{C}),$$

where if δ is the ordinary connecting homomorphism of $H^*_{\Phi\cap\Psi\cap\Theta}$, then we put $\delta_1' = \delta$, $\delta_2' = (-1)^p\delta$, and $\delta_3' = (-1)^{p+q}\delta$ on $F^{p,q,r}$. The result follows immediately. □

If $\mathscr{R}$ is a sheaf of rings, then the product $\mathscr{R}\otimes\mathscr{R} \to \mathscr{R}$ together with the cup product makes $H^*_\Phi(X;\mathscr{R})$ into a ring (with unit if $\mathscr{R}$ has a unit and Φ consists of all closed sets) called the *cohomology ring* of X. Moreover, for any $\mathscr{R}$-module $\mathscr{A}$, the map $\mathscr{R}\otimes\mathscr{A} \to \mathscr{A}$ makes $H^*_\Phi(X;\mathscr{A})$ into an $H^*_\Psi(X;\mathscr{R})$-module for any $\Psi \supset \Phi$.

Let X and Y be spaces; $\mathscr{A}$ and $\mathscr{B}$ sheaves on X and Y respectively; and Φ and Ψ families of supports on X and Y respectively. Let $\pi_X : X\times Y \to X$ and $\pi_Y : X\times Y \to Y$ be the projections. We use the notation $\Phi\times Y = \pi_X^{-1}(\Phi)$ and $X\times\Psi = \pi_Y^{-1}(\Psi)$. Note that $\Phi\times\Psi = (\Phi\times Y)\cap(X\times\Psi)$. In Example 6.3 we defined a natural homomorphism

$$\pi_X^* : H^p_\Phi(X;\mathscr{A}) \to H^p_{\Phi\times Y}(X\times Y;\pi_X^*(\mathscr{A}))$$

and similarly with π_Y^*.

Following π_X^* and π_Y^* with the cup product, we obtain the homomorphism

$$\boxed{\times : H^p_\Phi(X;\mathscr{A})\otimes H^q_\Psi(Y;\mathscr{B}) \to H^{p+q}_{\Phi\times\Psi}(X\times Y;\mathscr{A}\widehat{\otimes}\mathscr{B}),}$$

called the *cross product.* Thus

$$\boxed{\alpha\times\beta = \pi_X^*(\alpha)\cup\pi_Y^*(\beta).}$$

We shall allow the reader to develop the properties of this product.

7.4. Let L be a ring with unit and let $\mathscr{A}$ and $\mathscr{B}$ be sheaves of L-modules on X. Let $\beta\in\Gamma(\mathscr{B}) = H^0(X;\mathscr{B})$. Then β induces a homomorphism

$$h_{\mathscr{A},\beta} : \mathscr{A} \to \mathscr{A}\otimes_L\mathscr{B}$$

by $h_{\mathscr{A},\beta}(a)(x) = a\otimes\beta(x)$ for $a\in\mathscr{A}_x$. Therefore we have the induced homomorphism

$$h^*_{\mathscr{A},\beta} : H^p_\Phi(X;\mathscr{A}) \to H^p_\Phi(X;\mathscr{A}\otimes_L\mathscr{B}). \tag{14}$$

It follows from 6.2 that

$$h^*_{\mathscr{A},\beta}(\alpha) = \alpha\cup\beta.$$

(The reader might investigate generalizing this to the case $\beta\in\Gamma_\Psi(\mathscr{B})$.) If $\mathscr{B} = L$ and $\beta = 1 \in \Gamma(L)$, then $h_{\mathscr{A},1}$ is the canonical isomorphism $\mathscr{A} \xrightarrow{\approx} \mathscr{A}\otimes_L L$ and so $h^*_{\mathscr{A},1}(\alpha) = \alpha$, whence

$$\boxed{\alpha\cup 1 = \alpha.}$$

(This also follows directly from 6.2.)

Note that (14) factors as

$$H^p_\Phi(X;\mathscr{A}) \xrightarrow{\approx} H^p_\Phi(X;\mathscr{A}\otimes_L L) \xrightarrow{(1\otimes\beta)^*} H^p_\Phi(X;\mathscr{A}\otimes_L \mathscr{B}),$$

where $1\otimes\beta : \mathscr{A}\otimes_L L \to \mathscr{A}\otimes_L \mathscr{B}$ is $(1\otimes\beta)(a_x\otimes\lambda_x) = a_x\otimes\lambda_x\beta(x)$. In particular, if the subsheaf $h_{L,\beta}(L) = \operatorname{Im}\{h_{L,\beta} : L \to L\otimes_L \mathscr{B}\approx\mathscr{B}\}$ of $\mathscr{B}$ is a direct summand, then $h^*_{\mathscr{A},\beta}$ is a monomorphism. Particularly note this in the case in which L is a field and the coefficient sheaves are the constant sheaves A and B (vector spaces over L). Then any element $0\neq b\in B$ induces a *monomorphism*

$$h^*_{A,b} : H^p_\Phi(X;A) \rightarrowtail H^p_\Phi(X;A\otimes_L B),$$

which is given by

$$h^*_{A,b}(\alpha) = \alpha\cup\beta,$$

where $\beta\in H^0(X;B)$ is the constant section with value b. It follows that $\alpha\cup\beta\neq 0$ when $\alpha\neq 0$ in this case.

In particular, if X is connected and A, B are vector spaces over the *field* L, then

$$\boxed{0\neq\alpha\in H^p_\Phi(X;A),\ 0\neq\beta\in H^0(X;B) \ \Rightarrow\ 0\neq\alpha\cup\beta\in H^p_\Phi(X;A\otimes_L B).}$$

7.5. We conclude this section with remarks concerning the "computation" of the cup product by means of resolutions.

Suppose that $\mathscr{A}\mapsto\mathscr{L}^*(\mathscr{A})$ (respectively, $\mathscr{N}^*$ and $\mathscr{M}^*$) are exact functors carrying a sheaf $\mathscr{A}$ into a resolution of $\mathscr{A}$ by Φ-acyclic (resp., Ψ-acyclic and $\Phi\cap\Psi$-acyclic) sheaves. Suppose, moreover, that we are given a functorial homomorphism of differential sheaves $\mathscr{L}^*(\mathscr{A})\otimes\mathscr{N}^*(\mathscr{B}) \to \mathscr{M}^*(\mathscr{A}\otimes\mathscr{B})$ (where the source has the total degree and differential $1\otimes\delta+\delta\otimes 1$, with the usual sign convention $(1\otimes\delta)(a\otimes b) = (-1)^{\deg a}a\otimes\delta b$; see, for example, [19, Chapter VI]).

Then we have the natural map

$$\Gamma_\Phi(\mathscr{L}^*(\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{N}^*(\mathscr{B})) \to \Gamma_{\Phi\cap\Psi}(\mathscr{L}^*(\mathscr{A})\otimes\mathscr{N}^*(\mathscr{B})) \to \Gamma_{\Phi\cap\Psi}(\mathscr{M}^*(\mathscr{A}\otimes\mathscr{B})),$$

which induces a product

$$H^p(\Gamma_\Phi(\mathscr{L}^*(\mathscr{A})))\otimes H^q(\Gamma_\Psi(\mathscr{N}^*(\mathscr{B}))) \to H^{p+q}(\Gamma_{\Phi\cap\Psi}(\mathscr{M}^*(\mathscr{A}\otimes\mathscr{B}))).$$

We claim that under the isomorphisms ρ of 5.15 this becomes the cup product. This is just a matter of easy verification of the axioms for the cup product, which will be left to the reader.

For example, with the notation of Section 2, the map

$$\cup : M^p(X;\mathscr{A})\otimes M^q(X;\mathscr{B}) \to M^{p+q}(X;\mathscr{A}\otimes\mathscr{B})$$

defined by

$$(f\cup g)(x_0,\dots,x_{p+q}) = S(f(x_0,\dots,x_p))(x_{p+q})\otimes g(x_p,\dots,x_{p+q})$$

is easily seen to induce a functorial product

$$\cup : \mathscr{F}^p(X;\mathscr{A}) \otimes \mathscr{F}^q(X;\mathscr{B}) \to \mathscr{F}^{p+q}(X;\mathscr{A}\otimes\mathscr{B})$$

[with $d(\alpha\cup\beta) = d\alpha\cup\beta + (-1)^p\alpha\cup d\beta$], which must induce the cup product in cohomology by the preceding remarks.[12]

Our next remarks will depend on some notions to be defined later in this book. Let L be a principal ideal domain considered as a ground ring. We also consider L to be a constant sheaf on X. Let Φ and Ψ be paracompactifying families of supports on X, and let $\mathscr{L}^*$ be a resolution of L consisting of sheaves that are torsion free and both Φ-fine and Ψ-fine (varieties of acyclic sheaves to be defined in Section 9), and for which there is a homomorphism $h : \mathscr{L}^* \otimes \mathscr{L}^* \to \mathscr{L}^*$ of differential sheaves. (We shall see in Chapter III that this is the case for the Alexander-Spanier and the de Rham resolutions.)

Since $\mathscr{L}^*$ is torsion free, $\mathscr{L}^* \otimes \mathscr{A}$ is a resolution of $\mathscr{A}$ for any sheaf $\mathscr{A}$ of L-modules. In the preceding discussion, put $\mathscr{L}^*(\mathscr{A}) = \mathscr{L}^*\otimes\mathscr{A}$, $\mathscr{N}^*(\mathscr{B}) = \mathscr{L}^*\otimes\mathscr{B}$, and $\mathscr{M}^*(\mathscr{A}\otimes\mathscr{B}) = \mathscr{L}^*\otimes\mathscr{A}\otimes\mathscr{B}$. It will be shown in Section 9 that $\mathscr{L}^*\otimes\mathscr{A}$ is exact in $\mathscr{A}$ and is Φ- (and Ψ and $\Phi\cap\Psi$)-acyclic, and hence the preceding remarks apply to show that the cup product

$$H^p_\Phi(X;\mathscr{A})\otimes H^q_\Psi(X;\mathscr{B}) \to H^{p+q}_{\Phi\times\Psi}(X\times Y;\mathscr{A}\otimes\mathscr{B})$$

is induced by $\Gamma_\Phi(\mathscr{L}^*\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{L}^*\otimes\mathscr{B}) \to \Gamma_{\Phi\cap\Psi}(\mathscr{L}^*\otimes\mathscr{A}\otimes\mathscr{L}^*\otimes\mathscr{B}) \approx \Gamma_{\Phi\cap\Psi}(\mathscr{L}^*\otimes\mathscr{L}^*\otimes\mathscr{A}\otimes\mathscr{B}) \xrightarrow{h\otimes1\otimes1} \Gamma_{\Phi\cap\Psi}(\mathscr{L}^*\otimes\mathscr{A}\otimes\mathscr{B})$.

We conclude this section with another description of the cup product via resolutions. Using the fact from Section 2 that $\mathscr{C}^*(X;\mathscr{A})$ is pointwise homotopically trivial, it follows that the total differential sheaf $\mathscr{C}^*(X;\mathscr{A})\otimes\mathscr{C}^*(X;\mathscr{B})$ is a resolution of $\mathscr{A}\otimes\mathscr{B}$. Thus, by 5.15 we have the map

$$\rho : H^*(\Gamma_{\Phi\cap\Psi}(\mathscr{C}^*(X;\mathscr{A})\otimes\mathscr{C}^*(X;\mathscr{B}))) \to H^*_{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B}),$$

which, when combined with the natural map

$$\Gamma_\Phi(\mathscr{C}^*(X;\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{C}^*(X;\mathscr{B})) \to \Gamma_{\Phi\cap\Psi}(\mathscr{C}^*(X;\mathscr{A})\otimes\mathscr{C}^*(X;\mathscr{B}))$$

(and the map $H^p(A^*)\otimes H^q(B^*) \to H^{p+q}(A^*\otimes B^*)$ from ordinary homological algebra), yields a product satisfying the axioms for the cup product.

8 Maps of spaces

Let $f : X \to Y$ be a map and let $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ be an f-cohomomorphism from the sheaf $\mathscr{B}$ on Y to $\mathscr{A}$ on X.

[12] It is shown in [40] that a similar construction is possible with the canonical resolution $\mathscr{C}^0(X;\bullet)$. Also see 21.2.

For $U \subset Y$, we have the induced map $k_U^0 : C^0(U; \mathscr{B}) \to C^0(f^{-1}(U); \mathscr{A})$ defined by taking a serration $s : U \dashrightarrow \mathscr{B}$ into the serration

$$k_U^0(s) : f^{-1}(U) \dashrightarrow \mathscr{A}$$

given by

$$k_U^0(s)(x) = k_x(s(f(x))).$$

These evidently form an f-cohomomorphism $\mathscr{C}^0(Y; \mathscr{B}) \rightsquigarrow \mathscr{C}^0(X; \mathscr{A})$ commuting with k and the canonical monomorphisms $\mathscr{A} \rightarrowtail \mathscr{C}^0(X; \mathscr{A})$ and $\mathscr{B} \rightarrowtail \mathscr{C}^0(X; \mathscr{B})$. This yields an f-cohomomorphism of the quotient sheaves $\mathscr{Z}^1(\bullet; \bullet)$ by I-4.3. By induction, we obtain an f-cohomomorphism

$$k^* : \mathscr{C}^*(Y; \mathscr{B}) \rightsquigarrow \mathscr{C}^*(X; \mathscr{A})$$

of resolutions (i.e., commuting with differentials).

It is clear that for any f-cohomomorphism $k : \mathscr{B} \rightsquigarrow \mathscr{A}$, the induced map $k_Y : \mathscr{B}(Y) \to \mathscr{A}(X)$ satisfies $|k_Y(s)| \subset f^{-1}(|s|)$. Consequently, if Φ and Ψ are families of supports on X and Y respectively with $f^{-1}\Psi \subset \Phi$, then

$$k_Y : \Gamma_\Psi(\mathscr{B}) \to \Gamma_\Phi(\mathscr{A}).$$

Thus k^* induces the chain map

$$k_Y^* : C_\Phi^*(Y; \mathscr{B}) \to C_\Psi^*(X; \mathscr{A})$$

and hence gives rise to a homomorphism

$$\boxed{k^* : H_\Psi^*(Y; \mathscr{B}) \to H_\Phi^*(X; \mathscr{A}) \quad \text{when } f^{-1}\Psi \subset \Phi.} \tag{15}$$

Noting that $\mathscr{C}^*(\bullet; \bullet)$ is functorial with respect to cohomomorphisms, we see that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps, $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ is an f-cohomomorphism, and $j : \mathscr{C} \rightsquigarrow \mathscr{B}$ is a g-cohomomorphism, then in cohomology with suitable supports we have that

$$\boxed{(k \circ j)^* = k^* \circ j^*.}$$

Recall that any f-cohomomorphism $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ factors in two ways:

$$\begin{array}{ccc} \mathscr{B} & \overset{f^*}{\rightsquigarrow} & f^*\mathscr{B} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} \\ f\mathscr{A} & \overset{f}{\rightsquigarrow} & \mathscr{A} \end{array}$$

where j and h are homomorphisms.[13] Thus the induced square in cohomology commutes, with composition k^*. Thus k^* is determined by either of the natural transformations of functors

$$\boxed{f^* : H_\Psi^*(Y; \mathscr{B}) \to H_\Phi^*(X; f^*\mathscr{B})} \tag{16}$$

[13]Recall that homomorphisms are special cases of cohomomorphisms.

and

$$\boxed{f^\dagger : H^*_\Psi(Y; f\mathscr{A}) \to H^*_\Phi(X; \mathscr{A}),} \tag{17}$$

which are induced by the f-cohomomorphisms $f^* : \mathscr{B} \rightsquigarrow f^*\mathscr{B}$ and $f : f\mathscr{A} \rightsquigarrow \mathscr{A}$ respectively. That is, we have

$$\boxed{k^* = h^* \circ f^* = f^\dagger \circ j^*.}$$

[Of course, f^* (with $\mathscr{B}$, and hence $f^*\mathscr{B}$, constant) is the one of these most familiar to readers.]

Since the functor $\mathscr{B} \mapsto f^*\mathscr{B}$ is exact, we see easily from the definition that (16) commutes with connecting homomorphisms. Thus, by 6.2, it coincides with the version of f^* defined in 6.3.

In case $\mathscr{A} = f^*\mathscr{B}$, we have $h = 1$ and $j = \beta : \mathscr{B} \to ff^*\mathscr{B}$, the canonical homomorphism of I-4. Thus

$$\boxed{f^* = f^\dagger \circ \beta^*,} \tag{18}$$

which is the composition

$$\boxed{f^* : H^*_\Psi(Y; \mathscr{B}) \xrightarrow{\beta^*} H^*_\Psi(Y; ff^*\mathscr{B}) \xrightarrow{f^\dagger} H^*_\Phi(X; f^*\mathscr{B}).}$$

8.1. Suppose now that $\mathscr{B}^*$ and $\mathscr{A}^*$ are resolutions of $\mathscr{B}$ and $\mathscr{A}$ respectively and that

$$g^* : \mathscr{B}^* \rightsquigarrow \mathscr{A}^*$$

is an f-cohomomorphism of resolutions[14] extending $k : \mathscr{B} \rightsquigarrow \mathscr{A}$. Let $\mathscr{I}^*$ be an injective resolution of $\mathscr{B}$ and $\mathscr{L}^*$ an injective resolution of $\mathscr{A}$. Then $f^*\mathscr{I}^*$ is a resolution of $f^*\mathscr{B}$, since f^* is exact, and we can construct homomorphisms $\mathscr{B}^* \xrightarrow{\varphi} \mathscr{I}^*$, $\mathscr{A}^* \xrightarrow{\psi} \mathscr{L}^*$, and $f^*\mathscr{I}^* \xrightarrow{\gamma} \mathscr{L}^*$ of resolutions that are unique up to homotopy and where γ extends the canonical map $h : f^*\mathscr{B} \to \mathscr{A}$ induced by the f-cohomomorphism $k : \mathscr{B} \rightsquigarrow \mathscr{A}$. We have the diagram

$$\begin{array}{ccccc} \mathscr{B}^* & \xrightarrow{f^*} & f^*\mathscr{B}^* & \xrightarrow{h^*} & \mathscr{A}^* \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f^*(\varphi)} & & \downarrow{\scriptstyle\psi} \\ \mathscr{I}^* & \xrightarrow{f^*} & f^*\mathscr{I}^* & \xrightarrow{\gamma} & \mathscr{L}^* \end{array}$$

in which the composition along the top is g^*, the left-hand square commutes, and the right-hand square commutes up to chain homotopy. Taking sections, we obtain the square

$$\begin{array}{ccc} \Gamma_\Psi(\mathscr{B}^*) & \xrightarrow{g^*} & \Gamma_\Phi(\mathscr{A}^*) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \Gamma_\Psi(\mathscr{I}^*) & \xrightarrow{\eta} & \Gamma_\Phi(\mathscr{L}^*) \end{array}$$

[14] I.e., commuting with differentials and augmentations.

which commutes up to chain homotopy. By definition, φ and ψ induce the maps denoted by ρ in 5.15.

The special case in which $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$, $\mathscr{B}^* = \mathscr{C}^*(X;\mathscr{B})$, and $g^* = k^*$ shows that the map η induces the canonical homomorphism

$$k^* : H^*_\Psi(Y;\mathscr{B}) \to H^*_\Phi(X;\mathscr{A})$$

of (15). The general case then shows that the diagram

$$\begin{array}{ccc} H^*(\Gamma_\Psi(\mathscr{B}^*)) & \xrightarrow{g^*} & H^*(\Gamma_\Phi(\mathscr{A}^*)) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ H^*_\Psi(Y;\mathscr{B}) & \xrightarrow{k^*} & H^*_\Phi(X;\mathscr{A}) \end{array}$$

commutes. If $\mathscr{A}^*$ is Φ-acyclic and $\mathscr{B}^*$ is Ψ-acyclic, then the vertical maps in this diagram are isomorphisms. This shows that the canonical homomorphism k^* of (15) can be "computed" (via the isomorphisms ρ of 5.15) from any f-cohomomorphism $g^* : \mathscr{B}^* \rightsquigarrow \mathscr{A}^*$ of acyclic resolutions of $\mathscr{B}$ and $\mathscr{A}$ that extends the given f-cohomomorphism $k : \mathscr{B} \rightsquigarrow \mathscr{A}$.

8.2. We conclude this section by indicating the proof that (16) preserves cup products. Let Ψ_i, Φ_i, $i = 1, 2$, be support families on Y and X respectively with $f^{-1}(\Psi_i) \subset \Phi_i$ and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be sheaves on Y. The cohomomorphisms $\mathscr{B}_i \rightsquigarrow f^*\mathscr{B}_i$ induce a cohomomorphism of presheaves $\mathscr{B}_1(U) \otimes \mathscr{B}_2(U) \rightsquigarrow f^*\mathscr{B}_1(f^{-1}(U)) \otimes f^*\mathscr{B}_2(f^{-1}(U))$ and hence a cohomomorphism of the induced sheaves $\mathscr{B}_1 \otimes \mathscr{B}_2 \rightsquigarrow f^*\mathscr{B}_1 \otimes f^*\mathscr{B}_2$. There is the factorization of this: $\mathscr{B}_1 \otimes \mathscr{B}_2 \rightsquigarrow f^*(\mathscr{B}_1 \otimes \mathscr{B}_2) \to f^*\mathscr{B}_1 \otimes f^*\mathscr{B}_2$. The latter homomorphism is fairly clearly an isomorphism on stalks, and hence it is an isomorphism of sheaves. Then we have the commutative diagram

$$\begin{array}{ccccc} \mathscr{B}_1(Y) \otimes \mathscr{B}_2(Y) & & \longrightarrow & & f^*\mathscr{B}_1(X) \otimes f^*\mathscr{B}_2(X) \\ \downarrow & & & & \downarrow \\ (\mathscr{B}_1 \otimes \mathscr{B}_2)(Y) & \to & f^*(\mathscr{B}_1 \otimes \mathscr{B}_2)(X) & \xrightarrow{\approx} & (f^*\mathscr{B}_1 \otimes f^*\mathscr{B}_2)(X). \end{array} \tag{19}$$

We wish to show that the diagram

$$\begin{array}{ccc} H^*_{\Psi_1}(Y;\mathscr{B}_1) \otimes H^*_{\Psi_2}(Y;\mathscr{B}_2) & \xrightarrow{\cup} & H^*_{\Psi_1\cap\Psi_2}(Y;\mathscr{B}_1 \otimes \mathscr{B}_2) \\ \downarrow{\scriptstyle f^*\otimes f^*} & & \downarrow{\scriptstyle f^*} \\ H^*_{\Phi_1}(X;f^*\mathscr{B}_1) \otimes H^*_{\Phi_2}(X;f^*\mathscr{B}_2) & \xrightarrow{\cup} & H^*_{\Phi_1\cap\Phi_2}(X;f^*(\mathscr{B}_1 \otimes \mathscr{B}_2)) \end{array} \tag{20}$$

commutes. This is true in degree zero by (19) and follows in general from 6.2 (see the proof of 7.1) where we take $\mathscr{E}$ to be the class of pointwise split sequences so that the functors on the lower right-hand side of (20) will be $\mathscr{E}$-connected. This gives the formula

$$\boxed{f^*(\beta_1 \cup \beta_2) = f^*(\beta_1) \cup f^*(\beta_2)}$$

for $\beta_i \in H^*_{\Psi_1}(Y;\mathscr{B}_i)$, which converts immediately into

$$\boxed{f_1^*(\beta_1) \times f_2^*(\beta_2) = f_1^*(\beta_1) \times f_2^*(\beta_2)}$$

for $f_i : X_i \to Y_i$, and $\beta_i \in H^*_{\Psi_1}(Y_1;\mathscr{B}_i)$.

9 Φ-soft and Φ-fine sheaves

When Φ is a paracompactifying family of supports, the class of flabby sheaves can be extended to an important larger class of Φ-acyclic sheaves, the Φ-soft sheaves. We define this notion for general support families Φ but stress that they are not, in general, Φ-acyclic unless Φ is paracompactifying. (For example if $x \in \mathbb{R}^2$ and we let x denote the support family $\{\{x\},\varnothing\}$, then the constant sheaf $\mathbb{Z}$ is x-soft on $\mathbb{R}^2$ but is not x-acyclic. Indeed, $H^2_x(\mathbb{R}^2;\mathbb{Z}) \approx \mathbb{Z}$; see formula (41) on page 136.)

We shall use the abbreviation $\mathscr{A}(K) = (\mathscr{A}|K)(K) = \Gamma(\mathscr{A}|K)$ for arbitrary sets K and not just open sets.

9.1. Definition. *A sheaf $\mathscr{A}$ on X is called "Φ-soft" if the restriction map $\mathscr{A}(X) \to \mathscr{A}(K)$ is surjective for all $K \in \Phi$. If $\Phi = cld$ then $\mathscr{A}$ is simply called "soft."*

9.2. Proposition. *If $\mathscr{A}$ is a Φ-soft sheaf on X, then $\mathscr{A}|A$ is $\Phi|A$-soft for any subspace $A \subset X$.*

Proof. This is trivial, but note that in general, A must be locally closed in order that $\Phi|A$ be paracompactifying on A when Φ is paracompactifying on X. □

9.3. Proposition. *Let Φ be paracompactifying on X. Then the following three statements are equivalent:*

(i) *$\mathscr{A}$ is Φ-soft.*

(ii) *$\mathscr{A}|K$ is soft for every $K \in \Phi$.*

(iii) *$\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi|F}(\mathscr{A}|F)$ is surjective for all closed $F \subset X$.*

Proof. Statement (iii) implies (i) trivially, and (i) implies (ii) by 9.2. Assume (ii) and let $s \in \Gamma_{\Phi|F}(\mathscr{A}|F)$. Let $K = |s|$ and let $K' \in \Phi$ be a neighborhood of K. Let B be the boundary of K'. By (ii), the element of $\mathscr{A}((K' \cap F) \cup B)$ that is s on $K' \cap F$ and is 0 on B can be extended to some $s' \in \mathscr{A}(K')$. Then s' extends by zero to $s'' \in \mathscr{A}(X)$ with $|s''| \subset K'$, and so s'' is the desired extension of s. □

9.4. Example. Consider the sheaf $\mathscr{F}$ of germs of smooth real-valued functions on a differentiable manifold M^n. For a closed set $K \subset M$, an element of $\mathscr{F}(K)$ can be regarded as a real-valued function on K that extends *locally about each point* to a smooth function. By a smooth partition of unity argument, such a function extends globally to a smooth function on all of M.[15] This means that $\mathscr{F}$ is soft, indeed, Φ-soft for Φ paracompactifying.

Similarly, on any paracompact topological space X, the sheaf $\mathscr{C}$ of germs of continuous real-valued functions is soft. ◇

The following result is basic. (Also note Exercises 6 and 37.)

9.5. Theorem. *Let A be a subspace of X having a fundamental system of paracompact neighborhoods. Then for any sheaf $\mathscr{A}$ on X, we have $\mathscr{A}(A) = \varinjlim \mathscr{A}(U)$, where U ranges over the neighborhoods of A.*[16]

Proof. The canonical map $\varinjlim \mathscr{A}(U) \to \mathscr{A}(A)$ is injective since two sections that coincide on A must coincide on an open set containing A. Thus it suffices to show that any section $s \in \mathscr{A}(A)$ extends to some neighborhood of A. Cover A by open sets U_α such that there is an $s_\alpha \in \mathscr{A}(U_\alpha)$ with $s_\alpha|U_\alpha \cap A = s|U_\alpha \cap A$. By passing to a neighborhood of A, we may assume that X is paracompact and that $\{U_\alpha\}$ is a locally finite covering of X. Let $\{V_\alpha\}$ be a covering of X with $\overline{V}_\alpha \subset U_\alpha$ for all α. Put $W = \{x \in X \mid x \in \overline{V}_\alpha \cap \overline{V}_\beta \Rightarrow s_\alpha(x) = s_\beta(x)\}$. Let $J(x) = \{\alpha \mid x \in \overline{V}_\alpha\}$, a finite set. Then every $x \in X$ has a neighborhood $N(x)$ such that $y \in N(x) \Rightarrow J(y) \subset J(x)$.

If $x \in W$, the sections s_α for $\alpha \in J(x)$ coincide in a neighborhood of x, since $J(x)$ is finite. Thus W is open. Also, $A \subset W$. Now let $t \in \mathscr{A}(W)$ be defined by $t(x) = s_\alpha(x)$ when $x \in V_\alpha \cap W$. Then t is well-defined by the definition of W, and it is continuous since it coincides with s_α on $V_\alpha \cap W$. Therefore t is the desired extension of s. □

9.6. Corollary. *If Φ is paracompactifying, then every flabby sheaf on X is Φ-soft.* □

9.7. Corollary. *If X is hereditarily paracompact and $\mathscr{A}$ is a flabby sheaf on X, then $\mathscr{A}|A$ is flabby for every subspace $A \subset X$.*

Proof. Let $s \in \mathscr{A}(U \cap A)$ for some open set $U \subset X$. Then there is an open subset $V \supset U \cap A$ of X and a section $t \in \mathscr{A}(V)$ with $s = t|U \cap A$, by 9.5. Then t extends to X since $\mathscr{A}$ is flabby. The restriction of this to A gives the desired extension of s. □

Let us call a subspace $A \subset X$ *relatively Hausdorff* (in X) if any two points of A have disjoint open neighborhoods in X. By the same proof

[15] See, for example, [19, II-10.6].

[16] The condition on A is satisfied for any closed subset of a paracompact space X, and also for an arbitrary subspace A of a hereditarily paracompact space X.

that shows that a compact Hausdorff space is normal, a compact relatively Hausdorff subspace A of X has the property that any two disjoint closed sets in A have disjoint neighborhoods in X.

The following is a modification of 9.5 that we will find useful in dealing with the invariance of cohomology under homotopies.

9.8. Theorem. *Let A be a compact, relatively Hausdorff subspace of a space X. Then for any sheaf $\mathscr{A}$ on X, we have that $\mathscr{A}(A) = \varinjlim \mathscr{A}(U)$, where U ranges over the neighborhoods of A in X.*

Proof. Given $s \in \mathscr{A}(A)$ we can find a finite open covering $\{U_i\}$ of A in X and elements $s_i \in \mathscr{A}(U_i)$ with $s_i|A \cap U_i = s|A \cap U_i$. We can find compact sets $K_i \subset U_i \cap A$ such that $A = \bigcup K_i$. By a finite induction it suffices to prove the following assertion: If P_1 and P_2 are compact subsets of A with open neighborhoods V_1 and V_2, respectively, in X, and if $t_i \in \mathscr{A}(V_i)$ coincide on $P_1 \cap P_2$, then there is a section t over some neighborhood of $P_1 \cup P_2$ coinciding with t_i on P_i, $i = 1, 2$.

To prove this, notice that t_1 coincides with t_2 on some neighborhood V of $P_1 \cap P_2$ and that we may suppose that $V \subset V_1 \cap V_2$. The sets $P_1 - V$ and $P_2 - V$ are compact and disjoint, and since A is relatively Hausdorff, they have disjoint open neighborhoods $Q_i \supset P_i - V$ in X. Then the sections $t_1|Q_1$, $t_1|V = t_2|V$, and $t_2|Q_2$ coincide on their common domains and thereby provide the desired section t on $Q_1 \cup V \cup Q_2 \supset P_1 \cup P_2$. □

9.9. Theorem. *Let Φ be paracompactifying and suppose that*

$$0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$$

is exact with $\mathscr{A}'$ being Φ-soft. Then the sequence

$$0 \to \Gamma_\Phi(\mathscr{A}') \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}'') \to 0$$

is exact.

Proof. We may consider $\mathscr{A}'$ as a subsheaf of $\mathscr{A}$. Let $s \in \Gamma_\Phi(\mathscr{A}'')$ and let $K = |s| \in \Phi$. Let $K' \in \Phi$ be a neighborhood of K. Suppose that we can find an element $t \in \mathscr{A}(K')$ representing $s|K'$. Then on the boundary B of K', $t|B \in \mathscr{A}'(B)$ can be extended to $\mathscr{A}'(K')$. Subtracting this from t, we see that we may assume that $t|B = 0$. But then t can be extended by zero to X. Thus we may as well assume X to be paracompact and Φ to be the class *cld* of all closed subsets of X.

Let $s \in \mathscr{A}''(X)$ and let $\{U_\alpha\}$ be a locally finite covering of X with $s_\alpha \in \mathscr{A}(U_\alpha)$ representing $s|U_\alpha$. Let $\{V_\alpha\}$ be a covering of X with $\overline{V}_\alpha \subset U_\alpha$.

Assume that the indexing set $\{\alpha\}$ is well-ordered and put $F_\alpha = \bigcup_{\beta<\alpha} \overline{V}_\beta$. Since $\{U_\alpha\}$ is locally finite, F_α is closed for all α. We shall define inductively an element $t_\alpha \in \mathscr{A}(F_\alpha)$ representing $s|F_\alpha$ such that $t_\alpha|F_\beta = t_\beta$ for all

$\beta < \alpha$. Say that t_β has been defined for all $\beta < \alpha$. If α is a limit ordinal, then $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ and t_α is defined as the union of the previous t_β's. This is continuous since each $x \in X$ has a neighborhood meeting only finitely many of the $\overline{V}_\beta$'s. If α is the successor of α' then $t_{\alpha'}$ and s_α both represent s on $F_{\alpha'} \cap \overline{V}_\alpha$. The difference is a section of $\mathscr{A}'$ over $F_{\alpha'} \cap \overline{V}_\alpha$ and hence can be extended to $\mathscr{A}'(\overline{V}_\alpha)$. Therefore, $t_{\alpha'}$ can be extended to F_α representing $s|F_\alpha$, completing the induction. □

9.10. Proposition. *If Φ is paracompactifying and $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is exact with $\mathscr{A}'$ being Φ-soft, then $\mathscr{A}''$ is also Φ-soft $\Leftrightarrow$ $\mathscr{A}$ is Φ-soft.*

Proof. This follows from 9.3 and 9.9 by a simple diagram chase. □

9.11. Theorem. *A Φ-soft sheaf is Φ-acyclic if Φ is paracompactifying.*

Proof. Let $\mathscr{A}$ be a Φ-soft sheaf and consider the sequence

$$0 \to \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A}) \to \mathscr{Z}^1(X;\mathscr{A}) \to 0.$$

Since $\mathscr{C}^0(X;\mathscr{A})$ is flabby, we have that $H^p_\Phi(X;\mathscr{C}^0(X;\mathscr{A})) = 0$ for $p > 0$, and it follows from the associated cohomology sequence that

$$H^n_\Phi(X;\mathscr{A}) \approx H^{n-1}_\Phi(X;\mathscr{Z}^1(X;\mathscr{A})),$$

for $n > 1$. Also, $H^1_\Phi(X;\mathscr{A}) = 0$ by 9.9. By 9.10, $\mathscr{Z}^1(X;\mathscr{A})$ is also Φ-soft, so that the theorem follows by induction on n. □

9.12. Proposition. *Let Φ be a paracompactifying family of supports on X and let $A \subset X$ be locally closed. Then for a sheaf $\mathscr{B}$ on A,*

$$\boxed{\mathscr{B} \text{ is } \Phi|A\text{-soft on } A \quad \Leftrightarrow \quad \mathscr{B}^X \text{ is } \Phi\text{-soft on } X.}$$

Proof. The $\Leftarrow$ part follows from 9.2 since $\mathscr{B} = \mathscr{B}^X|A$. For the $\Rightarrow$ part, we note that $\Gamma_\Phi(\mathscr{B}^X) \approx \Gamma_{\Phi|A}(\mathscr{B})$ naturally by I-6.6, and similarly, for $F \subset X$ closed, $\Gamma_{\Phi|F}(\mathscr{B}^X) \approx \Gamma_{\Phi|A\cap X}(\mathscr{B}|A \cap F)$. The result now follows from 9.3iii. □

9.13. Corollary. *Let Φ be paracompactifying and $A \subset X$ locally closed. Then for a sheaf $\mathscr{A}$ on X,*

$$\boxed{\mathscr{A} \text{ is } \Phi\text{-soft} \quad \Rightarrow \quad \mathscr{A}_A \text{ is } \Phi\text{-soft.}}$$

Proof. $\mathscr{A}|A$ is $\Phi|A$-soft by 9.2, so that $\mathscr{A}_A = (\mathscr{A}|A)^X$ is Φ-soft by 9.12. □

The following result shows that softness is a "local" property. Recall that $E(\Phi) = \bigcup\{K \mid K \in \Phi\}$.

9.14. Lemma. *Let Φ be paracompactifying and $\mathscr{L}$ a sheaf on X. If each point $x \in E(\Phi)$ has a closed neighborhood N such that $\mathscr{L}|N$ is soft, then $\mathscr{L}$ is Φ-soft.*

Proof. By 9.3(ii), it suffices to consider the case in which X is paracompact and $\Phi = cld$. Let $\{U_\alpha\}$ be a locally finite open covering of X such that $\mathscr{L}|\overline{U}_\alpha$ is soft. Let $\{V_\alpha\}$ be an open covering of X with $\overline{V}_\alpha \subset U_\alpha$. Well order the indexing set and let $F_\alpha = \bigcup_{\beta<\alpha} \overline{V}_\beta$ (a closed set).

Let $K \subset X$ be closed and $s \in \mathscr{L}(K)$. We must extend s to X. By an easy transfinite induction we can define $t_\alpha \in \mathscr{L}(F_\alpha)$ such that $t_\alpha|F_\alpha \cap K = s|F_\alpha \cap K$ and $t_\alpha|F_\beta = t_\beta$ for $\beta < \alpha$. In the end we have the desired extension. □

9.15. Definition. *A sheaf $\mathscr{A}$ is said to be "Φ-fine" if $\mathscr{H}om(\mathscr{A},\mathscr{A})$ is Φ-soft.*

The following is another basic result.

9.16. Theorem. *Let Φ be paracompactifying. Then any module over a Φ-soft sheaf $\mathscr{R}$ of rings with unit is Φ-fine, and any Φ-fine sheaf is Φ-soft.*

Proof. Let $\mathscr{A}$ be an $\mathscr{R}$-module. We must show that $\mathscr{H}om(\mathscr{A},\mathscr{A})$ is Φ-soft. But $\mathscr{H}om(\mathscr{A},\mathscr{A})$ is an $\mathscr{R}$-module so that it suffices to show that every $\mathscr{R}$-module $\mathscr{A}$ is Φ-soft. [The last statement of the theorem will follow since a Φ-fine sheaf $\mathscr{B}$ is a module over the Φ-soft sheaf of rings $\mathscr{H}om(\mathscr{B},\mathscr{B})$.]

Thus, let $s \in \mathscr{A}(K)$ for some $K \in \Phi$. By 9.5 there is a neighborhood $K' \in \Phi$ of K and an $s' \in \mathscr{A}(K')$ extending s. Since $\mathscr{R}$ is Φ-soft, there is a section $t \in \mathscr{R}(K')$ that is zero on the boundary B of K' and 1 on K. The section $ts' : x \mapsto t(x) \cdot s'(x)$ in $\mathscr{A}(K')$ is zero on B and coincides with s on K. Therefore ts', and hence s, can be extended to X. □

9.17. Example. The sheaf $\mathscr{F}$ of germs of smooth real-valued functions on a differentiable manifold M^n is soft, as shown in Example 9.4. This is a sheaf of rings with unit, and so $\mathscr{F}$ is fine. Consequently, any $\mathscr{F}$-module is fine. This includes the sheaf of germs of differential p-forms on M, the sheaf of germs of vector fields on M, etc. ◇

Since $\mathscr{A} \otimes \mathscr{B}$ is a $\mathscr{H}om(\mathscr{A},\mathscr{A})$-module, we have the following consequence of 9.16:

9.18. Corollary. *If $\mathscr{A}$ is Φ-fine, Φ paracompactifying, then $\mathscr{A} \otimes \mathscr{B}$ is Φ-fine, and hence Φ-soft, for any sheaf $\mathscr{B}$.*[17] □

9.19. Corollary. *If Φ is paracompactifying and $\mathscr{L}^*$ is a torsion free Φ-fine*[18] *resolution of $\mathbb{Z}$ on X, then there is a natural isomorphism*

$$\boxed{H^p_\Phi(X; \mathscr{A}) \approx H^p(\Gamma_\Phi(\mathscr{L}^* \otimes \mathscr{A})).}$$

Proof. Since $\mathscr{L}^*$ is torsion free, $\mathscr{L}^* \otimes \mathscr{A}$ is a resolution of $\mathbb{Z} \otimes \mathscr{A} \approx \mathscr{A}$.[19] It is Φ-soft, and hence Φ-acyclic, by 9.18. The result follows from 4.1. □

9.20. Definition. *Let $A \subset X$. A family Φ of supports on X will be said to be "paracompactifying for the pair (X, A)" if Φ is paracompactifying and if each $K \cap A$, for $K \in \Phi$, has a fundamental system of paracompact neighborhoods in X.*

Note that these conditions imply that $\Phi \cap A$ is a paracompactifying family of supports on A. Also note that if A is closed, then Φ is paracompactifying for the pair $(X, A) \Leftrightarrow \Phi$ is paracompactifying. Moreover, for A open and $\Phi = cld$, Φ is paracompactifying for the pair $(X, A) \Leftrightarrow$ both X and A are paracompact.

9.21. Proposition. *If Φ is paracompactifying for the pair (X, A), then for any sheaf $\mathscr{A}$ on X, the map*

$$\theta : \varinjlim \Gamma_{\Phi \cap U}(\mathscr{A}|U) \to \Gamma_{\Phi \cap A}(\mathscr{A}|A)$$

(U ranging over the neighborhoods of A) is bijective. Moreover, if $\mathscr{A}$ is flabby, then $\mathscr{A}|A$ is $(\Phi \cap A)$-soft.

Proof. θ is clearly injective. Let $s \in \Gamma_{\Phi \cap A}(\mathscr{A}|A)$ with $|s| = K \cap A$ where $K \in \Phi$. Let $K' \in \Phi$ be a neighborhood of K. By 9.5 there is a neighborhood V of $K' \cap A$ in K' and an element $s' \in \mathscr{A}(V)$ extending $s|K' \cap A$. Let L be the closure in X of $|s'|$. Then $s'|(V - L) = 0$ and so s' on V and 0 on $X - L$ combine to form a section s'' over the open set $U = V \cup (X - L)$. Now $V \supset K' \cap A$ and $K' \supset L$, so that $A - V \subset X - K' \subset X - L$. Thus $A \subset V \cup (X - L) = U$. Both s'' and s vanish outside V and coincide on $V \cap A$, and so $s''|A = s$. Also, $|s''| = |s'| \subset U \cap L \in U \cap \Phi$ since $L \subset K' \in \Phi$. Therefore, s'' induces an element of $\varinjlim \Gamma_{\Phi \cap U}(\mathscr{A}|U)$ mapping to s under θ, proving the first statement.

For the last statement, let $t \in \mathscr{A}(A \cap K)$ for some $K \in \Phi$. By 9.5, t can be extended to X since $\mathscr{A}$ is flabby, and a fortiori to A. □

Using 9.8 instead of 9.5 in the last paragraph gives:

[17] Also see 16.31.
[18] Or just Φ-soft by 16.31.
[19] This is a stalkwise assertion, so it follows from standard homological algebra.

9.22. Proposition. *If A is a compact relatively Hausdorff subspace of X, then $\mathscr{A}|A$ is soft for any flabby sheaf $\mathscr{A}$ on X.* □

For more facts concerning soft sheaves see Section 16.

9.23. Let L be a ring with unit, let G be an L-module, and let Φ be a paracompactifying family of supports on X. Then the singular and Alexander-Spanier sheaves $\mathscr{S}^0(X;L)$ and $\mathscr{A}^0(X;L)$ are the same as $\mathscr{C}^0(X;L)$ and so they are Φ-soft. Also, they are sheaves of rings, and $\mathscr{S}^n(X;G)$ and $\mathscr{A}^n(X;G)$ are modules over them, and so they are also Φ-soft. As remarked in Section 1, $\mathscr{A}^*(X;G)$ is always a resolution of G, and $\mathscr{S}^*(X;G)$ is a resolution of G if X is *HLC*. Therefore, by 4.1 or 5.15,

$$_{AS}H^*_\Phi(X;G) \approx H^*_\Phi(X;G)$$

for Φ-paracompactifying, and

$$_\Delta H^*_\Phi(X;G) \approx H^*_\Phi(X;G)$$

for Φ-paracompactifying and X *HLC*. Similarly, by 9.17, if X is a smooth manifold then the de Rham sheaf $\Omega^*(X)$ is a Φ-fine resolution of $\mathbb{R}$, and so

$$_\Omega H^*_\Phi(X;\mathbb{R}) \approx H^*_\Phi(X;\mathbb{R})$$

for Φ paracompactifying.

In Chapter III we shall study these isomorphisms in more detail. We shall extend them to more general coefficients, show they are natural in X as well as in the coefficients, and show that they preserve cup products. We also take up the relative case there. Except for the latter, that chapter can be read at this point.

10 Subspaces

In this section we study relationships between the cohomology of a space and that of a subspace, with coefficients in sheaves related to the subspace. The main theorem 10.6 relates the cohomology of a subspace to that of its neighborhoods. This will be of central importance throughout the book.

10.1. Theorem. *Suppose either that Φ is a paracompactifying family of supports on X and that $A \subset X$ is locally closed, or that Φ is arbitrary and $A \subset X$ is closed. Then there is a natural isomorphism*

$$\boxed{H^*_\Phi(X;\mathscr{B}^X) \approx H^*_{\Phi|A}(A;\mathscr{B})}$$

of functors of sheaves on A, which preserves cup products.[20]

[20] Also see Exercise 1.

Proof. Note that the functor $\mathscr{B} \mapsto \mathscr{B}^X$ is exact. Thus we have the connected sequences of functors $F_1^p(\mathscr{B}) = H_\Phi^p(X; \mathscr{B}^X)$ and $F_2^p = H_{\Phi|A}^p(A; \mathscr{B})$, both of which are *fundamental* by 9.12 and 5.8. Now,

$$F_1^0(\mathscr{B}) = \Gamma_\Phi(\mathscr{B}^X) \qquad \text{and} \qquad F_2^0(\mathscr{B}) = \Gamma_{\Phi|A}(\mathscr{B}).$$

The restriction map $T^0 : \Gamma_\Phi(\mathscr{B}^X) \to \Gamma_{\Phi|A}(\mathscr{B})$ is an isomorphism of functors by I-6.6, whence the first part of the result follows from Theorem 6.2. The fact that cup products are preserved follows immediately from the axioms 7.1 for the cup product on the cohomology of A.[21] □

10.2. Corollary. *With the same hypotheses as in* 10.1 *we have the natural isomorphism*

$$\boxed{H_\Phi^*(X; \mathscr{A}_A) \approx H_{\Phi|A}^*(A; \mathscr{A}|A)}$$

of functors of sheaves $\mathscr{A}$ *on* A, *which preserves cup products.*

Proof. This follows from the fact that $\mathscr{A}_A = (\mathscr{A}|A)^X$. □

In particular we have the most important cases:

$$\boxed{H_\Phi^*(X; \mathscr{A}_F) \approx H_{\Phi|F}^*(F; \mathscr{A}|F) \quad \text{for } F \text{ closed and } \Phi \text{ arbitrary,}}$$

and

$$\boxed{H_\Phi^*(X; \mathscr{A}_U) \approx H_{\Phi|U}^*(U; \mathscr{A}|U) \quad \text{for } U \text{ open and } \Phi \text{ paracompactifying.}}$$

Note that $\mathscr{C}^*(X; \mathscr{A}_U)|U = \mathscr{C}^*(U; \mathscr{A}|U)$, so that there is the chain map

$$C_{\Phi|U}^*(U; \mathscr{A}|U) = \Gamma_{\Phi|U}(\mathscr{C}^*(U; \mathscr{A}|U)) \hookrightarrow \Gamma_\Phi(\mathscr{C}^*(X; \mathscr{A}_U)) = C_\Phi^*(X; \mathscr{A}_U),$$

which clearly induces the preceding isomorphism.

10.3. Suppose that $F \subset X$ is closed and $U = X - F$. If Φ is paracompactifying, then 10.2 together with the exact coefficient sequence $0 \to \mathscr{A}_U \to \mathscr{A} \to \mathscr{A}_F \to 0$ yields the fundamental exact cohomology sequence

$$\boxed{\cdots \to H_{\Phi|U}^p(U; \mathscr{A}) \to H_\Phi^p(X; \mathscr{A}) \to H_{\Phi|F}^p(F; \mathscr{A}) \to H_{\Phi|U}^{p+1}(U; \mathscr{A}) \to \cdots}$$

This sequence will be generalized in Section 12.

10.4. Let $A \subset X$ be an arbitrary subspace, Φ any family of supports on X, and $\mathscr{A}$ a sheaf on X. By 6.2 the restriction of sections $\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi \cap A}(\mathscr{A}|A)$ extends canonically to a homomorphism

$$\boxed{r_{A,X}^* : H_\Phi^*(X; \mathscr{A}) \to H_{\Phi \cap A}^*(A; \mathscr{A}|A)}$$

[21] This uses the obvious natural isomorphism $\mathscr{A}^X \otimes \mathscr{B}^X \approx (\mathscr{A} \otimes \mathscr{B})^X$.

called the *restriction* homomorphism. We also denote $r^*_{A,X}(\alpha)$ by $\alpha|A$.

Since $\Phi \cap A = i^{-1}(\Phi)$ and $\mathscr{A}|A = i^*(\mathscr{A})$ where $i : A \hookrightarrow X$, r^* is none other than the homomorphism induced by i; see Section 8. In particular (or directly from 7.1), r^* preserves cup products.

For a closed subspace $F \subset X$, $\Phi \cap F = \Phi|F$, and the restriction map

$$\boxed{r^*_{F,X} : H^*_\Phi(X;\mathscr{A}) \to H^*_{\Phi|F}(F;\mathscr{A}|F)}$$

is the same as the homomorphism $H^*_\Phi(X;\mathscr{A}) \to H^*_\Phi(X;\mathscr{A}_F)$ induced by the epimorphism $\mathscr{A} \to \mathscr{A}_F$ followed by the isomorphism $H^*_\Phi(X;\mathscr{A}_F) \xrightarrow{\approx} H^*_{\Phi|F}(F;\mathscr{A}|F)$ of 10.2. This follows from the uniqueness portion of 6.2.

We wish to relate the cohomology of a subspace to that of its neighborhoods. In order to deal with several cases at the same time, we make the following definition. It will also be useful in the study of relative cohomology. The term "taut" is borrowed from Spanier, who uses it in situations analogous to 10.6, but mainly in singular homology.

10.5. Definition. *Let Φ be a family of supports on X. Then a subspace $A \subset X$ is said to be "Φ-taut" if for every flabby sheaf F on X, the restriction $\Gamma_\Phi(\mathscr{F}) \to \Gamma_{\Phi\cap A}(\mathscr{F}|A)$ is surjective and $\mathscr{F}|A$ is $(\Phi \cap A)$-acyclic.*

The following five cases are examples of Φ-taut subspaces A of X:

(a) Φ arbitrary, A open.

(b) Φ paracompactifying for the pair (X, A).

(c) Φ paracompactifying, X hereditarily paracompact (e.g., metric), A arbitrary.

(d) Φ paracompactifying, A closed.

(e) $\Phi = cld$, A compact and relatively Hausdorff in X, e.g., a point.

Item (a) follows from 5.2; (b) follows from 9.21; (c) and (d) are special cases of (b); (e) follows from 9.8 and 9.22. Also see 12.1, 12.13, 12.14, 12.15, and Exercise 8.

The following result is fundamental and will be used often in the remainder of the book.

10.6. Theorem. *Let Φ be a family of supports on X and let A be a subspace of X. Let $\mathscr{N}$ be a collection of Φ-taut subspaces of X containing A and directed downwards by inclusion. Assume that for each $K \in \Phi|X - A$ there is an $N \in \mathscr{N}$ with $N \subset X - K$. Then A is Φ-taut $\Leftrightarrow$ the map*

$$\boxed{\theta : \varinjlim_{N\in\mathscr{N}} H^*_{\Phi\cap N}(N;\mathscr{A}|N) \to H^*_{\Phi\cap A}(A;\mathscr{A}|A),}$$

induced by restriction, is an isomorphism for every sheaf $\mathscr{A}$ on X.

Proof. For $\Leftarrow$, suppose that θ is an isomorphism and let $\mathscr{A}$ be flabby. For $N \in \mathscr{N}$, $\mathscr{A}|N$ is $(\Phi \cap N)$-acyclic, since N is Φ-taut. It follows that $\mathscr{A}|A$ is $(\Phi \cap N)$-acyclic. Moreover, $\Gamma_\Phi(\mathscr{A}) \to \varinjlim \Gamma_{\Phi\cap N}(\mathscr{A}|N) \xrightarrow{\approx} \Gamma_{\Phi\cap A}(\mathscr{A}|A)$ is surjective since each N is Φ-taut. Thus A is Φ-taut.

For $\Rightarrow$, suppose that A is Φ-taut and consider the functors $F_1^p(\mathscr{A}) = \varinjlim H^p_{\Phi\cap N}(N;\mathscr{A}|N)$ and $F_2^p(\mathscr{A}) = H^p_{\Phi\cap A}(A;\mathscr{A}|A)$. The tautness of A and of each $N \in \mathscr{N}$ implies that these are both fundamental connected sequences of functors. Moreover, in degree zero, $\theta : \varinjlim \Gamma_{\Phi\cap N}(N;\mathscr{A}|N) \to \Gamma_{\Phi\cap A}(\mathscr{A}|A)$ is clearly one-to-one for $\mathscr{A}$ arbitrary and is onto for $\mathscr{A}$ flabby, since A is Φ-taut. It follows from 6.2(c) that θ is an isomorphism in general. □

Remark: An important case of 10.6 is that for which $\mathscr{N}$ is a fundamental system of neighborhoods of A. However, as we shall see, the usefulness of this result is hardly limited to that case.

Remark: Let $f : X \to Y$ be Φ-closed (meaning $f(K)$ is closed for $K \in \Phi$) where Φ is a family of supports on X. Then putting $A = f^{-1}(y)$ for some $y \in Y$, we see that the family $\{f^{-1}(U) \mid U$ a neighborhood of $y\}$ of neighborhoods of A *refines* the family $\{X - K \mid K \in \Phi|(X - A)\}$. Thus 10.6 implies that $H^*_{\Phi\cap f^{-1}(y)}(f^{-1}(y);\mathscr{A}) = \varinjlim H^*_{\Phi\cap f^{-1}(U)}(f^{-1}(U);\mathscr{A})$ when $f^{-1}(y)$ is Φ-taut. (This holds, in particular, when Φ is paracompactifying, or when $\Phi = cld$ and $f^{-1}(y)$ is compact and relatively Hausdorff.) This will be of importance in Chapter IV. Note that in general, $\{f^{-1}(U)\}$ is not a fundamental system of neighborhoods of $f^{-1}(y)$.

10.7. Corollary. (Weak continuity.) *If $\{F_\alpha\}$ is a downward directed family of closed subspaces of the locally compact Hausdorff space X, then*

$$\boxed{H_c^*(\bigcap F_\alpha;\mathscr{A}) \approx \varinjlim H_c^*(F_\alpha;\mathscr{A}).}$$ □

This result will be generalized considerably in Section 14.

10.8. Corollary. (The minimality principle.) *Let X be a locally compact Hausdorff space and c the family of compact subsets of X. Then for any nonzero class $\alpha \in H_c^n(X;\mathscr{A})$, the collection of closed subspaces F of X such that $0 \neq \alpha|F \in H_c^n(F;\mathscr{A}|F)$ has a minimal element.* □

10.9. Example. Consider the "topologist's sine curve," which is the union

$$X = \{(x,y) \mid y = \sin \pi/x; 0 < x \le 1\} \cup \{0\}\times[-1,2] \cup [0,1]\times\{2\} \cup \{1\}\times[0,2]$$

or any of its variants. This space has the singular cohomology of a point. However, it is a decreasing intersection of spaces homeomorphic to an annulus, and the inclusion maps are homotopy equivalences. Therefore, by 10.7, $H^*(X;\mathbb{Z}) \approx H^*(\mathbb{S}^1;\mathbb{Z})$. This is a typical example of the difference between singular theory and "Čech type" theories such as sheaf-theoretic cohomology. ◇

10.10. Example. Let X be the union in $\mathbb{R}^3$ of spheres of radius $1/n$ all tangent to the xy-plane at the origin. This is a decreasing intersection of spaces of the homotopy type of a finite one-point union of spheres, and hence

$$H^2(X;\mathbb{Z}) \approx \bigoplus_{i=1}^{\infty} \mathbb{Z},$$

and $H^i(X;\mathbb{Z}) = 0$ for $i \neq 0, 2$. This contrasts with singular theory for which the cohomology of this space is nonzero in arbitrarily large degrees; see [3]. ◇

10.11. Example. Let M be any abelian group. An open interval U in $\mathbb{R}$ is contractible, and it will be shown in the next section that this implies that $H^p(U;M) = 0$ for all $p > 0$. Since an open subset of $\mathbb{R}$ is a topological sum of open intervals, this also holds for any open set $U \subset \mathbb{R}$. By 10.6 it follows that $H^p(A;M) = 0$ for all $p > 0$ and all subspaces $A \subset \mathbb{R}$, since $\mathbb{R}$ is hereditarily paracompact. In 16.28 and the solution to V-Exercise 26 the much more difficult fact is shown that $H^p(U;M) = 0$ for $p \geq n$ and any open set $U \subset \mathbb{R}^n$. Thus it will follow from 10.6 that $H^p(A;M) = 0$ for $p \geq n$ and any subspace A of $\mathbb{R}^n$. ◇

11 The Vietoris mapping theorem and homotopy invariance

Let $f : X \to Y$ be a *closed* map. Let $\mathscr{A}$ be a sheaf on X and Ψ a family of supports on Y. Assume further that each $f^{-1}(y)$, for $y \in Y$, is taut in X. This holds, for instance, when X is paracompact *or* when each $f^{-1}(y)$ is compact and relatively Hausdorff in X. The f-cohomomorphism $f : f\mathscr{A} \rightsquigarrow \mathscr{A}$ induces an f-cohomomorphism $\mathscr{C}^*(Y; f\mathscr{A}) \rightsquigarrow \mathscr{C}^*(X;\mathscr{A})$, which has the factorization

$$\mathscr{C}^*(Y; f\mathscr{A}) \to f\mathscr{C}^*(X;\mathscr{A}) \rightsquigarrow \mathscr{C}^*(X;\mathscr{A}).$$

Now, assume that $H^p(f^{-1}(y);\mathscr{A}) = 0$ for all $p > 0$ and all $y \in Y$. Then the derived sheaf of the differential sheaf $f\mathscr{C}^*(X;\mathscr{A})$ has stalks

$$\begin{aligned} \mathscr{H}^p(f\mathscr{C}^*(X;\mathscr{A}))_y &= \varinjlim H^p(C^*(f^{-1}(U);\mathscr{A})) \\ &= \varinjlim H^p(f^{-1}(U);\mathscr{A}) \\ &= H^p(f^{-1}(y);\mathscr{A}) = 0 \end{aligned}$$

by 10.6 for $p \neq 0$ (where U ranges over the open neighborhoods of y), while the exact sequence

$$0 \to f\mathscr{A} \to f\mathscr{C}^0(X;\mathscr{A}) \to f\mathscr{C}^1(X;\mathscr{A})$$

(since $\mathscr{A} \mapsto f\mathscr{A}$ is left exact) yields an isomorphism

$$f\mathscr{A} \approx \mathscr{H}^0(f\mathscr{C}^*(X;\mathscr{A})).$$

It follows that $f\mathscr{C}^*(X;\mathscr{A})$ is a resolution of $f\mathscr{A}$. Moreover, $f\mathscr{C}^*(X;\mathscr{A})$ is flabby by 5.7. Thus the chain map $C^*_\Psi(Y;f\mathscr{A}) \to \Gamma_\Psi(f\mathscr{C}^*(X;\mathscr{A}))$ induces an isomorphism in cohomology by 4.2. Moreover, by Exercise I-8, the map $\Gamma_\Psi(f\mathscr{C}^*(X;\mathscr{A})) \to \Gamma_{f^{-1}\Psi}(\mathscr{C}^*(X;\mathscr{A})) = C^*_{f^{-1}\Psi}(X;\mathscr{A})$ is an isomorphism.

Combining these facts, we obtain the following very general version of the Vietoris mapping theorem:

11.1. Theorem. *Let $f : X \to Y$ be a closed map, $\mathscr{A}$ a sheaf on X, and Ψ a family of supports on Y. Suppose that $H^p(f^{-1}(y);\mathscr{A}) = 0$ for all $p > 0$ and all $y \in Y$, and that each $f^{-1}(y)$ is taut in X. Then the natural map*

$$f^\dagger : H^*_\Psi(Y;f\mathscr{A}) \to H^*_{f^{-1}\Psi}(X;\mathscr{A}),$$

induced by the f-cohomomorphism $f : f\mathscr{A} \rightsquigarrow \mathscr{A}$, is an isomorphism. □

Note the case of an inclusion $i : F \hookrightarrow X$ of a closed subspace. In this case $i\mathscr{A} = \mathscr{A}^X$, and we retrieve 10.1.

11.2. Let us specialize, for the moment, to the case of a closed map $f : X \to Y$ that is finite-to-one (e.g., a covering map with finitely many sheets, or the orbit map of a finite group of transformations). Let $\mathscr{B}$ be a sheaf on Y and put $\mathscr{A} = f^*\mathscr{B}$. Then

$$(f\mathscr{A})_y = \bigoplus_{x\in f^{-1}(y)} \mathscr{A}_x = \bigoplus_{x\in f^{-1}(y)} \mathscr{B}_y = \underbrace{\mathscr{B}_y \oplus \cdots \oplus \mathscr{B}_y}_{n \text{ times}}$$

(where n is the number of points in $f^{-1}(y)$). The f-cohomomorphism $\mathscr{B} \rightsquigarrow \mathscr{A}$ induces the homomorphism $\beta : \mathscr{B} \to f\mathscr{A}$, which on the stalks at y, is the diagonal map $\mathscr{B}_y \to \mathscr{B}_y \oplus \cdots \oplus \mathscr{B}_y$. The composition

$$f^\dagger\beta^* : H^*_\Psi(Y;\mathscr{B}) \to H^*_\Psi(Y;f\mathscr{A}) \xrightarrow{\approx} H^*_{f^{-1}\Psi}(X;\mathscr{A})$$

is just $f^* : H^*_\Psi(Y;\mathscr{B}) \to H^*_{f^{-1}\Psi}(X;\mathscr{A})$. (This is, of course, a general fact that we discussed in Section 8.)

Now suppose that f is a covering map with n sheets. Then the map $\sigma : f\mathscr{A} \to \mathscr{B}$, defined by $(f\mathscr{A})_y = \mathscr{B}_y \oplus \cdots \oplus \mathscr{B}_y \to \mathscr{B}_y$ where

$$\sigma(s_1, \ldots, s_n) = \sum s_i,$$

is continuous, since it is induced from

$$(f\mathscr{A})(U) = \mathscr{A}(f^{-1}U) = \mathscr{A}(U_1) \oplus \cdots \oplus \mathscr{A}(U_n) \xrightarrow{\Theta} \mathscr{B}(U) \oplus \cdots \oplus \mathscr{B}(U) \xrightarrow{\Sigma} \mathscr{B}(U)$$

on the presheaf level, where U is a connected evenly covered neighborhood of y and the U_i are the components of $f^{-1}(U)$, and where Θ is the direct sum of the inverses of the isomorphisms

$$(f|U_i)^* : \mathscr{B}(U) \xrightarrow{\approx} (f^*\mathscr{B})(U_i) = \mathscr{A}(U_i).$$

Thus we have the homomorphisms

$$\mathscr{B} \underset{\sigma}{\overset{\beta}{\rightleftarrows}} f\mathscr{A}$$

with $\sigma\beta = n$, which induce, via 11.1,

$$\boxed{H^*_\Psi(Y;\mathscr{B}) \underset{\mu}{\overset{f^*}{\rightleftarrows}} H^*_{f^{-1}\Psi}(X;f^*\mathscr{B}),}$$

with $\mu f^*(\alpha) = n\alpha$. The map μ is called the *transfer*. A similar "transfer homomorphism" is considered in Section 19, and Exercise 26 generalizes both.

This implies, for example, that if $f : X \to Y$ is a covering map with $n < \infty$ sheets, then $\dim H^k_\Phi(Y;\mathbb{Q}) \le \dim H^k_{f^{-1}\Phi}(X;\mathbb{Q})$ for each k.

11.3. Example. Consider the covering map $f : \mathbb{S}^n \to \mathbb{RP}^n$ and let $\mathscr{A} = f\mathbb{Z}$. This has stalks $\mathbb{Z}\oplus\mathbb{Z}$ and is "twisted" via the automorphism $(n,k) \mapsto (k,n)$; see I-3.6. By 11.1 we have $H^*(\mathbb{RP}^n;\mathscr{A}) \approx H^*(\mathbb{S}^n;\mathbb{Z})$. We have the inclusion $\beta : \mathbb{Z} \hookrightarrow \mathscr{A}$ as the diagonal, and the composition

$$H^*(\mathbb{RP}^n;\mathbb{Z}) \xrightarrow{\beta^*} H^*(\mathbb{RP}^n;\mathscr{A}) \xrightarrow{\approx} H^*(\mathbb{S}^n;\mathbb{Z})$$

is f^* by the remarks in 11.2. The quotient sheaf $\mathscr{A}/\mathbb{Z} = \mathbb{Z}^t$ has stalks $\mathbb{Z}$ twisted by $n \mapsto -n$. The exact sequence $0 \to \mathbb{Z} \to \mathscr{A} \to \mathbb{Z}^t \to 0$ and 11.2 give

$$H^i(\mathbb{RP}^n;\mathbb{Z}^t) \approx \begin{cases} \mathbb{Z}_2, & i \text{ odd}, 0 < i \le n, \\ \mathbb{Z}, & i = n \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that in case n is even, $\mathbb{RP}^n$ is nonorientable and $\mathbb{Z}^t$ is its orientation sheaf. In this case, Poincaré duality (see Chapter V or IV-2.9) yields $H^i(\mathbb{RP}^n;\mathbb{Z}^t) \approx H_{n-i}(\mathbb{RP}^n;\mathbb{Z})$, giving another calculation of this. Also, $H_i(\mathbb{RP}^n;\mathbb{Z}^t) \approx H^{n-i}(\mathbb{RP}^n;\mathbb{Z})$. For n odd, duality gives $H^i(\mathbb{RP}^n;\mathbb{Z}^t) \approx H_{n-i}(\mathbb{RP}^n;\mathbb{Z}^t)$, and so our calculations give these homology groups. ◇

11.4. Example. Let $f : \mathbb{S}^1 \to I = [-1,1]$ be the projection, and consider the sheaf $f\mathbb{Z}$ on I; see I-3.3. It is not hard to check that $f\mathbb{Z} \approx \mathbb{Z} \oplus \mathbb{Z}_{I-\partial I}$. Hence, by 11.1, we have

$$\begin{aligned} H^p(I;f\mathbb{Z}) &\approx H^p(I;\mathbb{Z}) \oplus H^p(I;\mathbb{Z}_{I-\partial I}) \\ &\approx H^p(I;\mathbb{Z}) \oplus H^p_c(I-\partial I;\mathbb{Z}) && \text{by 10.2} \\ &\approx H^p(\star;\mathbb{Z}) \oplus H^p(I,\partial I;\mathbb{Z}) && \text{as will be seen later,} \end{aligned}$$

which is $\mathbb{Z}$ for $p = 0,1$ and is 0 otherwise. This agrees with 11.1, which gives $H^*(I;f\mathbb{Z}) \approx H^*(\mathbb{S}^1;\mathbb{Z})$. ◇

11.5. Example. Consider the 3-fold covering map $f : \mathbb{S}^1 \to \mathbb{S}^1$; see I-3.5. The sheaf $f\mathbb{Z}$ has stalks $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ twisted by the cyclic automorphism. By 11.1, $H^*(\mathbb{S}^1; f\mathbb{Z}) \approx H^*(\mathbb{S}^1; \mathbb{Z})$. Let $\beta : \varDelta \hookrightarrow f\mathbb{Z}$ be the "diagonal" subsheaf, which is constant with stalk $\mathbb{Z}$. Then by 11.2, the composition $H^1(\mathbb{S}^1; \mathbb{Z}) \approx H^1(\mathbb{S}^1; \varDelta) \xrightarrow{\beta^*} H^1(\mathbb{S}^1; f\mathbb{Z}) \xrightarrow{\approx} H^1(\mathbb{S}^1; \mathbb{Z})$ is just f^*, which is multiplication by 3. Let $\mathscr{A} = (f\mathbb{Z})/\varDelta$, which has stalks $\mathbb{Z} \oplus \mathbb{Z}$ twisted by the essentially unique automorphism of period 3. The exact coefficient sequence $0 \to \varDelta \to f\mathbb{Z} \to \mathscr{A} \to 0$ induces the exact sequence

$$0 = \Gamma(\mathscr{A}) \to H^1(\mathbb{S}^1; \varDelta) \xrightarrow{\beta^*} H^1(\mathbb{S}^1; f\mathbb{Z}) \to H^1(\mathbb{S}^1; \mathscr{A}) \to 0,$$

and it follows that

$$H^p(\mathbb{S}^1; \mathscr{A}) \approx \begin{cases} \mathbb{Z}_3, & \text{for } p = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$\diamond$

11.6. Example. Let $\mathbb{N}$ be the positive integers with the non-Hausdorff topology in which $\mathbb{N}$ and the initial segments $U_n = \{1, \ldots, n\}$ are the open sets; see Exercise 27. Let $A_1 \leftarrow A_2 \leftarrow \cdots$ be an inverse sequence of abelian groups. According to the exercise, this is equivalent to the sheaf $\mathscr{A}$ on $\mathbb{N}$ where $\mathscr{A}(U_n) = A_n$. Also by the exercise, $H^0(\mathbb{N}; \mathscr{A}) = \varprojlim A_i$ and $H^1(\mathbb{N}; \mathscr{A}) = \varprojlim^1 A_i$, the derived functor of $\varprojlim$. Let $0 = i_0 < i_1 < i_2 < \cdots$ be a sequence of integers and let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = k$ for $i_{k-1} < n \leq i_k$. Then f is continuous and closed. Each $f^{-1}(k)$ is finite, and Exercise 58 shows that $\mathscr{A}|f^{-1}(k)$ is acyclic for all sheaves $\mathscr{A}$ on $\mathbb{N}$ and that $f^{-1}(y)$ is taut in $\mathbb{N}$. Now $(f\mathscr{A})(U_k) = \mathscr{A}(f^{-1}U_k) = \mathscr{A}(U_{i_k}) = A_{i_k}$, so that $f\mathscr{A}$ is just the inverse subsequence given by this sequence of indices. By 11.1 we have that $H^*(\mathbb{N}; f\mathscr{A}) \approx H^*(\mathbb{N}; \mathscr{A})$. This means that passage to a subsequence does not change $\varprojlim$ or $\varprojlim^1$. $\diamond$

An important special case of 11.1 is that for which each $f^{-1}(y)$ is connected, and $\mathscr{A} = f^*\mathscr{B}$ for some sheaf $\mathscr{B}$ on Y, particularly a constant sheaf. In this case we have:

11.7. Theorem. (Vietoris mapping theorem.) *Let $f : X \to Y$ be a closed surjection, let $\mathscr{B}$ be a sheaf on Y, and let Ψ be a family of supports on Y. Also assume that each $f^{-1}(y)$ is connected and taut in X and that $H^p(f^{-1}(y); \mathscr{B}_y) = 0$ for $p > 0$ and all $y \in Y$. Then*

$$f^* : H^*_\Psi(Y; \mathscr{B}) \to H^*_{f^{-1}\Psi}(X; f^*\mathscr{B})$$

is an isomorphism.

Proof. Consider the monomorphism $\beta : \mathscr{B} \rightarrowtail ff^*\mathscr{B}$ of I-4 which is induced by the f-cohomomorphism $\mathscr{B} \rightsquigarrow f^*\mathscr{B}$ via the composition

$$\mathscr{B}(U) \to (f^*\mathscr{B})(f^{-1}(U)) = (ff^*\mathscr{B})(U).$$

On the stalks at $y \in Y$ this becomes

$$\mathscr{B}_y \to (f^*\mathscr{B})(f^{-1}(y)) \xrightarrow{\approx} (ff^*\mathscr{B})_y$$

since $f^{-1}(y) \neq \varnothing$ is taut. This is an isomorphism, since $(f^*\mathscr{B})|f^{-1}(y)$ is constant with stalks $\mathscr{B}_y$ and since $f^{-1}(y)$ is connected. It follows that $\beta : \mathscr{B} \to ff^*\mathscr{B}$ is an isomorphism. Since $f^* = f^\dagger \circ \beta^* : H^*_\Psi(Y;\mathscr{B}) \to H^*_{f^{-1}\Psi}(X; f^*\mathscr{B})$ by (18) of Section 8, the theorem follows from 11.1. □

Note that the inverse image of a constant sheaf is constant.

11.8. Corollary. *Let X be a space and $\mathscr{B}$ a sheaf on X. Let T be a compact, connected Hausdorff space that is acyclic for any constant coefficient sheaf [it suffices that $H^p(T;\mathscr{B}_x) = 0$ for $p > 0$ and all $x \in X$]. If $\pi : X \times T \to X$ is the projection, put $\mathscr{B} \times T = \pi^*\mathscr{B}$. For $t \in T$, let $i_t : X \to X \times T$ be the inclusion $x \mapsto (x,t)$. Then*

$$i_t^* : H^*_{\Phi\times T}(X \times T; \mathscr{B} \times T) \to H^*_\Phi(X;\mathscr{B})$$

is an isomorphism that is the inverse of π^* and hence is independent of $t \in T$.

Proof. This makes sense, since $i_t^*(\mathscr{B} \times T) = i_t^*(\pi^*\mathscr{B}) = 1^*\mathscr{B} = \mathscr{B}$. In cohomology we have $i_t^* \circ \pi^* = (\pi \circ i_t)^* = 1^* = 1$. Each $\{x\} \times T$ is compact and relatively Hausdorff, whence taut, in $X \times T$. By 11.7, π^* is an isomorphism, and hence $i_t^* = (\pi^*)^{-1}$ is independent of t. □

We now prove a strengthened version of 11.8 valid for locally compact spaces:

11.9. Theorem. *Let X be a locally compact Hausdorff space and let T be a compact connected Hausdorff space. Then with the notation of* 11.8,

$$i_t^* : H^*_c(X \times T; \mathscr{B} \times T) \to H^*_c(X;\mathscr{B})$$

is independent of $t \in T$.[22]

Proof. The point is that here, T need not be acyclic. Let $\pi : X \times T \to X$ be the projection. For any $\alpha \in H^*_c(X \times T; \mathscr{B} \times T)$, let $K(\alpha) = \{t \in T \mid \alpha \in \operatorname{Ker} i_t^*\}$. By 10.6, $t \in K(\alpha) \Rightarrow \alpha|(X \times N) = 0$ for some neighborhood N of $t \in T$. Then $N \subset K(\alpha)$, and it follows that $K(\alpha)$ is open for any α.

Now let $t \in T$ and put $i_t^*(\alpha) = \beta$. Then $i_t^*(\alpha - \pi^*\beta) = 0$, so that $t \in K(\alpha - \pi^*\beta)$. Thus, for all s near t, we have that $s \in K(\alpha - \pi^*\beta)$, which implies that

$$0 = i_s^*(\alpha - \pi^*\beta) = i_s^*(\alpha) - (i_s^*\pi^*)i_t^*(\alpha) = i_s^*(\alpha) - i_t^*(\alpha).$$

Thus the *value* of $i_t^*(\alpha)$ is locally constant in t and hence is constant, since T is connected. □

[22]See, however, Example 14.8.

11.10. Corollary. *The unit interval* $\mathbb{I}$ *is acyclic for any constant coefficients.*

Proof. Consider the map $\mu : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ given by $\mu(s,t) = st$. By 11.9 we have that $(\mu \circ i_0)^* = (\mu \circ i_1)^* : H^*(\mathbb{I}; G) \to H^*(\mathbb{I}; G)$. But $\mu \circ i_1$ is the identity map while $\mu \circ i_0$ is the constant map to $0 \in \mathbb{I}$, and the result follows. □

Remark: Corollary 11.10 can be proved in other ways. See, for example, Exercise 2. It also follows from 9.23, assuming the result for singular theory as known. However, we find it somewhat amusing to use a type of homotopy invariance to prove the acyclicity of $\mathbb{I}$ instead of the other way around.

11.11. Corollary. *If* G *is a compact connected topological group acting on the locally compact Hausdorff space* X, *then* G *acts trivially on* $H_c^*(X; L)$ *for any constant coefficient group* L. □

We remark that there are compact connected groups (e.g., inverse limits of circle groups, called *solenoids*) that are not arcwise connected. Thus such actions need not be homotopically trivial.

If Φ and Ψ are given families of supports on X and Y respectively, we shall say that a map $f : X \to Y$ is *proper* (with respect to Φ and Ψ) if $f^{-1}\Psi \subset \Phi$. If that is the case, then $f^* : H_\Psi^*(Y) \to H_\Phi^*(X)$ is defined. A homotopy $X \times \mathbb{I} \to Y$ is *proper* if it is so with respect to the families $\Phi \times \mathbb{I}$ and Ψ. For locally compact spaces, "proper" means proper with respect to compact supports unless otherwise indicated.

11.12. Theorem. *Any two properly homotopic maps (with respect to* Φ *and* Ψ*) of a space* X *into a space* Y *induce identical homomorphisms*

$$H_\Psi^*(Y; G) \to H_\Phi^*(X; G),$$

where G *is any constant coefficient group.* □

Note the special cases:

(a) $\Phi = cld = \Psi$. In this case, "properly homotopic" is the same as "homotopic."

(b) X, Y locally compact Hausdorff, $\Phi = c = \Psi$.

(c) $A \subset X$, $B \subset Y$, $\Phi = cld|X - A$, $\Psi = cld|Y - B$, with the homotopy taking A through B; see Section 12.

The foregoing results of this section will be considerably generalized in Chapter IV.

11.13. Corollary. *For constant coefficients in* G, $H^*(\bullet\,; G)$ *is an invariant of homotopy type for arbitrary topological spaces and maps.*

Also, $H_c^*(\bullet\,; G)$ *is an invariant of proper homotopy type for locally compact Hausdorff spaces and proper maps.* □

11.14. Example. This example shows that the condition that f be closed in 11.7 cannot be removed. In fact, in this example, f is an open map from a locally compact space to the unit interval, and each fiber $f^{-1}(y)$ is homeomorphic to the unit interval except at one point, $y = 0$, for which it is a real line. Moreover, X has the homotopy type of a circle, so that 11.7 will not hold for any nontrivial constant sheaf of coefficients.

Let X be obtained from the square $\mathbb{I} \times \mathbb{I}$ by deleting the point $\langle 0, \frac{1}{2} \rangle$ and identifying the points $\langle 0, 0 \rangle$ and $\langle 0, 1 \rangle$. Let $f : X \to \mathbb{I}$ be induced by $\langle x, y \rangle \mapsto x$. The reader may verify the properties claimed for this example. ◇

11.15. Example. Let us do some elementary calculations involving the cross product, even though they will follow trivially from later results. Let X be an arbitrary space and Φ a family of supports on X. Let L be a fixed base ring with unit and let $\mathscr{A}$ be a sheaf of L-modules. Consider the product $\mathbb{I} \times X$ and the constant sheaf L on $\mathbb{I}$. The epimorphism $L \widehat{\otimes} \mathscr{A} \twoheadrightarrow L_{\{0\}} \widehat{\otimes} \mathscr{A}$ induces an isomorphism

$$H^n_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L \widehat{\otimes} \mathscr{A}) \xrightarrow{\approx} H^n_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{\{0\}} \widehat{\otimes} \mathscr{A})$$

by 10.2 and 11.8. The exact sequence induced by the coefficient sequence $0 \to L_{(0,1]} \widehat{\otimes} \mathscr{A} \to L \widehat{\otimes} \mathscr{A} \to L_{\{0\}} \widehat{\otimes} \mathscr{A} \to 0$ then shows that

$$H^n_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{(0,1]} \widehat{\otimes} \mathscr{A}) = 0 \text{ for all } n.$$

Then the coefficient sequence $0 \to L_{(0,1)} \widehat{\otimes} \mathscr{A} \to L_{(0,1]} \widehat{\otimes} \mathscr{A} \to L_{\{1\}} \widehat{\otimes} \mathscr{A} \to 0$ shows that

$$H^n_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{\{1\}} \widehat{\otimes} \mathscr{A}) \xrightarrow{\delta} H^{n+1}_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{(0,1)} \widehat{\otimes} \mathscr{A})$$

is an isomorphism for all n. In particular,

$$H^0(\{1\}; L) \approx H^0(\mathbb{I}; L_{\{1\}}) \xrightarrow{\delta} H^1(\mathbb{I}; L_{(0,1)})$$

is an isomorphism. There are the cross products

$$H^0(\mathbb{I}; L_{\{1\}}) \otimes H^n_\Phi(X; \mathscr{A}) \to H^n_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{\{1\}} \widehat{\otimes} \mathscr{A}),$$
$$H^1(\mathbb{I}; L_{(0,1)}) \otimes H^n_\Phi(X; \mathscr{A}) \to H^{n+1}_{\mathbb{I} \times \Phi}(\mathbb{I} \times X; L_{(0,1)} \widehat{\otimes} \mathscr{A}).$$

The first of these is equivalent to the cross product

$$H^0(\{1\}; L) \otimes H^n_\Phi(X; \mathscr{A}) \to H^n_{\mathbb{I} \times \Phi}(\{1\} \times X; L \widehat{\otimes} \mathscr{A})$$

by 10.2, and this is equivalent to the composition

$$L \otimes H^n_\Phi(X; \mathscr{A}) \xrightarrow{\varepsilon^* \otimes 1} H^0(X; L) \otimes H^n_\Phi(X; \mathscr{A}) \xrightarrow{\cup} H^n_\Phi(X; \mathscr{A}),$$

which is an isomorphism taking $1 \otimes a \mapsto 1 \cup a = a$. Letting 1 denote the generator of $H^0(\mathbb{I}; L_{\{1\}}) \approx H^0(\{1\}; L) \approx L$ we have, for $a \in H^n_\Phi(X; \mathscr{A})$,

$$\delta(1 \times a) = \iota \times a,$$

where
$$\iota = \delta(1) \in H^1(\mathbb{I}; L_{(0,1)}).$$
Consequently, we have the isomorphism
$$H^n_\Phi(X;\mathscr{A}) \xrightarrow{\approx} H^{n+1}_{\mathbb{I}\times\Phi}(\mathbb{I}\times X; L_{(0,1)}\widehat{\otimes}\mathscr{A}),$$
given by $a \mapsto \iota \times a$.

In particular we have the isomorphism
$$H^p(\mathbb{I}^n; L_U) \approx \begin{cases} L, & p = n, \\ 0, & p \neq n, \end{cases}$$
where $U = (0,1)^n$ and where $H^n(\mathbb{I}^n; L_U)$ is generated by $\iota \times \cdots \times \iota$. This is isomorphic to $H^p_c((0,1)^n; L) \approx H^p_c(\mathbb{R}^n; L)$ by 10.2, so that
$$H^p_c(\mathbb{R}^n; L) \approx \begin{cases} L, & p = n, \\ 0, & p \neq n. \end{cases}$$
Regarding $\mathbb{S}^n$, $n > 0$, as the one-point compactification of $\mathbb{R}^n$, the exact coefficient sequence $0 \to L_{\mathbb{R}^n} \to L \to L_{\{\infty\}} \to 0$ on $\mathbb{S}^n$ gives
$$H^p(\mathbb{S}^n; L) \approx \begin{cases} L, & p = 0, n, \\ 0, & p \neq 0, n. \end{cases}$$

Now consider the sheaf L_V on $\mathbb{S}^n$ where $V = \mathbb{S}^n - \{x\}$ for some point $x \in \mathbb{S}^n$. Let $f : \mathbb{I}^n \to \mathbb{S}^n$ be the identification of $\partial\mathbb{I}^n$ to the point x. By 11.7,
$$f^* : H^n(\mathbb{S}^n; L_V) \to H^n(\mathbb{I}^n; L_U)$$
is an isomorphism. Let $\alpha_n \in H^n(\mathbb{S}^n; L_V)$ be such that $f^*(\alpha_n) = \iota \times \cdots \times \iota$. Let $u_n = j^*(\alpha_n) \in H^n(\mathbb{S}^n; L)$, where $j^* : H^n(\mathbb{S}^n; L_V) \xrightarrow{\approx} H^n(\mathbb{S}^n; L)$. Similarly, by 11.7,
$$(f\times 1)^* : H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L_V\widehat{\otimes}\mathscr{A}) \to H^p_{\mathbb{I}^n\times\Phi}(\mathbb{I}^n\times X; L_U\widehat{\otimes}\mathscr{A})$$
is an isomorphism for all p that carries $\alpha_n \times \beta \mapsto \iota \times \cdots \times \iota \times \beta$. By the previous remarks, $\beta \mapsto \iota\times\cdots\times\iota\times\beta$ of $H^n_\Phi(X;\mathscr{A}) \to H^{n+p}_{\mathbb{I}^n\times\Phi}(\mathbb{I}^n\times X; L_U\widehat{\otimes}\mathscr{A})$ is an isomorphism. Therefore we have the isomorphism
$$H^n_\Phi(X;\mathscr{A}) \xrightarrow{\approx} H^{n+p}_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L_V\widehat{\otimes}\mathscr{A})$$
given by $\beta \mapsto \alpha_n \times \beta$. By naturality, the map
$$H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L_V\widehat{\otimes}\mathscr{A}) \to H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L\widehat{\otimes}\mathscr{A})$$
takes $\alpha_n \times \beta$ to $u_n \times \beta$. The exact coefficient sequence $0 \to L_V\widehat{\otimes}\mathscr{A} \to L\widehat{\otimes}\mathscr{A} \to L_{\{x\}}\widehat{\otimes}\mathscr{A} \to 0$ induces the exact sequence
$$H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L_V\widehat{\otimes}\mathscr{A}) \to H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L\widehat{\otimes}\mathscr{A}) \to H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n\times X; L_{\{x\}}\widehat{\otimes}\mathscr{A}),$$

in which the last map can be identified with the map $H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n \times X; L\widehat{\otimes}\mathscr{A}) \to H^p_\Phi(X;\mathscr{A})$ induced by the inclusion $i_x : X \hookrightarrow \mathbb{S}^n \times X$. This sequence is split by $\pi^* : H^p_\Phi(X;\mathscr{A}) \to H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n \times X; L\widehat{\otimes}\mathscr{A})$, noting that $L\widehat{\otimes}\mathscr{A} \approx \pi^*\mathscr{A}$. It follows that the map

$$\eta : H^{p-n}_\Phi(X;\mathscr{A}) \oplus H^p_\Phi(X;\mathscr{A}) \to H^p_{\mathbb{S}^n\times\Phi}(\mathbb{S}^n \times X; L\widehat{\otimes}\mathscr{A})$$

given by

$$\eta(a,b) = u_n \times a + 1 \times b$$

is an isomorphism, since $1 \times b = \pi^*(b)$. In particular, since $u_n^2 = 0$ for $n > 0$,

$$H^*(\mathbb{S}^n \times X; L) \approx \Lambda_L(v_n) \otimes H^*(X;L),$$

where $v_n = u_n \times 1 \in H^n(\mathbb{S}^n \times X; L)$ is an algebra isomorphism for $n > 0$, where $\Lambda_L(v_n)$ is the exterior algebra over L on v_n. An induction shows that for all $n_i > 0$,

$$H^*(\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}; L) \approx \Lambda_L(w_1, \ldots, w_n),$$

where $w_i = \pi_i^*(u_{n_i})$, $\pi_i : \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k} \to \mathbb{S}^{n_i}$ being the ith projection. $\diamond$

12 Relative cohomology

In this section, we establish a sheaf-theoretic *relative* cohomology theory. For a closed subspace $F \subset X$ and for Φ paracompactifying, we will have, by 12.3, that $H^*_\Phi(X, F;\mathscr{A}) = H^*_{\Phi|X-F}(X-F;\mathscr{A}|X-F)$ and so in this case, we already have the long exact cohomology sequence of the pair (X, F) by 10.3. This is also enough to conclude a very strong excision property. Hence, for closed paracompact pairs, closed supports, and constant coefficients, this gives us all the Eilenberg-Steenrod axioms for cohomology. The Milnor additivity axiom also holds obviously. Consequently, we can conclude, in this case, that this cohomology theory agrees with singular theory on CW-complexes. (Chapter III, most of which can be read at this point, also establishes this in a different manner.) For most purposes this suffices, and so we recommend that first-time readers skip this somewhat technical section, after making note of the formulas in 12.1 and 12.3 for the purpose of understanding the relative notation.

Let $i : A \hookrightarrow X$ and let Φ be a family of supports on X. For any sheaf $\mathscr{A}$ on X we have the natural i-cohomomorphism (see Section 8)

$$\mathscr{C}^*(X;\mathscr{A}) \rightsquigarrow \mathscr{C}^*(A;\mathscr{A}|A).$$

Equivalently, we have the homomorphism

$$\boxed{i^* : \mathscr{C}^*(X;\mathscr{A}) \to i\mathscr{C}^*(A;\mathscr{A}|A)}$$

of sheaves on X.

In order to define relative cohomology, we shall show that i^* is surjective and has a flabby kernel. We introduce the notation

$$\boxed{\operatorname{Ker} i^* = \mathscr{C}^*(X, A; \mathscr{A}),}$$

$$\boxed{C^*_\Phi(X, A; \mathscr{A}) = \Gamma_\Phi(\mathscr{C}^*(X, A; \mathscr{A})),}$$

and

$$\boxed{H^*_\Phi(X, A; \mathscr{A}) = H^*(C^*_\Phi(X, A; \mathscr{A})).}$$

Since $\operatorname{Ker} i^*$ is flabby and since $\Gamma_\Phi(i\mathscr{B}) = \Gamma_{\Phi\cap A}(\mathscr{B})$ by I-Exercise 8, we will obtain the induced short exact sequence

$$\boxed{0 \to C^*_\Phi(X, A; \mathscr{A}) \to C^*_\Phi(X; \mathscr{A}) \to C^*_{\Phi\cap A}(A; \mathscr{A}|A) \to 0} \tag{21}$$

and hence the long exact cohomology sequence

$$\boxed{\cdots\to H^p_\Phi(X, A; \mathscr{A})\to H^p_\Phi(X; \mathscr{A})\to H^p_{\Phi\cap A}(A; \mathscr{A}|A)\to H^{p+1}_\Phi(X, A; \mathscr{A})\to\cdots} \tag{22}$$

Now $\mathscr{C}^*(X, A; \mathscr{A})$ and $C^*_\Phi(X, A; \mathscr{A})$ are exact functors[23] of $\mathscr{A}$, so that a short exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ of sheaves will induce the long exact sequence

$$\boxed{\cdots\to H^p_\Phi(X,A;\mathscr{A}')\to H^p_\Phi(X,A;\mathscr{A})\to H^p_\Phi(X,A;\mathscr{A}'')\to H^{p+1}_\Phi(X,A;\mathscr{A}')\to\cdots} \tag{23}$$

compatible with (22).

We now proceed to verify our contention. If $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X and A respectively and $k : \mathscr{A} \rightsquigarrow \mathscr{B}$ is an i-cohomomorphism, we shall say that k is *surjective* if each $k_x : \mathscr{A}_x \to \mathscr{B}_x$ is surjective for $x \in A$; that is, if the induced homomorphism $\mathscr{A}|A \to \mathscr{B}$ is surjective.

If $k : \mathscr{A} \rightsquigarrow \mathscr{B}$ is surjective and $U \subset X$ is open, then we have the exact sequence

$$0 \to \prod_{x\in U-A} \mathscr{A}_x \times \prod_{x\in U\cap A} \operatorname{Ker}(k_x) \longrightarrow \prod_{x\in U} \mathscr{A}_x \xrightarrow{\ \prod k_x\ } \prod_{x\in U\cap A} \mathscr{B}_x \to 0.$$

Thus $k^0_U : C^0(U; \mathscr{A}) \to C^0(U \cap A; \mathscr{B})$ is surjective. [Note that this is the map $\mathscr{C}^0(X; \mathscr{A})(U) \to (i\mathscr{C}^0(A; \mathscr{B}))(U)$.] Moreover, the kernel of the surjection $\mathscr{C}^0(X; \mathscr{A}) \twoheadrightarrow i\mathscr{C}^0(X; \mathscr{B})$ is the flabby sheaf

$$U \mapsto \prod_{x\in U-A} \mathscr{A}_x \times \prod_{x\in U\cap A} \operatorname{Ker}(k_x).$$

It follows immediately that the induced i-cohomomorphism

$$k^0 : \mathscr{C}^0(X; \mathscr{A}) \rightsquigarrow \mathscr{C}^0(A; \mathscr{B})$$

[23]This follows from the exactness of the absolute versions by an easy diagram chase.

is surjective in our sense. Passing to quotient sheaves, we see that

$$\mathscr{Z}^1(X;\mathscr{A}) \rightsquigarrow \mathscr{Z}^1(A;\mathscr{B})$$

is also surjective.

By induction, it follows that the homomorphisms

$$\mathscr{C}^n(X;\mathscr{A}) \to i\mathscr{C}^n(A;\mathscr{B})$$

are all surjective and have flabby kernels. Taking $\mathscr{B} = \mathscr{A}|A$ yields our original contention.

Note that

$$0 \to \Gamma_{\Phi|X-A}(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi\cap A}(\mathscr{A}|A)$$

is always exact, so that

$$\boxed{H^0_\Phi(X,A;\mathscr{A}) = \Gamma_{\Phi|X-A}(\mathscr{A}) = H^0_{\Phi|X-A}(X;\mathscr{A}).}$$

Thus the derived functors of $H^0_\Phi(X,A;\mathscr{A})$ are $H^*_{\Phi|X-A}(X;\mathscr{A})$. The following result shows that Φ-tautness is a necessary and sufficient condition for the $H^*_\Phi(X,A;\mathscr{A})$ to be the derived functors of $H^0_\Phi(X,A;\mathscr{A})$.

12.1. Theorem. *If A is a Φ-taut subspace of X, then there is a natural isomorphism*

$$\boxed{H^*_\Phi(X,A;\mathscr{A}) \approx H^*_{\Phi|X-A}(X;\mathscr{A}).}$$

Conversely, if $H^p_\Phi(X,A;\mathscr{A}) = 0$ for $p > 0$ and every flabby sheaf $\mathscr{A}$ on X, then A is Φ-taut.[24]

Proof. Let $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$. By Definition 10.5 the following sequence, which is always left exact, is exact when A is Φ-taut:

$$0 \to \Gamma_{\Phi|X-A}(\mathscr{A}^*) \to \Gamma_\Phi(\mathscr{A}^*) \to \Gamma_{\Phi\cap A}(\mathscr{A}^*|A) \to 0.$$

We also have the natural map $\mathscr{A}^*|A \to \mathscr{C}^*(A;\mathscr{A}|A)$ induced by the natural cohomomorphism $\mathscr{A}^* \rightsquigarrow \mathscr{C}^*(A;\mathscr{A}|A)$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \Gamma_{\Phi|X-A}(\mathscr{A}^*) & \to & \Gamma_\Phi(\mathscr{A}^*) & \to & \Gamma_{\Phi\cap A}(\mathscr{A}^*|A) & \to & 0 \\
 & & \downarrow f & & \downarrow = & & \downarrow g & & \\
0 & \to & C^*_\Phi(X,A;\mathscr{A}) & \to & C^*_\Phi(X;\mathscr{A}) & \to & C^*_{\Phi\cap A}(A;\mathscr{A}|A) & \to & 0.
\end{array}$$

Since A is Φ-taut, whence $\mathscr{A}^*|A$ is $\Phi\cap A$-acyclic, g induces an isomorphism in cohomology. By the 5-lemma, f also induces an isomorphism. The

[24] Also see Exercise 18.

second statement follows directly from (22) and the definition of tautness. □

Note that the proof also shows that in the situation of 12.1, the exact sequence of the pair (X, A) is equivalent to the long exact sequence induced by the short exact sequence

$$0 \to \Gamma_{\Phi|X-A}(\mathscr{A}^*) \to \Gamma_{\Phi}(\mathscr{A}^*) \to \Gamma_{\Phi\cap A}(\mathscr{A}^*|A) \to 0$$

of cochain complexes.

We remark that by (21), $C^*_{\Phi|X-A}(X, A; \mathscr{A}) = C^*_{\Phi|X-A}(X; \mathscr{A})$ because $(\Phi|X-A)\cap A = 0$, and that the isomorphism of 12.1 is induced by inclusion of supports: $H^*_{\Phi|X-A}(X; \mathscr{A}) = H^*_{\Phi|X-A}(X, A; \mathscr{A}) \to H^*_{\Phi}(X, A; \mathscr{A})$.

12.2. The cup product in relative cohomology will be discussed briefly at the end of this section. However, in some special cases it can be produced based on 12.1 as follows. If Φ and Ψ are families of supports on X, then there is the cup product

$$\cup : H^p_{\Phi|X-A}(X; \mathscr{A}) \otimes H^q_{\Psi|X-B}(X; \mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi|X-A\cup B}(X; \mathscr{A} \otimes \mathscr{B}).$$

Therefore, if A is Φ-taut, B is Ψ-taut, and $A \cup B$ is $(\Phi \cap \Psi)$-taut, then by 12.1, this gives a relative cup product

$$\cup : H^p_{\Phi}(X, A; \mathscr{A}) \otimes H^q_{\Psi}(X, B; \mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi}(X, A \cup B; \mathscr{A} \otimes \mathscr{B}).$$

In particular, this holds for arbitrary Φ and Ψ if A and B are open, and it also holds for paracompactifying families when A and B are closed. [Such a product for A and B closed and *arbitrary* families can also be based on 12.3.] Various compatibility formulas follow directly from 7.1. For example, if $U \supset V$ are open in X and $j^* : H^*_{\Psi}(X, V; \mathscr{A}) \to H^*_{\Psi}(X; \mathscr{A})$ is the canonical map (induced, for example, by the inclusion of supports $\Phi|X - V \hookrightarrow \Phi$), $\alpha \in H^p_{\Phi}(X, U; \mathscr{A})$, and $\beta \in H^q_{\Psi}(X, V; \mathscr{B})$, then

$$\alpha \cup j^*(\beta) = \alpha \cup \beta \in H^{p+q}_{\Phi\cap\Psi}(X, U; \mathscr{A} \otimes \mathscr{B}).$$

Similarly, if $\alpha \in H^p_{\Phi}(U; \mathscr{A})$ and $\beta \in H^q_{\Psi}(X; \mathscr{B})$ and δ is the connecting homomorphism for the pair (X, U), then

$$\delta(\alpha \cup \beta) = \delta\alpha \cup \beta \in H^{p+q+1}_{\Phi\cap\Psi}(X, U; \mathscr{A} \otimes \mathscr{B}).$$

Many other such formulas are self-evident.

For the case of closed subspaces, relative cohomology has the following "single space" interpretation:

12.3. Proposition. *If $F \subset X$ is closed, then there is a natural isomorphism*

$$\boxed{H^p_{\Phi}(X, F; \mathscr{A}) \approx H^p_{\Phi}(X; \mathscr{A}_{X-F})}$$

valid for any sheaf $\mathscr{A}$ on X and any family Φ of supports. If, in addition, Φ is paracompactifying,[25] *then*

$$\boxed{H^p_\Phi(X,F;\mathscr{A}) \approx H^p_{\Phi|X-F}(X-F;\mathscr{A}).}$$

Proof. Since $\mathscr{A}_{X-F}|F = 0$, the cohomology sequence of the pair (X,F) with coefficients in $\mathscr{A}_{X-F}$ shows that the map

$$j^* : H^*_\Phi(X,F;\mathscr{A}_{X-F}) \to H^*_\Phi(X;\mathscr{A}_{X-F})$$

is an isomorphism.

The map $H^*_\Phi(X;\mathscr{A}_F) \to H^*_{\Phi\cap F}(F;\mathscr{A}|F)$ is an isomorphism by 10.2. Thus the cohomology sequence of (X,F) with coefficients in $\mathscr{A}_F$ shows that

$$H^*_\Phi(X,F;\mathscr{A}_F) = 0.$$

It follows from the sequence (23), with $\mathscr{A}' = \mathscr{A}_{X-F}$ and $\mathscr{A}'' = \mathscr{A}_F$, that

$$h^* : H^*_\Phi(X,F;\mathscr{A}_{X-F}) \to H^*_\Phi(X,F;\mathscr{A})$$

is an isomorphism. Thus j^* and h^* provide the required isomorphism. The last statement is a special case of 10.2, included here for the benefit of browsers. □

Clearly, the last part of 12.3 is a very strong type of "excision" isomorphism in cases where it applies. In the case of locally compact Hausdorff spaces and compact supports this is often stated as "invariance under relative homeomorphism." A *relative homeomorphism* is a closed map of pairs $(X,A) \to (Y,B)$ such that $A = f^{-1}B$ and the induced map $X-A \to Y-B$ is a homeomorphism.

12.4. Corollary. *Let (X,A) and (Y,B) be closed pairs. Let Ψ and Φ be paracompactifying families of supports on X and Y respectively. Let $f : (X,A) \to (Y,B)$ be a relative homeomorphism such that $\Psi|X-A = f^{-1}(\Phi|Y-B)$ (e.g., if $\Psi = f^{-1}\Phi$). Then*

$$f^* : H^*_\Phi(Y,B;\mathscr{A}) \to H^*_\Psi(X,A;f^*\mathscr{A})$$

is an isomorphism for any sheaf $\mathscr{A}$ on Y.

Proof. This follows directly from 12.3 except for the parenthetical condition. To see that the condition is sufficient, let $K \in \Psi|X-A$. Then $K \subset f^{-1}L$ for some $L \in \Phi$ and so $f(K) \subset L$. Since $f(K)$ is closed, it follows that $f(K) \in \Phi|Y-B$. The converse is immediate. □

This has the following two special cases as the main cases of interest:

[25] Also see 12.10.

12.5. Corollary. *If (X, A) and (Y, B) are closed paracompact pairs and $f : (X, A) \to (Y, B)$ is a relative homeomorphism, then*

$$f^* : H^*(Y, B; \mathscr{A}) \to H^*(X, A; f^*\mathscr{A})$$

is an isomorphism for any sheaf $\mathscr{A}$ on Y. □

12.6. Corollary. *If (X, A) and (Y, B) are closed locally compact Hausdorff pairs and $f : (X, A) \to (Y, B)$ is a proper relative homeomorphism, then*

$$f^* : H^*_c(Y, B; \mathscr{A}) \to H^*_c(X, A; f^*\mathscr{A})$$

is an isomorphism for any sheaf $\mathscr{A}$ on Y. □

12.7. If $f : (X, A) \to (Y, B)$ is a map of pairs and $\mathscr{A}$, $\mathscr{B}$ are sheaves on X and Y respectively with a given f-cohomomorphism $k : \mathscr{B} \rightsquigarrow \mathscr{A}$, then there is the induced commutative diagram (assuming that $f^{-1}\Psi \subset \Phi$)

$$\begin{array}{ccccccccc}
0 & \to & C^*_\Psi(Y, B; \mathscr{B}) & \to & C^*_\Psi(Y; \mathscr{B}) & \to & C^*_{\Psi\cap B}(B; \mathscr{B}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^*_\Phi(X, A; \mathscr{A}) & \to & C^*_\Phi(X; \mathscr{A}) & \to & C^*_{\Phi\cap A}(A; \mathscr{A}) & \to & 0
\end{array}$$

and hence a corresponding diagram of cohomology groups.

For the particular case of the inclusion $(A, B) \hookrightarrow (X, B)$ where $B \subset A \subset X$, we have the commutative diagram (with coefficients in $\mathscr{A}$ understood)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C^*_\Phi(X, A) & \to & C^*_\Phi(X, B) & \to & C^*_{\Phi\cap A}(A, B) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C^*_\Phi(X, A) & \to & C^*_\Phi(X) & \to & C^*_{\Phi\cap A}(A) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C^*_{\Phi\cap B}(B) & = & C^*_{\Phi\cap B}(B) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

in which the columns and second row are exact. Simple diagram chasing yields the fact that the first row is exact. Thus we have the induced exact "cohomology sequence of a triple (X, A, B)":

$$\boxed{\cdots \to H^p_\Phi(X, A; \mathscr{A}) \to H^p_\Phi(X, B; \mathscr{A}) \to H^p_{\Phi\cap A}(A, B; \mathscr{A}) \to H^{p+1}_\Phi(X, A; \mathscr{A}) \to \cdots} \quad (24)$$

Note that when A and B are closed then this is induced by the exact coefficient sequence $0 \to \mathscr{A}_{X-A} \to \mathscr{A}_{X-B} \to \mathscr{A}_{A-B} \to 0$ via 12.3 and 10.2.

Most of the remainder of this section is devoted to generalizing the foregoing results. For example, we wish to see how much the "paracompactifying" assumption can be weakened in the last part of 12.3. We also

want to investigate more general types of subspaces. Consequently, the rest of this section is rather technical and can be skipped with little loss.

For closed subspaces, we have the following "excision" theorem for arbitrary support families:

12.8. Theorem. *If $F \subset X$ is closed and $U \subset F$ is open in X, then the restriction*

$$i^* : H^*_\Phi(X, F; \mathscr{A}) \to H^*_{\Phi\cap(X-U)}(X - U, F - U; \mathscr{A})$$

is an isomorphism for any sheaf $\mathscr{A}$ on X and any family Φ of supports.

Proof. By 12.3 it suffices to show that the homomorphism

$$H^*_\Phi(X; \mathscr{A}_{X-F}) \to H^*_{\Phi\cap(X-U)}(X - U; \mathscr{A}_{X-F})$$

is an isomorphism. By (22) it suffices to show that

$$H^*_\Phi(X, X - U; \mathscr{A}_{X-F}) = 0,$$

but by 12.3, this group is isomorphic to $H^*_\Phi(X; \mathscr{A}_{(X-F)\cap U})$, which is zero since $(X - F) \cap U = \varnothing$. □

The following result is the basic excision theorem for nonclosed subspaces:

12.9. Theorem. *If A and B are subspaces of X with $\overline{B} \subset \operatorname{int}(A)$, then the inclusion of pairs $i : (X - B, A - B) \hookrightarrow (X, A)$ induces an isomorphism*

$$\mathscr{C}^*(X, A; \mathscr{A}) \xrightarrow{\approx} i\mathscr{C}^*(X - B, A - B; \mathscr{A}),$$

and hence the restriction map

$$i^* : H^*_\Phi(X, A; \mathscr{A}) \to H^*_{\Phi\cap(X-B)}(X - B, A - B; \mathscr{A})$$

is an isomorphism for any family of supports Φ on X.

Proof. Since $\mathscr{C}^*(X, A; \mathscr{A})$ is zero on $\operatorname{int}(A)$, $\mathscr{C}^*(X-B, A-B; \mathscr{A})$ is zero on $\operatorname{int}(A) - B$. It follows that $i\mathscr{C}^*(X - B, A - B; \mathscr{A})$ is zero on $\operatorname{int}(A)$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathscr{C}^*(X, A; \mathscr{A}) & \to & \mathscr{C}^*(X; \mathscr{A}) & \to & i\mathscr{C}^*(A; \mathscr{A}) & \to 0 \\
& \downarrow f & & \downarrow g & & \downarrow h & \\
0 \to & i\mathscr{C}^*(X - B, A - B; \mathscr{A}) & \to & i\mathscr{C}^*(X - B; \mathscr{A}) & \to & i\mathscr{C}^*(A - B; \mathscr{A}) & \to 0
\end{array}$$

with exact rows,[26] where i is used for all inclusion maps into X. Now g and h are isomorphisms on the stalk at any point $x \in X - \overline{B}$, and the same

[26] Exactness of the bottom row is by 5.4 and 5.7.

fact follows for f by the 5-lemma. Thus f is an isomorphism, as claimed, since both sheaves vanish on $\operatorname{int}(A) \supset \overline{B}$. □

In some situations, one can prove stronger excision theorems, such as the following:

12.10. Theorem. *If $B \subset A \subset X$ and Φ is a family of supports on X such that A is Φ-taut and $X - B$ is $(\Phi|X - A)$-taut in X and such that $A - B$ is $(\Phi|X - B)$-taut in $X - B$ (e.g., if A is open and B is closed and Φ-taut) then there is a natural isomorphism*

$$H^*_\Phi(X, A; \mathscr{A}) \approx H^*_{\Phi|X-B}(X - B, A - B; \mathscr{A}).$$

Thus, for F closed and Φ-taut in X, we have (by taking $A = B = F$)

$$H^*_\Phi(X, F; \mathscr{A}) \approx H^*_{\Phi|X-F}(X - F; \mathscr{A}).$$

Proof. Since $\Phi|X - A = (\Phi|X - A) \cap (X - B)$ we have the exact sequence (coefficients in $\mathscr{A}$)

$$\cdots \to H^*_{\Phi|X-A}(X, X - B) \to H^*_{\Phi|X-A}(X) \to H^*_{\Phi|X-A}(X - B) \to \cdots$$

Since $X - B$ is $\Phi|X - A$ taut and since $(\Phi|X - A)|B = 0$, the first term is zero by 12.1. Since A is Φ-taut, the second term is isomorphic to $H^p_\Phi(X, A)$ by 12.1. Since $(\Phi|X - A) = (\Phi|X - B)|(X - B) - (A - B)$, the third term is isomorphic to $H^*_{\Phi|X-B}(X - B, A - B)$ by 12.1. □

Note that in 12.10, $\Phi|X - B$ can be replaced by any family Ψ on $X - B$ such that $A - B$ is Ψ-taut and $\Psi|X - A = \Phi|X - A$.

12.11. Corollary. *Let $A \subset X$ be closed, let $\mathscr{A}$ be a sheaf on X/A, and let Φ be a family of supports on X/A such that $\{\star\}$ is Φ-taut in X/A where $\star = \{A\}$; e.g., either some member of Φ is a neighborhood of $\star$ or $\{\star\} \notin \Phi$. Let $\Psi = f^{-1}\Phi$ where $f : X \to X/A$ is the collapsing map. If A is Ψ-taut in X then*

$$f^* : H^*_\Phi(X/A, \star; \mathscr{A}) \to H^*_\Psi(X, A; f^*\mathscr{A})$$

is an isomorphism. In particular, if X is paracompact, then

$$f^* : H^*(X/A, \star; \mathscr{A}) \to H^*(X, A; f^*\mathscr{A})$$

is an isomorphism. Similarly, if A is compact and X is locally compact Hausdorff, then

$$f^* : H^*_c(X/A, \star; \mathscr{A}) \to H^*_c(X, A; f^*\mathscr{A})$$

is an isomorphism.

Proof. It follows from the proof of 12.4 that $\Psi|X-A = f^{-1}(\Phi|X/A-\star)$. Since $\{\star\}$ is Φ-taut, the result is an immediate consequence of the last part of 12.10.[27] □

12.12. Example. This example shows that the tautness assumption in the first part of 12.11 (or the paracompactness assumption in the second part) is necessary. Let $[0,\Omega]$ denote the compactified "long ray" in the compactified long line; see Exercise 3. Let $Y = [0,\Omega] \times [0,1]$ and $B = Y - (0,\Omega) \times (0,1)$, the "boundary" of Y. Let $X = Y - \{\langle\Omega,1\rangle\}$ and $A = X \cap B$. Standard arguments show that any neighborhood of A in X is the intersection with X of a neighborhood of B in Y. It follows that $X/A \approx Y/B$. However, it is not hard to show, with the aid of Exercise 3, that X, Y, and A are all acyclic while B has the cohomology of a circle. It follows that $H^2(X,A;\mathbb{Z}) = 0$ while $H^2(X/A;\mathbb{Z}) \approx H^2(Y/B;\mathbb{Z}) \approx H^2(Y,B;\mathbb{Z}) \approx \mathbb{Z}$. ◇

We shall now show how to simplify the hypotheses of 12.10 in the case in which B is closed.

12.13. Proposition. *Let $B \subset A \subset X$ and let Φ be a family of supports on X. Assume that B is closed. Then $A-B$ is $(\Phi|X-B)$-taut in $X-B$ $\Leftrightarrow$ A is $(\Phi|X-B)$-taut in X.*

Proof. Let $\Psi = \Phi|X-B$. Then $(\Psi \cap A)|B = 0$ and, since $A-B$ is open (hence taut) in A, the restriction (arbitrary coefficients)

$$H^*_{\Psi\cap A}(A) \to H^*_{\Psi\cap(A-B)}(A-B)$$

is an isomorphism by 12.1. Similarly, for an open neighborhood U of A, the restriction

$$H^*_{\Psi\cap U}(U) \to H^*_{\Psi\cap(U-B)}(U-B)$$

is an isomorphism. The result now follows from 10.6. □

12.14. Proposition. *If A and B are subspaces of X with $B \subset A$, and if B is Φ-taut in X, then B is $(\Phi \cap A)$-taut in A.*

Proof. Let $\mathscr{A}$ be a flabby sheaf on A. Then $i\mathscr{A}$ is flabby by 5.7, where $i : A \hookrightarrow X$ is the inclusion. Since B is Φ-taut in X, the map $\Gamma_\Phi(i\mathscr{A}) \to \Gamma_{\Phi\cap B}(i\mathscr{A}|B)$ is surjective. But $\Gamma_\Phi(i\mathscr{A}) = \Gamma_{\Phi\cap A}(\mathscr{A})$ and $\mathscr{A}|B = (i\mathscr{A})|B$, so that $\Gamma_{\Phi\cap A}(\mathscr{A}) \to \Gamma_{\Phi\cap B}(\mathscr{A}|B)$ is surjective. Also, $\mathscr{A}|B = (i\mathscr{A})|B$ is $(\Phi \cap B)$-acyclic since B is Φ-taut in X and $i\mathscr{A}$ is flabby. But $\Gamma_{\Phi\cap A}(\mathscr{A}) \to \Gamma_{\Phi\cap B}(\mathscr{A}|B)$ being surjective and $\mathscr{A}|B$ being $(\Phi \cap B)$-acyclic for all flabby sheaves $\mathscr{A}$ on A is the definition of B being $(\Phi \cap A)$-taut in A. □

12.15. Proposition. *If A and B are Φ-taut subspaces of X with $B \subset A$, then A is $(\Phi|X-B)$-taut in X.*

[27] The second and third parts are also immediate consequences of the corollaries to 12.4.

Proof. Let $\mathscr{A}$ be a flabby sheaf on X. Note that

$$(\Phi|X-B)\cap A = (\Phi\cap A)|A-B$$

since if $K\cap A \in (\Phi\cap A)|A-B$, then $\overline{K\cap A}\cap B = \overline{K\cap A}\cap A\cap B = K\cap A\cap B = \varnothing$. Consider the commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\Gamma_{\Phi|X-B}(\mathscr{A}) & \xrightarrow{f} & \Gamma_{(\Phi|X-B)\cap A}(\mathscr{A}|A) & & \\
\downarrow & & \downarrow & & \\
\Gamma_{\Phi}(\mathscr{A}) & \longrightarrow & \Gamma_{\Phi\cap A}(\mathscr{A}|A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\Gamma_{\Phi\cap B}(\mathscr{A}|B) & \xrightarrow{=} & \Gamma_{\Phi\cap B}(\mathscr{A}|B) & &
\end{array}$$

with exact row and columns (since A is Φ-taut). Diagram chasing reveals that f is onto. Thus it suffices to show that $H^p_{(\Phi\cap A)|A-B}(A;\mathscr{A}|A) = 0$ for $p>0$. But this is isomorphic to $H^p_{\Phi\cap A}(A,B;\mathscr{A})$ by 12.1, since B is $(\Phi\cap A)$-taut by 12.14. By (24) it now suffices to show that

$$H^*_\Phi(X,B;\mathscr{A}) = 0 = H^*_\Phi(X,A;\mathscr{A})$$

for $p>0$. But $H^*_\Phi(X,B;\mathscr{A}) \approx H^*_{\Phi|X-B}(X;\mathscr{A}) = 0$ by 12.1 and since $\mathscr{A}$ is flabby. Similarly, $H^*_\Phi(X,A;\mathscr{A}) = 0$. □

12.16. Corollary. *If $B\subset A\subset X$ with B closed and A and B Φ-taut in X, then*

$$H^*_\Phi(X,A;\mathscr{A}) \approx H^*_{\Phi|X-B}(X-B,A-B;\mathscr{A}). \qquad \square$$

Now we will show that relative cohomology does not depend on the use of the canonical resolution in its definition. The proof will rely on the fact that for any resolution $\mathscr{L}^*$ of a sheaf $\mathscr{A}$, the differential sheaf $\mathscr{C}^*(X;\mathscr{L}^*)$ is also a resolution of $\mathscr{A}$. Here $\mathscr{C}^*(X;\mathscr{L}^*)$ is given the total gradation $\mathscr{C}^*(X;\mathscr{L}^*)^n = \bigoplus_{p+q=n}\mathscr{C}^p(X;\mathscr{L}^q)$ and total differential $d = d'+d''$, where d' is the differential of $\mathscr{C}^*(X;\mathscr{L}^q)$ and d'' is, on $\mathscr{C}^p(X;\mathscr{L}^*)$, $(-1)^p$ times the homomorphism induced by the differential of $\mathscr{L}^*$. This fact follows from the pointwise homotopy triviality of the canonical resolution $\mathscr{C}^*(X;\bullet)$; see Exercise 48.

12.17. Theorem. *Let $i: A\hookrightarrow X$ and let $\mathscr{A}$ be a sheaf on X. Let $\mathscr{L}^*$ and $\mathscr{N}^*$ be resolutions of $\mathscr{A}$ and $\mathscr{A}|A$ respectively for which there is a surjective i-cohomomorphism $k:\mathscr{L}^*\rightsquigarrow\mathscr{N}^*$ of resolutions. Suppose, moreover, that $\mathscr{L}^*$ consists of Φ-acyclic sheaves and $\mathscr{N}^*$ of $(\Phi\cap A)$-acyclic sheaves, and that $k_X:\Gamma_\Phi(\mathscr{L}^*)\to\Gamma_{\Phi\cap A}(\mathscr{N}^*)$ is onto. Let $\mathscr{K}^*$ be the kernel of the associated homomorphism $\mathscr{L}^*\to i\mathscr{N}^*$ [so that $\Gamma_\Phi(\mathscr{K}^*) = \operatorname{Ker} k_X$]. Then there is a natural isomorphism*

$$H^*(\Gamma_\Phi(\mathscr{K}^*)) \approx H^*_\Phi(X,A;\mathscr{A}).$$

Proof. Note that there are the homomorphisms of resolutions $\mathscr{L}^* \to \mathscr{C}^*(X;\mathscr{L}^*) \leftarrow \mathscr{C}^*(X;\mathscr{A})$ and similarly for $\mathscr{N}^*$.

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \Gamma_\Phi(\mathscr{K}^*) & \to & \Gamma_\Phi(\mathscr{L}^*) & \to & \Gamma_{\Phi\cap A}(\mathscr{N}^*) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K^* & \to & C^*_\Phi(X;\mathscr{L}^*) & \to & C^*_{\Phi\cap A}(A;\mathscr{N}^*) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & C^*_\Phi(X,A;\mathscr{A}) & \to & C^*_\Phi(X;\mathscr{A}) & \to & C^*_{\Phi\cap A}(A;\mathscr{A}|A) & \to 0 \end{array}$$

in which the rows are exact (hence defining K^*). The vertical maps on the right and the middle induce isomorphisms in cohomology [since $\mathscr{C}^*(X;\mathscr{L}^*)$ is a flabby resolution of $\mathscr{A}$, etc.]. By the 5-lemma, the vertical maps on the left also induce isomorphisms. □

Now we shall apply 12.17 to the case of the resolution $\mathscr{F}^*(X;\mathscr{A})$ defined at the end of Section 2. We shall use the notation introduced in that section.

Let $M^p(X,A;\mathscr{A})$ be the collection of all $f \in M^p(X;\mathscr{A})$ such that $f(x_0,\ldots,x_p) = 0$ when all $x_i \in A$. Let $F^p(X,A;\mathscr{A}) = \psi_p(M^p(X,A;\mathscr{A}))$. We claim that

$$F^p(X,A;\mathscr{A}) = \operatorname{Ker}\{i^* : F^p(X;\mathscr{A}) \to F^p(A;\mathscr{A}|A)\}.$$

In fact, if $g \in M^p(X;\mathscr{A})$ and $\psi_p(g) \in \operatorname{Ker} i^*$, define $h \in M^p(X;\mathscr{A})$ by

$$h(x_0,\ldots,x_p) = \begin{cases} g(x_0,\ldots,x_p), & \text{if } x_i \in A \text{ for all } i \\ 0, & \text{otherwise.} \end{cases}$$

Then $h \in \operatorname{Ker}\psi_p$ and $f = g - h \in M^p(X,A;\mathscr{A})$, whence $\psi_p(g) = \psi_p(f) \in F^p(X,A;\mathscr{A})$.

Passing to open subsets of X, we see that the presheaf $U \mapsto F^p(U, A\cap U;\mathscr{A})$ is a flabby sheaf [denoted by $\mathscr{F}^p(X,A;\mathscr{A})$] which is the kernel of the restriction $\mathscr{F}^p(X;\mathscr{A}) \to i\mathscr{F}^p(A;\mathscr{A}|A)$. Let

$$F^p_\Phi(X,A;\mathscr{A}) = \Gamma_\Phi(\mathscr{F}^p(X,A;\mathscr{A})).$$

Then it follows that

$$0 \to F^p_\Phi(X,A;\mathscr{A}) \to F^p_\Phi(X;\mathscr{A}) \to F^p_{\Phi\cap A}(A;\mathscr{A}) \to 0$$

is exact, and by 12.17, that

$$\boxed{H^p_\Phi(X,A;\mathscr{A}) \approx H^p(F^*_\Phi(X,A;\mathscr{A})).}$$

The cup product on $M^*(X;\bullet)$ defined at the end of Section 7 induces a homomorphism

$$\mathscr{F}^p(X;\mathscr{A}) \otimes \mathscr{F}^q(X,A;\mathscr{B}) \to \mathscr{F}^{p+q}(X,A;\mathscr{A}\otimes\mathscr{B}), \tag{25}$$

which in turn induces

$$\cup : F^p_\Phi(X;\mathscr{A}) \otimes F^q_\Psi(X,A;\mathscr{B}) \to F^{p+q}_{\Phi\cap\Psi}(X,A;\mathscr{A}\otimes\mathscr{B})$$

and

$$\boxed{\cup : H^p_\Phi(X;\mathscr{A}) \otimes H^q_\Psi(X,A;\mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi}(X,A;\mathscr{A}\otimes\mathscr{B}).}$$

The reader may develop properties of this product as well as show that it coincides with the "single space" cup product

$$\cup : H^p_\Phi(X;\mathscr{A}) \otimes H^q_{\Psi|X-A}(X;\mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi|X-A}(X;\mathscr{A}\otimes\mathscr{B}),$$

via 12.1, when A is Ψ-taut and $(\Phi\cap\Psi)$-taut in X. We shall return to this subject at the end of Section 13.

13 Mayer-Vietoris theorems

In this section we shall first derive the two exact sequences of the Mayer-Vietoris type that one encounters most frequently. We shall then endeavor to generalize these sequences through the use of the relative cohomology groups introduced in the preceding section.

First, let $\mathscr{A}$ be a sheaf on X and Φ a family of supports on X. Let X_1 and X_2 be *closed* subspaces of X with $X = X_1 \cup X_2$ and put $A = X_1 \cap X_2$. We have the surjections

$$r_i : \mathscr{A} \twoheadrightarrow \mathscr{A}_{X_i} \quad \text{and} \quad s_i : \mathscr{A}_{X_i} \twoheadrightarrow \mathscr{A}_A.$$

Considering stalks, we see that the sequence

$$0 \to \mathscr{A} \xrightarrow{\alpha} \mathscr{A}_{X_1} \oplus \mathscr{A}_{X_2} \xrightarrow{\beta} \mathscr{A}_A \to 0$$

is exact, where $\alpha = (r_1, r_2)$ and $\beta = s_1 - s_2$. Thus, by 10.2, we have the exact Mayer-Vietoris sequence (for X_i closed and arbitrary Φ)

$$\boxed{\cdots\to H^p_\Phi(X;\mathscr{A})\to H^p_{\Phi\cap X_1}(X_1;\mathscr{A})\oplus H^p_{\Phi\cap X_2}(X_2;\mathscr{A})\to H^p_{\Phi\cap A}(A;\mathscr{A})\to H^{p+1}_\Phi(X;\mathscr{A})\to\cdots} \quad (26)$$

since $\Phi \cap X_1 = \Phi|X_1$, etc.

Second, assume instead that Φ is paracompactifying and let U_1 and U_2 be open sets with $U = U_1 \cap U_2$ and $X = U_1 \cup U_2$. We have the inclusions

$$i_k : \mathscr{A}_U \rightarrowtail \mathscr{A}_{U_k} \quad \text{and} \quad j_k : \mathscr{A}_{U_k} \rightarrowtail \mathscr{A}.$$

Again the sequence

$$0 \to \mathscr{A}_U \xrightarrow{\alpha} \mathscr{A}_{U_1} \oplus \mathscr{A}_{U_2} \xrightarrow{\beta} \mathscr{A} \to 0$$

is exact, where $\alpha = (i_1, i_2)$ and $\beta = j_1 - j_2$. Thus we obtain from 10.2 the exact Mayer-Vietoris sequence

$$\boxed{\cdots\to H^p_{\Phi|U}(U;\mathscr{A})\to H^p_{\Phi|U_1}(U_1;\mathscr{A})\oplus H^p_{\Phi|U_2}(U_2;\mathscr{A})\to H^p_\Phi(X;\mathscr{A})\to H^{p+1}_{\Phi|U}(U;\mathscr{A})\to\cdots} \quad (27)$$

when Φ is paracompactifying. One can generalize this to the case of *arbitrary* supports on a *normal* space $X = U_1 \cup U_2$ as follows. In fact it is enough to have a separation for the sets $A = X - U_1$ and $B = X - U_2$; say $V \supset A$ is open and $X - \overline{V} \supset B$, i.e., $\overline{V} \subset U_2$. Let $\mathscr{A}^*$ be a flabby resolution of $\mathscr{A}$ on X. Then the sequence

$$0 \to \Gamma_\Phi(\mathscr{A}^*_U) \to \Gamma_\Phi(\mathscr{A}^*_{U_1} \oplus \mathscr{A}^*_{U_2}) \to \Gamma_\Phi(\mathscr{A}^*)$$

is exact, and this is equivalent to the sequence

$$0 \to \Gamma_{\Phi|U}(\mathscr{A}^*|U) \xrightarrow{\alpha} \Gamma_{\Phi|U_1}(\mathscr{A}^*|U_1) \oplus \Gamma_{\Phi|U_2}(\mathscr{A}^*|U_2) \xrightarrow{\beta} \Gamma_\Phi(\mathscr{A}^*).$$

The claimed sequence will follow from this and the fact that $H^*_{\Phi|U}(U;\mathscr{A}) \approx H^*(\Gamma_{\Phi|U}(\mathscr{A}^*|U))$ etc., if we can show that β is surjective. For this, suppose that $f \in \Gamma_\Phi(\mathscr{A}^*)$ and let $h \in \Gamma(\mathscr{A}^*)$ be an extension of f on V and 0 on $X - (|f| \cap \overline{V})$. Then $|h| \subset |f| \cap \overline{V} \in \Phi|U_2$. Also, $g = f - h$ is zero on V and on $X - |f|$, so that $|g| \subset |f| \cap (X - V) \in \Phi|U_1$. Thus $f = g + h = \beta(g, -h)$ is the desired decomposition.

Conversely, if the sequence (27) always holds for $\Phi = cld$ on a space X, then the disjoint closed sets $A = X - U_1$ and $B = X - U_2$ can be separated. To see this, let $\mathscr{A} = \mathscr{C}^0(X;\mathbb{Z})$ and let $f \in \Gamma_\Phi(\mathscr{A})$ be the section that is the constant serration of $\mathbb{Z}$ with value 1. Since $\mathscr{A}|U$ is flabby, we have $H^1_{\Phi|U}(U;\mathscr{A}|U) = 0$, and so the assumed exactness of the sequence implies the existence of a decomposition $f = g + h$ with $g \in \Gamma_{\Phi|U_1}(\mathscr{A}|U_1)$ and $h \in \Gamma_{\Phi|U_2}(\mathscr{A}|U_2)$. Then $X - |g| \supset A$, $X - |h| \supset B$ and $(X - |g|) \cap (X - |h|) = X - (|g| \cup |h|) \subset X - |f| = \varnothing$.[28]

13.1. Example. For an arbitrary space X, the cone CX on X is the quotient space $X \times \mathbb{I}/X \times \{0\}$. Since this is contractible, it is acyclic for any constant coefficients and closed supports. The (unreduced) *suspension* of a space X is $\Sigma X = CX/X \times \{1\}$. This is the union along $X \times \{\frac{1}{2}\}$ of two cones. The Mayer-Vietoris sequence (26) applies to show that $\tilde{H}^k(\Sigma X) \approx \tilde{H}^{k-1}(X)$ for all k. If $\star \in X$ is a base point, then ΣX contains the arc I, which is the image of $\star \times \mathbb{I}$ and is a closed subspace of ΣX. The *reduced suspension* of X is $SX = \Sigma X/I$. Now the collapsing map $\Sigma X \to SX$ is closed and I is connected, acyclic, and taut (since it is compact and relatively Hausdorff in ΣX), so the Vietoris mapping theorem 11.7 applies to show that $\tilde{H}^k(SX) \approx \tilde{H}^k(\Sigma X) \approx \tilde{H}^{k-1}(X)$ for all k. Note that this does not hold in this generality for singular theory where one must assume the space to be "well pointed," that is, that the base point is nondegenerate. ◇

13.2. Example. We shall use the Mayer-Vietoris sequence (26) to compute a rather interesting cohomology group. Let $S = \{0, 1, 1/2, 1/3, \ldots\}$ on the x-axis of the plane. Let $K = \{(x,y) \in \mathbb{R}^2 \mid x \in S,\ -1 \le y \le 1\}$,

[28] Compare Exercise 6.

which is compact. Let $X = K - \{(0,0)\}$, which is locally compact. We wish to compute $H^1(X;\mathbb{Z})$. Let X_1 be the part of X with coordinate $y \geq 0$ and X_2 that with $y \leq 0$. Then $X_1 \cap X_2 = \{1, 1/2, 1/3, \ldots\}$ is discrete, and so $H^0(X_1 \cap X_2) \approx \prod_{i=1}^{\infty} \mathbb{Z}$ is uncountable. However, X_1 has $S \times \{1\}$ as a deformation retract and so $H^0(X_1) \approx H^0(S) \approx \bigoplus_{i=1}^{\infty} \mathbb{Z}$ since it is just the locally constant integer-valued functions on S. (This also follows easily from weak continuity, 10.7.) This group is countable. Similarly for X_2. The Mayer-Vietoris sequence has the segment

$$H^0(X_1) \oplus H^0(X_2) \to H^0(X_1 \cap X_2) \to H^1(X),$$

and it follows that $H^1(X)$ is uncountable.[29] Note that by Exercise 40, $H^1(X) \approx [X; \mathbb{S}^1]$, the group of homotopy classes of maps $X \to \mathbb{S}^1$. Thus, there exist an uncountable number of homotopically distinct maps $X \to \mathbb{S}^1$. This might seem quite unintuitive, and the reader is encouraged to construct some such homotopically nontrivial maps. (Note that they cannot extend to all of K, since $[K; \mathbb{S}^1] \approx H^1(K) = 0$ because $K \simeq S$.) ◇

In the remainder of this section we shall generalize these sequences to more general subspaces. For instance, one may want such a sequence for a pair of subspaces, one of which is open and the other closed. However, this is rare and the rest of this section is rather technical, so we recommend that first-time readers skip to the next section.

The following terminology will be convenient:

13.3. Definition. *Let Φ be a family of supports on the space X. A pair (X_1, X_2) of subspaces of X will be said to be "Φ-excisive" if the inclusion $(X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ induces an isomorphism*

$$H^*_{\Phi \cap (X_1 \cup X_2)}(X_1 \cup X_2, X_2; \mathscr{A}) \xrightarrow{\approx} H^*_{\Phi \cap X_1}(X_1, X_1 \cap X_2; \mathscr{A})$$

for every sheaf $\mathscr{A}$ on X.

Proposition 13.4 will show that this property does not depend on the order in which we take X_1 and X_2.

In order to characterize excisive pairs it will be convenient to work with the resolutions $\mathscr{F}^*(X; \mathscr{A})$ of Section 2. However, the resolutions $\mathscr{C}^*(X; \mathscr{A})$ could be employed in much the same way.

Define

$$M^p(X, X_1, X_2; \mathscr{A}) = M^p(X, X_1; \mathscr{A}) \cap M^p(X, X_2; \mathscr{A}),$$

that is, the set of all $f \in M^p(X; \mathscr{A})$ such that $f(x_0, \ldots, x_p) = 0$ whenever all the x_i are in X_1 or all are in X_2. Let

$$F^p(X, X_1, X_2; \mathscr{A}) = \psi_p(M^p(X, X_1, X_2; \mathscr{A})).$$

[29]In V-9.13, another method is given for doing this computation. It shows that any subspace of the plane looking very roughly like this one has similar properties.

We claim that

$$F^p(X, X_1, X_2; \mathscr{A}) = F^p(X, X_1; \mathscr{A}) \cap F^p(X, X_2; \mathscr{A}).$$

In fact, if $\psi_p(g)$ is in the right-hand side, define

$$h(x_0, \ldots, x_p) = \begin{cases} g(x_0, \ldots, x_p), & \text{if all } x_i \in X_1 \text{ or if all } x_i \in X_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_p(h) = 0$ and $g - h \in M^p(X, X_1, X_2; \mathscr{A})$, so that

$$\psi_p(g) = \psi_p(g - h) \in F^p(X, X_1, X_2; \mathscr{A})$$

as claimed.

The presheaf $\mathscr{F}^p(X, X_1, X_2; \mathscr{A}) : U \mapsto F^p(U, X_1 \cap U, X_2 \cap U; \mathscr{A})$ is the intersection of the subsheaves $\mathscr{F}^p(X, X_i; \mathscr{A})$ of $\mathscr{F}^p(X; \mathscr{A})$, and hence it is a sheaf. It is obviously flabby since $F^p(U, X_1 \cap U, X_2 \cap U; \mathscr{A})$ is the image under ψ_p of

$$\{f \in M^p(U; \mathscr{A}) \mid f(x_0, \ldots, x_p) = 0 \text{ if all } x_i \in X_1 \text{ or all } x_i \in X_2\},$$

and such an f extends to all of X. Let

$$F^p_\Phi(X, X_1, X_2; \mathscr{A}) = \Gamma_\Phi(\mathscr{F}^p(X, X_1, X_2; \mathscr{A}))$$

and

$$H^p_\Phi(X, X_1, X_2; \mathscr{A}) = H^p(F^*_\Phi(X, X_1, X_2; \mathscr{A})).$$

Consider the commutative diagram (28), which has exact rows and columns (coefficients in $\mathscr{A}$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^*_\Phi(X, X_1, X_2) & \to & F^*_\Phi(X, X_2) & \longrightarrow & F^*_{\Phi \cap X_1}(X_1, X_1 \cap X_2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^*_\Phi(X, X_1) & \to & F^*_\Phi(X) & \xrightarrow{i_1} & F^*_{\Phi \cap X_1}(X_1) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} & & \\
0 & \to & F^*_{\Phi \cap X_2}(X_2, X_1 \cap X_2) & \to & F^*_{\Phi \cap X_2}(X_2) & \xrightarrow{j_2} & F^*_{\Phi \cap X_1 \cap X_2}(X_1 \cap X_2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{28}$$

Taking $X = X_1 \cup X_2$ in (28) we obtain immediately, from the cohomology sequence of the first row, the following criterion for Φ-excisiveness:

13.4. Proposition. *(X_1, X_2) is Φ-excisive $\Leftrightarrow$ $H^p_\Psi(X_1 \cup X_2, X_1, X_2; \mathscr{A}) =$ 0 for all p and all sheaves $\mathscr{A}$ on $X_1 \cup X_2$, where $\Psi = \Phi \cap (X_1 \cup X_2)$.* □

For general $X \supset X_1 \cup X_2$ we also have the exact sequence

$$F^*_\Phi(X, X_1 \cup X_2; \mathscr{A}) \rightarrowtail F^*_\Phi(X, X_1, X_2; \mathscr{A}) \twoheadrightarrow F^*_{\Phi\cap(X_1\cup X_2)}(X_1 \cup X_2, X_1, X_2; \mathscr{A})$$

and it follows that *for* (X_1, X_2) Φ*-excisive*, there is a natural isomorphism

$$H^*_\Phi(X, X_1 \cup X_2; \mathscr{A}) \approx H^*_\Phi(X, X_1, X_2; \mathscr{A}). \tag{29}$$

From (28) we derive the exact sequence (coefficients in $\mathscr{A}$)

$$\begin{aligned} 0 \to F^*_\Phi(X, X_1, X_2) \to F^*_\Phi(X) &\xrightarrow{(i_1, i_2)} F^*_{\Phi\cap X_1}(X_1) \oplus F^*_{\Phi\cap X_2}(X_2) \\ &\xrightarrow{j_1 - j_2} F^*_{\Phi\cap X_1\cap X_2}(X_1 \cap X_2) \to 0. \end{aligned} \tag{30}$$

For the moment let $G^*_\Phi(X_1, X_2) = F^*_\Phi(X_1 \cup X_2)/F^*_\Phi(X_1 \cup X_2, X_1, X_2)$. Then $H^*_\Phi(X_1 \cup X_2; \mathscr{A}) \approx H^*(G^*_\Phi(X_1, X_2))$ for (X_1, X_2) Φ-excisive, and from (30) we have the exact sequence

$$G^*_\Phi(X_1, X_2) \rightarrowtail F^*_{\Phi\cap X_1}(X_1) \oplus F^*_{\Phi\cap X_2}(X_2) \twoheadrightarrow F^*_{\Phi\cap X_1\cap X_2}(X_1 \cap X_2). \tag{31}$$

For (X_1, X_2) Φ-excisive, (31) induces the exact Mayer-Vietoris sequence (26), where $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$.

Now let (A_1, A_2) be a pair of subspaces of X with $A_i \subset X_i$. Let

$$G^*_\Phi(X_1, X_2; A_1, A_2) = \operatorname{Ker}\{G^*_\Phi(X_1, X_2) \to G^*_{\Phi\cap(A_1\cup A_2)}(A_1, A_2)\}.$$

Then for (X_1, X_2) and (A_1, A_2) both Φ-excisive, we have

$$H^*_\Phi(X_1 \cup X_2, A_1 \cup A_2; \mathscr{A}) \approx H^*(G^*_\Phi(X_1, X_2; A_1, A_2)).$$

Consider the following commutative diagram with exact rows and columns (in which we have omitted the obvious supports):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & G^*(X_1, X_2; A_1, A_2) & \to & F^*(X_1, A_1) \oplus F^*(X_2, A_2) & \to & F^*(X_1 \cap X_2, A_1 \cap A_2) & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & G^*(X_1, X_2) & \to & F^*(X_1) \oplus F^*(X_2) & \to & F^*(X_1 \cap X_2) & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & G^*(A_1, A_2) & \to & F^*(A_1) \oplus F^*(A_2) & \to & F^*(A_1 \cap A_2) & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From this we derive that when (X_1, X_2) and (A_1, A_2) are both Φ-excisive with $A_i \subset X_i$, there is the following exact Mayer-Vietoris sequence (with coefficients in $\mathscr{A}$):

$$\boxed{\begin{aligned} \cdots \to H^p_\Phi(X_1 \cup X_2, A_1 \cup A_2) \to H^p_{\Phi\cap X_1}(X_1, A_1) \oplus H^p_{\Phi\cap X_2}(X_2, A_2) \\ \to H^p_{\Phi\cap X_1\cap X_2}(X_1 \cap X_2, A_1 \cap A_2) \to \cdots \end{aligned}} \tag{32}$$

This exact sequence generalizes both (26) and (27).

In both of the following cases, the pair (X_1, X_2) is Φ-excisive for all Φ:

(a) $X_1 \cup X_2 = \operatorname{int}(X_1) \cup \operatorname{int}(X_2)$, interiors relative to $X_1 \cup X_2$, by 12.9.

(b) X_1 and X_2 both closed in $X_1 \cup X_2$, by 12.8.[30]

In particular, (32) is always valid when the X_i and A_i are all closed. Consequently, (27) is valid by 12.10 whenever the closed sets $X - U_1$, $X - U_2$, and $X - U$ are all Φ-taut (and $X = U_1 \cup U_2$).

The following result gives another sufficient condition for Φ-excisiveness:

13.5. Proposition. *Let $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$. If Φ is a family of supports on X with*

$$(\Phi \cap X_1)|(X_1 - A) = \Phi|(X - X_2) \subset \Phi|\operatorname{int}(X_1)$$

and such that X_2 is Φ-taut in X and A is $(\Phi \cap A)$-taut in X_1, then (X_1, X_2) is Φ-excisive.

Proof. Since $\mathscr{C}^*(X, X_1; \mathscr{A})$ vanishes on $\operatorname{int}(X_1)$ and since

$$\Phi|X - X_2 \subset \Phi|\operatorname{int}(X_1),$$

we have that

$$H^*_{\Phi|X-X_2}(X, X_1; \mathscr{A}) = 0.$$

By 12.1 it follows that X_1 is $(\Phi|X - X_2)$-taut. Let $\mathscr{A}^*$ be a flabby resolution of $\mathscr{A}$ on X. Since $(\Phi|X - X_2) = (\Phi|X - X_2) \cap X_1$, the restriction

$$f : \Gamma_{\Phi|X-X_2}(\mathscr{A}^*) \to \Gamma_{\Phi|X-X_2}(\mathscr{A}^*|X_1)$$

is surjective. Its kernel is $\Gamma_{(\Phi|X-X_2)|X-X_1}(\mathscr{A}^*) = 0$. Since X_1 is $(\Phi|X - X_2)$-taut and $\Phi|X - X_2 = (\Phi \cap X_1)|X_1 - A$, f induces an isomorphism

$$f^* : H^*_{\Phi|X-X_2}(X; \mathscr{A}) \xrightarrow{\approx} H^*_{(\Phi \cap X_1)|X_1 - A}(X_1; \mathscr{A}|X_1),$$

which by tautness and 12.1 can be canonically identified with

$$H^*_\Phi(X, X_2; \mathscr{A}) \xrightarrow{\approx} H^*_{\Phi \cap X_1}(X_1, A; \mathscr{A}),$$

as was to be shown. □

13.6. Corollary. *If $X = X_1 \cup \operatorname{int}(X_2) = \operatorname{int}(X_1) \cup X_2$ and if X_1, X_2, and $X_1 \cap X_2$ are all Φ-taut in X, then (X_1, X_2) is Φ-excisive.*

[30] This contrasts with singular theory, where such a weak condition falls far short of sufficing.

Proof. This follows easily from 12.14 and 13.5. □

We note that with the hypotheses of 13.6, the Mayer-Vietoris sequence (26) can be derived directly as the cohomology sequence of the short exact sequence

$$0 \to \Gamma_\Phi(\mathscr{A}^*) \xrightarrow{(r_1,r_2)} \Gamma_{\Phi\cap X_1}(\mathscr{A}^*|X_1) \oplus \Gamma_{\Phi\cap X_2}(\mathscr{A}^*|X_2) \xrightarrow{j_1-j_2} \Gamma_{\Phi\cap A}(\mathscr{A}^*|A) \to 0$$

where $\mathscr{A}^*$ is a flabby resolution of $\mathscr{A}$.

13.7. Let A and B be subspaces of X and let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X. The cup product on $M^*(X;\bullet)$ defined in Section 7 clearly induces a product

$$M^p(X,A;\mathscr{A}) \otimes M^q(X,B;\mathscr{B}) \to M^{p+q}(X,A,B;\mathscr{A}\otimes\mathscr{B})$$

and hence also

$$\mathscr{F}^p(X,A;\mathscr{A}) \otimes \mathscr{F}^q(X,B;\mathscr{B}) \to \mathscr{F}^{p+q}(X,A,B;\mathscr{A}\otimes\mathscr{B}),$$

$$F^p_\Phi(X,A;\mathscr{A}) \otimes F^q_\Psi(X,B;\mathscr{B}) \to F^{p+q}_{\Phi\cap\Psi}(X,A,B;\mathscr{A}\otimes\mathscr{B}),$$

and

$$H^p_\Phi(X,A;\mathscr{A}) \otimes H^q_\Psi(X,B;\mathscr{B}) \to H^{p+q}_{\Phi\cap\Psi}(X,A,B;\mathscr{A}\otimes\mathscr{B}).$$

Thus, by (29), when (A,B) is $(\Phi\cap\Psi)$-excisive, we obtain the cup product

$$\boxed{\cup : H^p_\Phi(X,A;\mathscr{A}) \otimes H^q_\Psi(X,B;\mathscr{B}) \to H^{p+q}_{\Phi\times\Psi}(X,A\cup B;\mathscr{A}\otimes\mathscr{B}).} \qquad (33)$$

Also see Exercises 19 and 20 at the end of the chapter.

If $A \subset X$ and $B \subset Y$ and if Φ and Ψ are families of supports on X and Y, respectively, such that $(X\times B, A\times Y)$ is $(\Phi\times\Psi)$-excisive, then the cup product (33) induces the cross product[31]

$$\boxed{\times : H^p_\Phi(X,A;\mathscr{A}) \otimes H^q_\Psi(X,B;\mathscr{B}) \to H^{p+q}_{\Phi\times\Psi}((X,A)\times(Y,B);\mathscr{A}\widehat{\otimes}\mathscr{B}),}$$

by $\alpha\times\beta = \pi_X^*(\alpha)\cup\pi_Y^*(\beta)$, π_X and π_Y being the projections $(X\times Y, A\times B) \to (X,A)$ and $(X\times Y, A\times B) \to (Y,B)$ respectively. In particular, the cross product is defined when either A or B is empty or when A and B are both closed or both open.

14 Continuity

Let $D = \{\lambda,\mu,\ldots\}$ be a *directed* set. Let $\{X_\lambda; f_{\lambda,\mu}\}$ be an inverse system of spaces based on D (that is, $f_{\lambda,\mu} : X_\mu \to X_\lambda$ is defined for $\mu > \lambda$ and satisfies $f_{\lambda,\mu}\circ f_{\mu,\nu} = f_{\lambda,\nu}$), and let $X = \varprojlim X_\lambda$. For each $\lambda \in D$ let $\mathscr{A}_\lambda$ be a sheaf

[31] Where $(X,A)\times(Y,B)$ denotes the pair $(X\times Y, X\times B\cup A\times Y)$.

on X_λ, and for $\mu > \lambda$ assume that we are given an $f_{\lambda,\mu}$-cohomomorphism $k_{\lambda,\mu} : \mathscr{A}_\mu \rightsquigarrow \mathscr{A}_\lambda$ such that for $\nu > \mu > \lambda$, $k_{\nu,\mu} \circ k_{\mu,\lambda} = k_{\nu,\lambda}$. (That is, $\{\mathscr{A}_\lambda; k_{\mu,\lambda}\}$ is a *direct* system of sheaves and cohomomorphisms.)

Let $f_\lambda : X \to X_\lambda$ be the canonical projection. Note that $k_{\mu,\lambda}$ induces a homomorphism

$$h_{\mu,\lambda} : f_\lambda^* \mathscr{A}_\lambda \to f_\mu^* \mathscr{A}_\mu$$

(regard $f_\lambda^*\mathscr{A}_\lambda$ as $f_\mu^* f_{\lambda,\mu}^* \mathscr{A}$) with $h_{\nu,\mu} \circ h_{\mu,\lambda} = h_{\nu,\lambda}$. Let $\mathscr{A} = \varinjlim f_\lambda^* \mathscr{A}_\lambda$.

14.1. We have, for $x \in X$, the natural commutative diagram

$$\begin{array}{ccccc} \mathscr{A}_\lambda(X_\lambda) & \longrightarrow & (f_\lambda^*\mathscr{A}_\lambda)(X) & \longrightarrow & \mathscr{A}(X) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathscr{A}_\lambda)_{f_\lambda(x)} & \xrightarrow{\approx} & (f_\lambda^*\mathscr{A}_\lambda)_x & \longrightarrow & \mathscr{A}_x. \end{array}$$

The horizontal map on the lower right becomes an isomorphism upon passage to the limit over λ (since $\mathscr{A} = \varinjlim f_\lambda^*\mathscr{A}_\lambda$). Thus we obtain the commutative diagram

$$\begin{array}{ccc} \varinjlim \mathscr{A}_\lambda(X_\lambda) & \xrightarrow{\theta} & \mathscr{A}(X) \\ \downarrow & & \downarrow \\ \varinjlim (\mathscr{A}_\lambda)_{f_\lambda(x)} & \underset{\approx}{\xrightarrow{\theta_x}} & \mathscr{A}_x. \end{array}$$

Note that for suitable supports, $H^*(X_\lambda; \mathscr{A}_\lambda)$ forms (via $k_{\mu,\lambda}^*$) a direct system of groups and that there are compatible maps $H^*(X_\lambda; \mathscr{A}_\lambda) \to H^*(X; f_\lambda^*\mathscr{A}_\lambda) \to H^*(X; \mathscr{A})$, so that θ generalizes to a map

$$\boxed{\theta : \varinjlim H^*(X_\lambda; \mathscr{A}_\lambda) \to H^*(X; \mathscr{A}).}$$

14.2. Lemma. *If each X_λ (and hence X) is compact Hausdorff, then the canonical map*

$$\theta : \varinjlim \mathscr{A}_\lambda(X_\lambda) \to \mathscr{A}(X)$$

is an isomorphism.

Proof. First, we show that θ is one-to-one. Let $s_\lambda \in \mathscr{A}_\lambda(X_\lambda)$ and for $\mu > \lambda$, let $s_\mu = k_{\mu,\lambda}(s_\lambda) \in \mathscr{A}_\mu(X_\mu)$. Let $s \in \varinjlim(\mathscr{A}_\lambda(X_\lambda))$ be the element defined by s_λ and assume that $\theta(s) = 0$.

Now, $\theta(s)(x) = \theta_x(\varinjlim s_\mu(f_\mu(x)))$ by 14.1. Thus, for each $x \in X$ there is an element $\mu > \lambda$ of D such that $s_\mu(f_\mu(x)) = 0$. But then $s_\mu(f_\mu(y)) = 0$ for all y sufficiently close to x, because a section of a sheaf meets the zero section in an *open* set. By the compactness of X and the *directedness* property of D, there is a $\mu > \lambda$ such that $s_\mu(f_\mu(x)) = 0$ for all $x \in X$. Thus s_μ vanishes in a *neighborhood* N of $f_\mu(X) \subset X_\mu$. Since the X_ν are compact, there is a $\nu > \mu$ such that $f_{\mu,\nu}(X_\nu) \subset N$ (this follows from the fact that the inverse limit of nonempty compact sets is nonempty), and it follows that $s_\nu = k_{\nu,\mu}(s_\mu) = 0$ on all of X_ν. Thus θ is injective.

We now show that θ is onto. Let $s \in \mathscr{A}(X)$. For each $x \in X$, there is a $\lambda = \lambda(x) \in D$ and an element of $(\mathscr{A}_\lambda)_{f_\lambda(x)}$ mapping onto $s(x)$ by 14.1. Let $U_\lambda \subset X_\lambda$ be a neighborhood of $f_\lambda(x)$ and let $s_\lambda \in \mathscr{A}_\lambda(U_\lambda)$ be such that $s_\lambda(f_\lambda(x))$ maps to $s(x)$. Then, using k_λ to also denote the map $\mathscr{A}_\lambda(U_\lambda) \to \mathscr{A}(f_\lambda^{-1}(U_\lambda))$, $k_\lambda(s_\lambda)$ coincides with s in some neighborhood of x. We may assume that $k_\lambda(s_\lambda) = s|f_\lambda^{-1}(U_\lambda)$, since sets of the form $f_\mu^{-1}(U_\mu)$, μ varying, form a neighborhood basis in X.

Now, since X is compact, there are a finite number of such $f_\lambda^{-1}(U_\lambda)$ that cover X, and by the directedness property of D, we may assume that for some *fixed* λ, there are a finite number of open subsets $U_1, \dots, U_n$ of X_λ and elements $s_i \in \mathscr{A}_\lambda(U_i)$ such that

$$k_\lambda(s_i) = s|f_\lambda^{-1}(U_i) \quad \text{for } i = 1, \dots, n,$$

and such that the $f_\lambda^{-1}(U_i)$ cover X. Passing to a larger λ, it may even be assumed that the U_i cover X_λ.

Let $\{V_i \,|\, i = 1, \dots, n\}$ be an open covering of X_λ such that $\overline{V}_i \subset U_i$. Then, for each pair (i,j), k_λ maps $s_i|\overline{V}_{i,j} - s_j|\overline{V}_{i,j}$ to zero, where $V_{i,j} = V_i \cap V_j$. Now $f_\lambda^{-1}(\overline{V}_{i,j}) = \varprojlim f_{\lambda,\mu}^{-1}(\overline{V}_{i,j})$ (over $\mu > \lambda$), and similarly for the restrictions of the sheaves to these sets, so that the fact that θ is a monomorphism implies that there is a $\mu = \mu(i,j) > \lambda$ such that $k_{\mu,\lambda}(s_i|\overline{V}_{i,j}) = k_{\mu,\lambda}(s_j|\overline{V}_{i,j})$. We may assume that μ is independent of (i,j), and replacing λ by μ, we see that we can assume that s_i and s_j coincide on $V_i \cap V_j$ for all i,j. Thus there is an $s_\lambda \in \mathscr{A}_\lambda(X_\lambda)$ (λ large) restricting on V_i to s_i. Hence $k_\lambda s_\lambda = s$ and the lemma follows, since θ is induced by the k_λ. □

14.3. Corollary. *If each X_λ is compact Hausdorff and each $\mathscr{A}_\lambda$ is soft, then $\mathscr{A}$ is soft.*[32]

Proof. For any closed set $F \subset X$, $F = \varprojlim(f_\lambda F)$, so that by 14.2, $\mathscr{A}(F) = \varinjlim \mathscr{A}_\lambda(f_\lambda F)$, and the result follows immediately. □

14.4. Theorem. (Continuity.) *If each X_λ is locally compact Hausdorff and each $f_{\lambda,\mu}$ is proper, then the homomorphism*

$$\theta : \varinjlim H_c^*(X_\lambda; \mathscr{A}_\lambda) \to H_c^*(X; \mathscr{A})$$

is an isomorphism.

Proof. Embed each X_λ, and hence X, in its one-point compactification X_λ^+ and extend each $\mathscr{A}_\lambda$ by zero to a sheaf on X_λ^+. This does not alter the cohomology with *compact* supports because $H_c^*(X; \mathscr{A}) \approx H^*(X^+; \mathscr{A}^{X^+})$ by 10.1. Therefore it suffices to treat the case in which each X_λ is compact.

[32] Also see 16.30 and Exercise 29.

Now let $\mathscr{L}_\lambda^* = \mathscr{C}^*(X_\lambda;\mathscr{A}_\lambda)$. The cohomomorphism $k_{\mu,\lambda} : \mathscr{A}_\lambda \rightsquigarrow \mathscr{A}_\mu$ induces a cohomomorphism

$$k_{\mu,\lambda}^* : \mathscr{L}_\lambda^* \rightsquigarrow \mathscr{L}_\mu^*.$$

Note that $f_\lambda^*\mathscr{L}_\lambda^*$ is a resolution of $f_\lambda^*\mathscr{A}_\lambda$ since f_λ^* is an exact functor. Thus, $\mathscr{L}^* =_{\text{def}} \varinjlim f_\lambda^*\mathscr{L}_\lambda^*$ is a resolution of $\varinjlim f_\lambda^*\mathscr{A}_\lambda = \mathscr{A}$. By 14.3, $\mathscr{L}^*$ is soft.

The cohomomorphism $k_\lambda^* : \mathscr{L}_\lambda^* = \mathscr{C}^*(X_\lambda;\mathscr{A}_\lambda) \rightsquigarrow \mathscr{C}^*(X;\mathscr{A}) = \mathscr{A}^*$ admits the factorization

$$\mathscr{L}_\lambda^* \rightsquigarrow f_\lambda^*\mathscr{L}_\lambda^* \to \mathscr{A}^*$$

inducing

$$H^*(X_\lambda;\mathscr{A}_\lambda) = H^*(\mathscr{L}_\lambda^*(X_\lambda)) \to H^*(f_\lambda^*\mathscr{L}_\lambda^*(X)) \to H^*(\mathscr{A}^*(X)) = H^*(X;\mathscr{A}),$$

which induces θ upon passage to the limit. However,

$$\varinjlim H^*(\mathscr{L}_\lambda^*(X_\lambda)) = H^*(\varinjlim \mathscr{L}_\lambda^*(X_\lambda)) = H^*(\mathscr{L}^*(X))$$

by 14.2, and the homomorphism $\mathscr{L}^* \to \mathscr{A}^*$ of resolutions induces an isomorphism $H^*(\mathscr{L}^*(X)) \xrightarrow{\approx} H^*(\mathscr{A}^*(X))$ by 4.2, since $\mathscr{L}^*$ is soft. □

There are two special cases of 14.4 of individual importance. First, when each $X_\lambda = X$, we obtain:

14.5. Corollary. *Let $\{\mathscr{A}_\lambda\}$ be a direct system of sheaves on a locally compact Hausdorff space X and let $\mathscr{A} = \varinjlim \mathscr{A}_\lambda$. Then the canonical map*

$$\theta : \varinjlim H_c^*(X;\mathscr{A}_\lambda) \to H_c^*(X;\mathscr{A})$$

is an isomorphism.[33] □

Second, if each $\mathscr{A}_\lambda$ is the constant sheaf G, we obtain:

14.6. Corollary. *Let $\{X_\lambda\}$ be an inverse system of locally compact Hausdorff spaces and proper maps, and let $X = \varprojlim X_\lambda$. Then for constant coefficients in G, the canonical map*

$$\theta : \varinjlim H_c^*(X_\lambda;G) \to H_c^*(X;G)$$

is an isomorphism. □

14.7. Example. For each integer $n > 0$ let $X_n = \mathbb{S}^1$ and let $\pi_n : X_{n+1} \to X_n$ be the covering map of degree p. Then the space $\Sigma_p = \varprojlim X_n$ is called the "p-adic solenoid." By 14.6 we see that $H^1(\Sigma_p;\mathbb{Z}) \approx \mathbb{Q}_p$, the group of rational numbers with denominators a power of p.

[33] Contrast Exercise 4.

If instead, we take π_n to be of degree n, then the solenoid $\Sigma = \varprojlim X_n$ has $H^1(\Sigma;\mathbb{Z}) \approx \mathbb{Q}$. ◇

14.8. Example. Let Y_n be the union of the unit circle $C = \mathbb{S}^1$ with the line segment $I = [-1,1] \times \{0\}$. Let $\pi_n : Y_{n+1} \to Y_n$ be the covering map of degree 3 on $X_{n+1} = C$ together with the identity on I. Put $Y = \varprojlim Y_n$, which is compact and contains the compact subspace $X = \Sigma_3$. Also let $a, b \in Y$ be the points corresponding to the end points of the interval I. Then Y is just the mapping cone of the inclusion $\{a,b\} \hookrightarrow X$. In simplicial homology with integer coefficients take the basis of $H^1(Y_n)$ represented by the counterclockwise cycle in C and the cycle given by I in the direction from 1 to -1 followed by the counterclockwise lower semicircle in C. Take the Kronecker dual basis $\{\alpha_n, \beta_n\}$ in $H^1(Y_n)$. Let $g_n : Y_n \twoheadrightarrow C$ be the projection collapsing X_n to a point and let $\gamma \in H^1(C)$ be the class dual to the counterclockwise circle. Then one can compute that

$$\begin{aligned} \pi_n^*(\alpha_n) &= 3\alpha_{n+1} + \beta_{n+1} \\ \pi_n^*(\beta_n) &= \beta_{n+1} \\ g_n^*(\gamma) &= \beta_n. \end{aligned}$$

It follows from continuity that in terms of generators and relations,

$$H^1(Y) \approx \{\beta, \alpha_1, \alpha_2, \dots \mid \alpha_n = 3\alpha_{n+1} + \beta\}.$$

The map $g : Y \to Y/X \approx C$ induces $g^*(\gamma) = \beta$. Since $H^2(Y,X) \approx H_c^2(Y - X) \approx H^2(I, \partial I) = 0$ and $\tilde{H}^0(X) = 0$, there is the exact sequence

$$0 \to H^1(Y,X) \to H^1(Y) \to H^1(X) \to 0,$$

which has the form

$$0 \to \mathbb{Z} \to H^1(Y) \to \mathbb{Q}_3 \to 0.$$

We claim that this does not split. Indeed, a splitting map $f : H^1(Y) \to \mathbb{Z}$ would have the form

$$\begin{aligned} f(\alpha_n) &= s_n, \\ f(\beta) &= 1, \end{aligned}$$

for some integers s_n, and the relations imply that

$$s_n = 3s_{n+1} + 1.$$

It is easy to see that no such sequence of integers can exist.

For a space K consider the Mayer-Vietoris sequence (26) of the pair $(K \times X, K \times I)$ of closed subspaces of $K \times Y$. Let $j : K \times X \hookrightarrow K \times Y$ be the inclusion. Identifying the cohomology of $K \times I$ with that of K and that of $K \times \{a,b\}$ with the direct sum of two copies of that of K, this sequence has a segment of the form

$$H^1(K \times Y) \to H^1(K \times X) \oplus H^1(K) \xrightarrow{\varphi} H^1(K) \otimes H^1(K),$$

and the homomorphism φ is given by

$$\varphi\langle s,t\rangle = \langle i_a^*(s)+t,\ -i_b^*(s)-t\rangle,$$

where $i_x : K \to K\times X$ is the inclusion $i_x(k) = \langle k, x\rangle$. We wish to ask, as in 11.9, whether $i_a^* = i_b^*$. If this is the case, then for any $s \in H^1(K\times X)$ there is a $t \in H^1(K)$, namely $t = -i_a^*(s) = -i_b^*(s)$, such that $\varphi\langle s,t\rangle = 0$. This implies that $s \in \operatorname{Im}\left(j^* : H^1(K\times Y) \to H^1(K\times X)\right)$. Thus the condition $i_a^* = i_b^*$ implies that j^* is surjective. By Exercise 40 this is equivalent to $j^\# : [K\times Y; C] \to [K\times X; C]$ being onto. By the exponential law in homotopy theory this implies that $\hat{\jmath} : [K; C^Y] \to [K; C^X]$ is onto. Now take $K = C^X$, which is metrizable and hence paracompact. If $\hat{\jmath}$ is onto, then there is a map $\lambda : C^X \to C^Y$ such that the composition $C^X \to C^Y \to C^X$ is homotopic to the identity. Then $\lambda_\# : [X;C] \to [Y;C]$ splits the surjection $H^1(Y) \approx [Y;C] \to [X;C] \approx H^1(X)$. We have seen that such a splitting does not exist in the present example, and so we conclude that for $K = C^X$ where $X = \Sigma_3$, the epimorphisms $i_x^* : H^1(K\times X) \twoheadrightarrow H^1(K)$ are *not* independent of $x \in X$. ◇

14.9. Example. Consider Example I-2.7, and retain the notation used there. The sheaves $\mathscr{L}_n$ there form a direct system, of which an increasing union is a special case, with direct limit $\mathscr{L}$. We wish to illustrate 14.5 by calculating the cohomology of X with coefficients in $\mathscr{L}$. Note that $\mathscr{L}_n/\mathscr{L}_{n-1} \approx \mathbb{Z}_{A_n}$ for $n>1$, and $\mathscr{L}_1 \approx \mathbb{Z}_{U_1}$. Thus, for $n>1$,

$$\begin{aligned}
H^p(X;\mathscr{L}_n/\mathscr{L}_{n-1}) &\approx H^p(X;\mathbb{Z}_{A_n}) & \\
&\approx H^p_{cld|A_n}(A_n;\mathbb{Z}) & \text{by 10.2}\\
&\approx H^p_c(A_n;\mathbb{Z}) & \text{since } cld|A_n = c\\
&\approx H^p_c(\mathbb{S}^{n-1}\times(0,1];\mathbb{Z}) & \text{homeomorphism}\\
&\approx H^p(\mathbb{S}^{n-1}\times([0,1],\{0\});\mathbb{Z}) & \text{by 12.3}\\
&\approx 0 & \text{by 11.12.}
\end{aligned}$$

By the exact cohomology sequence of the coefficient sequence $0 \to \mathscr{L}_n \to \mathscr{L}_{n+1} \to \mathscr{L}_{n+1}/\mathscr{L}_n \to 0$, we conclude that $H^p(X;\mathscr{L}_n) \to H^p(X;\mathscr{L}_{n+1})$ is an isomorphism. By 14.5 we get $H^p(X;\mathscr{L}) \approx H^p(X;\mathscr{L}_1) \approx H^p(X;\mathbb{Z}_{U_1}) \approx H^p_c(U_1;\mathbb{Z}) \approx H^p(\mathbb{D}^n,\mathbb{S}^{n-1};\mathbb{Z}) \approx \mathbb{Z}$ for $p=n$ and is 0 otherwise. ◇

14.10. Example. The direct system $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \cdots$ has limit $\mathbb{Q}$, and since tensor products and direct limits commute, tensoring this with a group A gives that the limit of $A \xrightarrow{2} A \xrightarrow{3} A \xrightarrow{4} \cdots$ is $A\otimes\mathbb{Q}$. Thus *if X is compact Hausdorff* then

$$H^p(X;\mathbb{Q}) \approx H^p(X;\varinjlim \mathbb{Z}) \approx \varinjlim H^p(X;\mathbb{Z}) \approx H^p(X;\mathbb{Z})\otimes\mathbb{Q}$$

by 14.5. This does not generally hold for a noncompact space. For example, if X_n is the mapping cone of the covering map $\mathbb{S}^1 \to \mathbb{S}^1$ of degree n, and $X = +X_n$, then $H^2(X;\mathbb{Q}) = \prod H^2(X_n;\mathbb{Q}) = 0$ while $H^2(X;\mathbb{Z})\otimes\mathbb{Q} \approx$

$(\prod \mathbb{Z}_n) \otimes \mathbb{Q} \neq 0$, since $\prod \mathbb{Z}_n$ is not all torsion. Thus the displayed formula, which will be generalized in Section 15, represents a limitation on which groups can appear as cohomology groups of compact spaces. ◇

Continuity does not extend to more general support families, as is shown by Exercise 4. However, somewhat restricted continuity type results can be shown in more generality, as we shall now see.

14.11. Definition. *A sheaf $\mathscr{S}$ on X is said to be concentrated on a subspace $A \subset X$ if $\mathscr{S}|X - A = 0$.*

14.12. Lemma. *If $\mathscr{S}$ is a sheaf on X that is concentrated on $A \subset X$ and if $\mathscr{T}$ is a torsion free sheaf on X, then the canonical map*

$$\theta : \varinjlim \Gamma_\Phi(\mathscr{S}' \otimes \mathscr{T}) \to \Gamma_\Phi(\mathscr{S} \otimes \mathscr{T}),$$

where $\mathscr{S}'$ ranges over the subsheaves of $\mathscr{S}$ concentrated on members of $\Phi|A$, is an isomorphism for any paracompactifying family Φ of supports on X.

Proof. Each $\mathscr{S}' \otimes \mathscr{T} \to \mathscr{S} \otimes \mathscr{T}$ is monomorphic since $\mathscr{T}$ is torsion free. Thus each $\Gamma_\Phi(\mathscr{S}' \otimes \mathscr{T}) \to \Gamma_\Phi(\mathscr{S} \otimes \mathscr{T})$ is monomorphic, so that θ is monomorphic since $\varinjlim$ is exact.

Let $s \in \Gamma_\Phi(\mathscr{S} \otimes \mathscr{T})$. Then $|s|$ has a neighborhood $K \in \Phi$. For $x \in |s|$ there is a neighborhood $U \subset K$ of x such that $s|U = \sum s_{\alpha,U} \otimes t_{\alpha,U}$ for some $s_{\alpha,U} \in \mathscr{S}(U)$ and $t_{\alpha,U} \in \mathscr{T}(U)$. Since K is paracompact, we can cover $|s|$ by a locally finite family $\{U\}$ of such sets U. By passing to a shrinking of this covering, we can assume that $\{\overline{U}\}$ is locally finite, and that the $s_{\alpha,U}$ extend to some $\bar{s}_{\alpha,U} \in \Gamma(\mathscr{S}|\overline{U})$. Let $B = \bigcup |\bar{s}_{\alpha,U}| \subset K \cap A$. The set B is closed since the collection $\{\overline{U}\}$ is locally finite, and so $B \in \Phi$ since $K \in \Phi$. The sections $s_{\alpha,U}$ generate a subsheaf $\mathscr{S}'$ of $\mathscr{S}$ that is concentrated on $B \in \Phi|A$, and clearly $s \in \Gamma_\Phi(\mathscr{S}' \otimes \mathscr{T})$. □

14.13. Theorem. *If $\mathscr{S}$ is a sheaf on X that is concentrated on $A \subset X$ and if Φ is a paracompactifying family of supports on X, then the canonical map*

$$\varinjlim H^p_\Phi(X; \mathscr{S}') \to H^p_\Phi(X; \mathscr{S})$$

is an isomorphism, where $\mathscr{S}'$ ranges over the subsheaves of $\mathscr{S}$ that are concentrated on members of $\Phi|A$.

Proof. Let $\mathscr{T}^*$ be a torsion-free Φ-fine resolution of $\mathbb{Z}$, such as the canonical resolution. Then by 9.19 and 14.12, we have that

$$\begin{aligned} H^p_\Phi(X; \mathscr{S}) &\approx H^p(\Gamma_\Phi(\mathscr{S} \otimes \mathscr{T}^*)) \approx H^p(\varinjlim \Gamma_\Phi(\mathscr{S}' \otimes \mathscr{T}^*)) \\ &\approx \varinjlim H^p(\Gamma_\Phi(\mathscr{S}' \otimes \mathscr{T}^*)) \approx \varinjlim H^p_\Phi(X; \mathscr{S}'), \end{aligned}$$

which is obviously induced by the inclusions $\mathscr{S}' \hookrightarrow \mathscr{S}$. □

This result means, intuitively, that when $\mathscr{S}$ is concentrated on a set A, then $H^p_\Phi(X;\mathscr{S})$ "depends" only on the subsets $B \subset A$ that are *closed* in X, in fact only on those sets $B \in \Phi|A$. Important applications of this will be given in IV-8.

14.14. Example. This example shows that the condition that Φ be paracompactifying in 14.13 is essential. Let $X = [0,1]$, $A = (0,1]$, and $\Phi = cld|\{0\}$. Then $H^p_\Phi(X;\mathbb{Z}) \approx H^p(X,A;\mathbb{Z}) = 0$, so that the cohomology sequence of $0 \to \mathbb{Z}_{(0,1]} \to \mathbb{Z} \to \mathbb{Z}_{\{0\}} \to 0$ shows that $H^1_\Phi(X;\mathbb{Z}_{(0,1]}) \approx H^0_{cld|\{0\}}(X;\mathbb{Z}_{\{0\}}) \approx H^0(\{0\};\mathbb{Z}) \approx \mathbb{Z}$. However, for any sheaf $\mathscr{L}$ concentrated on $K \subset A$ with K closed in X, we have $H^*_\Phi(X;\mathscr{L}) = 0$ since $\mathscr{C}^*(X;\mathscr{L})$ is also concentrated on K, whence $C^*_\Phi(X;\mathscr{L}) = 0$. ◇

14.15. Example. It might be thought that $H^*_\Phi(X,A;\mathscr{A}) = 0$ when $\mathscr{A}$ is concentrated on A. This is true for A closed since then $H^*_\Phi(X;\mathscr{A}) = H^*_\Phi(X;\mathscr{A}_A) \approx H^*_{\Phi\cap A}(A;\mathscr{A}|A)$, and the contention follows from the cohomology sequence of (X,A). However, this is far from true when A is not closed. For example, we shall compute $H^2(\mathbb{R}^2,U;\mathbb{Z}_U)$ for the open unit ball U about the origin. We have $H^2(\mathbb{R}^2;\mathbb{Z}_U) \approx H^2_c(U;\mathbb{Z}) \approx \mathbb{Z}$, as follows from 11.15 or from the Künneth formula in the next section or by noting that this is the same as $H^2(\mathbb{S}^2;\mathbb{Z})$ and using the fact that this is isomorphic to the singular cohomology by 9.23. Also, $H^p(U;\mathbb{Z}_U|U) = H^p(U;\mathbb{Z}) = 0$ for $p > 0$, since U is contractible. Thus the exact sequence of $(\mathbb{R}^2,U)$ with coefficients in $\mathbb{Z}_U$ shows that $H^2(\mathbb{R}^2,U;\mathbb{Z}_U) \approx \mathbb{Z}$.

However, it is true that $H^p(X;\mathscr{A}) \approx H^p(X,X-A;\mathscr{A})$ when $\mathscr{A}$ is concentrated on A, as follows immediately from the cohomology sequence of the pair $(X,X-A)$. ◇

15 The Künneth and universal coefficient theorems

In this section a principal ideal domain L will be taken as the ground ring. Thus tensor and torsion products are over L throughout the section.

Let X and Y be locally compact Hausdorff spaces. If $\mathscr{A}$ and $\mathscr{B}$ are sheaves of L-modules on X and Y respectively and if $U \subset X$ and $V \subset Y$ are open, we have the cross product $\mu : \mathscr{A}(U) \otimes \mathscr{B}(V) \to (\mathscr{A}\widehat{\otimes}\mathscr{B})(U \times V)$, which induces an isomorphism $\mathscr{A}_x \otimes \mathscr{B}_y \xrightarrow{\approx} (\mathscr{A}\widehat{\otimes}\mathscr{B})_{(x,y)}$ on stalks.

15.1. Proposition. *If $\mathscr{A}$ and $\mathscr{B}$ are c-fine sheaves on the locally compact Hausdorff spaces X and Y respectively, then the canonical map*

$$\mu : \Gamma_c(\mathscr{A}) \otimes \Gamma_c(\mathscr{B}) \to \Gamma_c(\mathscr{A}\widehat{\otimes}\mathscr{B})$$

is an isomorphism.

Proof. If X^+ is the one-point compactification of X and if $\mathscr{A}^+ = \mathscr{A}^{X^+}$, then $\Gamma_c(\mathscr{A}) = \Gamma(\mathscr{A}^+)$. For $i : X \hookrightarrow X^+$, we have $\mathscr{H}om(\mathscr{A}^+, \mathscr{A}^+)(U) = \operatorname{Hom}(\mathscr{A}^+|U, \mathscr{A}^+|U) \approx \operatorname{Hom}(\mathscr{A}|U \cap X, \mathscr{A}|U \cap X) = \operatorname{Hom}(\mathscr{A}, \mathscr{A})(U \cap X) = (i\mathscr{H}om(\mathscr{A}, \mathscr{A}))(U)$, whence $\mathscr{H}om(\mathscr{A}^+, \mathscr{A}^+) \approx i\mathscr{H}om(\mathscr{A}, \mathscr{A})$, so that $\mathscr{A}^+$ is c-fine since $i\mathscr{H}om(\mathscr{A}, \mathscr{A})$ is c-soft by Exercise 15. ($\mathscr{A}'$ can also be seen to be c-fine by direct application of Exercise 12.) It follows that we may assume that X and Y are compact.

We shall first show that $\mu : \mathscr{A}(X) \otimes \mathscr{B}(Y) \to (\mathscr{A}\widehat{\otimes}\mathscr{B})(X \times Y)$ is onto. Let $s \in (\mathscr{A}\widehat{\otimes}\mathscr{B})(X \times Y)$. There are finite coverings $\{U_i\}$ of X and $\{V_j\}$ of Y such that $s|U_i \times V_j = \mu(t_{i,j})$ for some element $t_{i,j} \in \mathscr{A}(U_i)\otimes\mathscr{B}(V_j)$. Let $\{g_i\}$ and $\{h_j\}$ be partitions of unity subordinate to $\{U_i\}$ and $\{V_j\}$ respectively [in $\operatorname{Hom}(\mathscr{A}, \mathscr{A})$ and $\operatorname{Hom}(\mathscr{B}, \mathscr{B})$ respectively]; see Exercise 13. Then the induced endomorphisms $g_i\widehat{\otimes}h_j \in \operatorname{Hom}(\mathscr{A}\widehat{\otimes}\mathscr{B}, \mathscr{A}\widehat{\otimes}\mathscr{B})$ form a partition of unity subordinate to $\{U_i \times V_j\}$. Suppose that for i, j fixed, $t_{i,j} = \sum_k a_k \otimes b_k$, where $a_k \in \mathscr{A}(U_i)$ and $b_k \in \mathscr{B}(V_j)$. Then $(g_i\widehat{\otimes}h_j)(s) = \mu\left(\sum_k g_i(a_k) \otimes h_j(b_k)\right)$. Now $g_i(a_k)$ vanishes outside some compact subset of U_i, so that it extends by zero to an element of $\mathscr{A}(X)$ and similarly for $h_j(b_k)$. Thus we obtain from $\sum_k g_i(a_k) \otimes h_j(b_k)$ an element $c_{i,j} \in \mathscr{A}(X) \otimes \mathscr{B}(Y)$ with $\mu(c_{i,j}) = (g_i\widehat{\otimes}h_j)(s)$. It follows that $s = \sum_{i,j}(g_i\widehat{\otimes}h_j)(s) = \sum_{i,j} \mu(c_{i,j}) = \mu\left(\sum_{i,j} c_{i,j}\right)$, so that μ is onto.

Similarly, if $c = \sum_k(a_k \otimes b_k)$ is an element of $\mathscr{A}(X) \otimes \mathscr{B}(Y)$ with $\mu(c) = 0$, then there are finite coverings $\{U_i\}$ of X and $\{V_j\}$ of Y such that $\sum_k(a_k|U_i) \otimes (b_k|V_j) = 0$ in $\mathscr{A}(U_i) \otimes \mathscr{B}(V_j)$. Again, let $\{g_i\}$ and $\{h_j\}$ be partitions of unity subordinate to these coverings. Then, since application of g_i followed by extension by zero defines a homomorphism $\mathscr{A}(U_i) \to \mathscr{A}(X)$, it follows that $\sum_k g_i(a_k) \otimes h_j(b_k) = 0$ for all i, j. Thus

$$c = \sum_k \left(\sum_i g_i(a_k)\right) \otimes \left(\sum_j h_j(b_k)\right) = \sum_{i,j}\sum_k g_i(a_k) \otimes h_j(b_k) = 0. \qquad \square$$

15.2. Theorem. (Künneth.) *If X and Y are locally compact Hausdorff spaces and if $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X and Y respectively with $\mathscr{A}\widehat{*}\mathscr{B} = 0$, then there is a natural exact sequence (over the principal ideal domain L as base ring)*

$$\boxed{\bigoplus_{p+q=n} H_c^p(X;\mathscr{A}) \otimes H_c^q(Y;\mathscr{B}) \rightarrowtail H_c^n(X \times Y;\mathscr{A}\widehat{\otimes}\mathscr{B}) \twoheadrightarrow \bigoplus_{p+q=n+1} H_c^p(X;\mathscr{A}) * H_c^q(Y;\mathscr{B})}$$

that splits nonnaturally.

Proof. Let $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$ and $\mathscr{B}^* = \mathscr{C}^*(Y;\mathscr{B})$. The differential sheaf $\mathscr{A}^*\widehat{\otimes}\mathscr{B}^*$ is c-fine by Exercise 14. It is also a resolution of $\mathscr{A}\widehat{\otimes}\mathscr{B}$, since $\mathscr{A}^*$ and $\mathscr{B}^*$ are pointwise homotopically trivial. (This also follows from the algebraic Künneth formula for differential sheaves; see Exercise 42.) Thus we have

$$H_c^*(X \times Y;\mathscr{A}\widehat{\otimes}\mathscr{B}) \approx H^*(\Gamma_c(\mathscr{A}^*\widehat{\otimes}\mathscr{B}^*)) \approx H^*(\Gamma_c(\mathscr{A}^*) \otimes \Gamma_c(\mathscr{B}^*)). \quad (34)$$

Since $\Gamma_c(\mathscr{A}^*) * \Gamma_c(\mathscr{B}^*) = 0$ by Exercise 36, the algebraic Künneth formula (which we assume to be known; see [54] or [75]), when applied to the right-hand side of (34), yields the result. □

15.3. Theorem. (Universal Coefficient Theorem.) *Let X be a locally compact Hausdorff space. Let $\mathscr{A}$ be a sheaf on X and let M be an L-module such that $\mathscr{A} * M = 0$. Then there is a natural exact sequence (over the principal ideal domain L as base ring)*

$$\boxed{0 \to H_c^n(X;\mathscr{A}) \otimes M \to H_c^n(X;\mathscr{A} \otimes M) \to H_c^{n+1}(X;\mathscr{A}) * M \to 0}$$

which splits (naturally in M but not in X).

Proof. With the notation of the proof of 15.2, this follows from the algebraic universal coefficient theorem applied to the formula

$$H_c^*(X;\mathscr{A} \otimes M) \approx H^*(\Gamma_c(\mathscr{A}^* \otimes M)) \approx H^*(\Gamma_c(\mathscr{A}^*) \otimes M),$$

by 15.1 applied to the case in which Y is a point. [Except for the naturality in M of the splitting, this also follows directly from 15.2 by taking Y to be a point and $\mathscr{B} = M$ to be an L-module.] □

In general these results do not extend to more general spaces. For example, if X is the topological sum of the lens spaces $L(p,1)$, p ranging over all primes, then $H^2(X;\mathbb{Q}) \approx \prod_p H^2(L(p,1);\mathbb{Q}) \approx \prod_p(\mathbb{Z}_p \otimes \mathbb{Q}) = 0$, while $H^2(X;\mathbb{Z}) \otimes \mathbb{Q} \approx \left(\prod_p \mathbb{Z}_p\right) \otimes \mathbb{Q} \neq 0$ since $\prod_p \mathbb{Z}_p$ is not all torsion. However, the following gives one case of the universal coefficient theorem that is valid for general spaces.

15.4. Theorem. *Let $\mathscr{A}$ be a sheaf of L-modules (L being a principal ideal domain) on the arbitrary space X and let M be a finitely generated L-module with $\mathscr{A} * M = 0$. Then for any family Φ of supports on X, there is a natural exact sequence*

$$\boxed{0 \to H_\Phi^n(X;\mathscr{A}) \otimes M \to H_\Phi^n(X;\mathscr{A} \otimes M) \to H_\Phi^{n+1}(X;\mathscr{A}) * M \to 0}$$

which splits (naturally in M but not in X).

Proof. By 5.14, $H_\Phi^n(X;\mathscr{A} \otimes M) \approx H^n(\Gamma_\Phi(\mathscr{C}^*(X;\mathscr{A})) \otimes M)$, so that the result follows from the algebraic universal coefficient theorem applied to the complex $\Gamma_\Phi(\mathscr{C}^*(X;\mathscr{A})) \otimes M$. □

See IV-7.6, IV-Exercise 18, and V-Exercise 25 for further results of the Künneth type on cohomology. Chapter V contains analogous results on homology.

The following result is an immediate corollary of 15.2:

15.5. Proposition. *If $\mathscr{A}$ and $\mathscr{B}$ are sheaves on the locally compact Hausdorff spaces X and Y respectively, then over a principal ideal domain as base ring, the canonical map $\mu : \Gamma_c(\mathscr{A}) \otimes \Gamma_c(\mathscr{B}) \to \Gamma_c(\mathscr{A} \widehat{\otimes} \mathscr{B})$ is an isomorphism in any of the following three cases:*

(a) *$\mathscr{A}$ arbitrary, $\mathscr{B}$ torsion free and c-acyclic.*

(b) *$\mathscr{A}$ and $\mathscr{B}$ both torsion free.*

(c) *$\mathscr{A} \widehat{*} \mathscr{B} = 0$ and $\mathscr{A}$ and $\mathscr{B}$ both c-acyclic.* □

15.6. Example. $H_c^p(\mathbb{R}; L) \approx H_c^p((0,1); L) \approx L$ for $p = 1$ and is zero otherwise, as follows from the cohomology sequence of $([0,1], \{0,1\})$. By 15.2 we deduce that

$$H_c^p(\mathbb{R}^n; L) \approx \begin{cases} L, & p = n, \\ 0, & p \neq n. \end{cases}$$

From the cohomology sequence of $(\mathbb{S}^n, \star)$ we also have that

$$H^p(\mathbb{S}^n; L) \approx \begin{cases} L, & p = 0, n, \\ 0, & p \neq 0, n. \end{cases}$$ ◇

16 Dimension

In this section we study the notion of cohomological dimension, which has a close relationship with the classical dimension theory of spaces.

The case of locally compact spaces is of the most importance to us, and the proofs for it are often much simpler than for more general results. Thus we shall redundantly state and prove some results in the locally compact case even though more general results are given later in the section.

Also, although we shall occasionally comment upon known items from classical dimension theory, our formally stated results, with minor exceptions, are based solely upon the theory developed in this book. This makes the present discussion, as well as its continuation in Section 8 of Chapter IV, essentially self-contained.

16.1. Proposition. *For a paracompactifying family Φ and a sheaf $\mathscr{L}$ on X, the following four statements are equivalent:*

(a) *$\mathscr{L}$ is Φ-soft.*

(b) *$\mathscr{L}_U$ is Φ-acyclic for all open $U \subset X$.*

(c) *$H^1_\Phi(X; \mathscr{L}_U) = 0$ for all open $U \subset X$.*

(d) *$H^1_{\Phi|U}(U; \mathscr{L}|U) = 0$ for all open $U \subset X$.*

Proof. Item (a) implies (b) by 9.13. Item (b) trivially implies (c), which is equivalent to (d) by 10.2. If (d) holds, $F \subset X$ is closed and $U = X - F$; then the exact sequence

$$0 \to \Gamma_{\Phi|U}(\mathscr{L}|U) \to \Gamma_\Phi(\mathscr{L}) \to \Gamma_{\Phi|F}(\mathscr{L}|F) \to H^1_{\Phi|U}(U;\mathscr{L}|U) = 0$$

of 10.3 shows that $\mathscr{L}$ is Φ-soft by 9.3. □

16.2. Theorem. *For a paracompactifying family Φ of supports on X and a sheaf $\mathscr{A}$ on X, the following four statements are equivalent:*

(a) *For every resolution $0 \to \mathscr{A} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^n \to 0$ of $\mathscr{A}$ of length n in which $\mathscr{L}^p$ is Φ-soft for $p < n$, $\mathscr{L}^n$ is also Φ-soft.*

(b) *$\mathscr{A}$ has a Φ-soft resolution of length n.*

(c) *$H^k_\Phi(X;\mathscr{A}_U) = H^k_{\Phi|U}(U;\mathscr{A}|U) = 0$ for all open $U \subset X$ and all $k > n$.*

(d) *$H^{n+1}_\Phi(X;\mathscr{A}_U) = 0$ for all open $U \subset X$.*

Proof. The implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are clear. Assume (d) and let $0 \to \mathscr{A} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^n \to 0$ be as in (a). Let $\mathscr{Z}^p = \operatorname{Ker}(\mathscr{L}^p \to \mathscr{L}^{p+1})$. Then for $U \subset X$ open, the exact sequences

$$0 \to \mathscr{Z}^p_U \to \mathscr{L}^p_U \to \mathscr{Z}^{p+1}_U \to 0$$

show that $H^1_\Phi(X;\mathscr{Z}^n_U) \approx H^2_\Phi(X;\mathscr{Z}^{n-1}_U) \approx \cdots \approx H^{n+1}_\Phi(X;\mathscr{A}_U) = 0$. Thus $\mathscr{L}^n = \mathscr{Z}^n$ is Φ-soft by 16.1. □

16.3. Definition. *Let Φ be a family of supports on X and let L be a fixed ground ring with unit. We let $\dim_{\Phi,L} X$ be the least integer n (or ∞) such that $H^k_\Phi(X;\mathscr{A}) = 0$ for all sheaves $\mathscr{A}$ of L-modules and all $k > n$.*

16.4. Theorem. *The following four statements are equivalent:*

(a) $\dim_{\Phi,L} X \le n$.

(b) *$H^{n+1}_\Phi(X;\mathscr{A}) = 0$ for all sheaves $\mathscr{A}$ of L-modules.*

(c) *For every sheaf $\mathscr{A}$ of L-modules, $\mathscr{Z}^n(X;\mathscr{A})$ is Φ-acyclic.*

(d) *Every sheaf $\mathscr{A}$ of L-modules has a Φ-acyclic resolution of length n.*

Moreover, if Φ is paracompactifying, then "Φ-acyclic" in (c) *and* (d) *can be replaced by "Φ-soft."*

Proof. Clearly (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (b), so assume (b). Then as in the proof of 16.2, we have [for $\mathscr{Z}^p = \mathscr{Z}^p(X;\mathscr{A})$ and $\mathscr{Z}^0 = \mathscr{A}$]

$$H^k_\Phi(X;\mathscr{Z}^n) \approx H^{k+1}_\Phi(X;\mathscr{Z}^{n-1}) \approx \cdots \approx H^{n+1}_\Phi(X;\mathscr{Z}^{k-1}) = 0$$

for all $k \geq 1$, proving (c). □

The following fact is an immediate consequence of 9.14 and shows that softness and dimension with respect to a paracompactifying family Φ of supports depends only on the extent $E(\Phi)$ of the family.

16.5. Proposition. *If Φ and Ψ are paracompactifying and $E(\Phi) \subset E(\Psi)$, then any Ψ-soft sheaf is Φ-soft and* $\dim_{\Phi,L} X \leq \dim_{\Psi,L} X$. □

We say that a space X is *locally paracompact* if it is Hausdorff and each point has a closed paracompact neighborhood. Such a space possesses a paracompactifying family Φ with $E(\Phi) = X$ (and conversely). Indeed, the set of all closed K such that K has a closed paracompact neighborhood in X is such a family.

16.6. Definition. *If X is locally paracompact then we define (using 16.5)* $\dim_L X = \dim_{\Phi,L} X$, *where Φ is paracompactifying and $E(\Phi) = X$.*

Note that $\dim_L X$ need not coincide with $\dim_{cld,L} X$ when X is not paracompact.

Suppose that Φ is paracompactifying on X and put $W = E(\Phi)$, an open set. Then $\Gamma_\Phi(\mathscr{B}) = \Gamma_{\Phi|W}(\mathscr{B}|W)$ for any sheaf $\mathscr{B}$ on X, since each member of Φ has a neighborhood in Φ. It follows that $H^p_\Phi(X;\mathscr{A}) \approx H^p_{\Phi|W}(W;\mathscr{A}|W)$, whence $\dim_{\Phi,L} X = \dim_{\Phi|W,L} W = \dim_L W$, since W is locally paracompact, $\Phi|W$ is paracompactifying, and $E(\Phi|W) = W$. Therefore, when Φ is paracompactifying, the study of $\dim_{\Phi,L} X$ reduces to the study of $\dim_L W$ for W *locally paracompact.*

The following is an immediate consequence of 9.3 and 16.2(a).

16.7. Proposition. *If Φ is a paracompactifying family of supports on X, then*

$$\boxed{\dim_{\Phi,L} X = \sup_{K\in\Phi}\{\dim_L K\}.}$$ □

16.8. Theorem. *Let X be locally paracompact (respectively, hereditarily paracompact). Then $\dim_L X \geq \dim_L A$ for any locally closed (resp., arbitrary) subspace $A \subset X$. Conversely, if each point $x \in X$ admits a locally closed (resp., arbitrary) neighborhood N with $\dim_L N \leq n$, then $\dim_L X \leq n$.*

Proof. Let Φ be a paracompactifying family with $E(\Phi) = X$. Then the first part follows from 16.2 and the facts that $\mathscr{A}|A$ is Φ-soft for $\mathscr{A}$ Φ-soft and that $\Phi|A$ is paracompactifying for A locally closed.

Now suppose that X is hereditarily paracompact and $A \subset X$ is arbitrary. Let $i : A \hookrightarrow X$ be the inclusion, let $\mathscr{A}$ be a sheaf on A, and recall that $\mathscr{A} \approx (i\mathscr{A})|A$. By 10.6 we have that $H^p(A;\mathscr{A}) \approx \varinjlim H^p(U; i\mathscr{A})$, where U ranges over the open neighborhoods of A in X, since A is taut in X. Now $\dim_L U \leq \dim_L X = n$, by the portion already proved. Therefore $H^p(A;\mathscr{A}) = 0$ for $p > n$, whence $\dim_L A \leq n$ as claimed.

For the converse, we may assume that N is closed in the statement of the theorem, in either case. Let Φ be paracompactifying with $E(\Phi) = X$ and let $\mathscr{L}^*$ be a resolution of $\mathscr{A}$ of length n with $\mathscr{L}^p$ Φ-soft for $p < n$. Then $\mathscr{L}^p|N$ is $\Phi|N$-soft for $p < n$. We may assume that $N \in \Phi$, so that $\mathscr{L}^p|N$ is soft for $p \leq n$ by 16.2. Thus $\mathscr{L}^n$ is Φ-soft by 9.14 and $\dim_L X \leq n$ by 16.2. □

In the nonparacompact case one still has the following monotonicity property of cohomological dimension, which is an immediate consequence of 10.1 and Definition 16.3:

16.9. Proposition. *If $F \subset X$ is closed, then* $\dim_{\Phi|F,L} F \leq \dim_{\Phi,L} X$. □

Generally, little can be said about $\dim_{\Phi,L} X$ when Φ is not paracompactifying. An exception is the next result concerning families of the form $\Phi|A$ and its corollary concerning arbitrary families. It will be used in Chapter V.

16.10. Theorem. *Let Φ be paracompactifying on the locally paracompact space X and let $A \subset X$ with $X - A$ locally closed, paracompact, and Φ-taut (e.g., A closed and $X - A$ paracompact). Then* $\dim_{\Phi|A,L} X \leq \dim_L X + 1$.

Proof. Let $n = \dim_L X$. Note that $\Phi \cap X - A$ is paracompactifying since $X - A$ is paracompact. By 16.8 we have that $\dim_L X - A \leq n$, so that $H^p_{\Phi \cap X-A}(X - A;\mathscr{A}) = 0$ for $p > n$ and any sheaf $\mathscr{A}$ of L-modules on X. The exact sequence of the pair $(X, X-A)$ shows that $H^p_\Phi(X, X-A;\mathscr{A}) = 0$ for $p > n+1$. By 12.1, $H^p_{\Phi|A}(X;\mathscr{A}) \approx H^p_\Phi(X, X-A;\mathscr{A}) = 0$ for $p > n+1$, whence $\dim_{\Phi|A,L} X \leq n+1$ by definition. □

16.11. Corollary. *If X is hereditarily paracompact and Φ is an arbitrary family of supports on X, then* $\dim_{\Phi,L} X \leq \dim_L X + 1$.[34]

Proof. Note that $\Gamma_\Phi(\mathscr{A}^*) = \varinjlim \Gamma_{\Phi|K}(\mathscr{A}^*)$, where K ranges over Φ and $\mathscr{A}^*$ is a flabby resolution of $\mathscr{A}$, whence $H^*_\Phi(X;\mathscr{A}) = \varinjlim H^*_{\Phi|K}(X;\mathscr{A})$. This implies that $\dim_{\Phi,L} X \leq \sup_{K\in\Phi} \dim_{\Phi|K,L} X \leq \dim_L X + 1$, the last

[34] By Exercise 25 this also holds when X is only *locally* hereditarily paracompact.

inequality resulting from 16.10 since $\Phi|K = cld|K$ and cld is paracompactifying on X. □

Exercise 23 shows that the +1 in 16.10 and 16.11 cannot be dropped.

16.12. Lemma. *Let $\mathfrak{S}$ be a class of sheaves of L-modules (L a ring with unit) on a space X satisfying the following three properties:*

(a) *If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is exact with $\mathscr{A}' \in \mathfrak{S}$, then $\mathscr{A} \in \mathfrak{S} \Leftrightarrow \mathscr{A}'' \in \mathfrak{S}$.*

(b) *If $\{\mathscr{A}_\alpha\}$ is an upward-directed family of subsheaves of $\mathscr{A}$ with each $\mathscr{A}_\alpha \in \mathfrak{S}$, then $\bigcup \mathscr{A}_\alpha \in \mathfrak{S}$.*

(c) *For any ideal I of L and open set $U \subset X$ we have $I_U \in \mathfrak{S}$.*

Then $\mathfrak{S}$ consists of all sheaves of L-modules.[35]

Proof. Assume that the sheaf $\mathscr{B} \notin \mathfrak{S}$. The class of subsheaves (of L-modules) of $\mathscr{B}$ that are in $\mathfrak{S}$ has a maximal element $\mathscr{A}$ by (b). If $0 \neq s \in (\mathscr{B}/\mathscr{A})(U)$ for some open $U \subset X$, then s defines a nontrivial homomorphism $h : L_U \to \mathscr{B}/\mathscr{A}$. The image of h is not in $\mathfrak{S}$ by (a), so that the kernel of h cannot be in $\mathfrak{S}$ by (a) and (c). Thus we see that it suffices to show that any subsheaf (of ideals) of L is in $\mathfrak{S}$.

Now let $\mathscr{I}$ be a subsheaf of L. Then any element of $\mathscr{I}$ is contained in a subsheaf of $\mathscr{I}$ of the form I_U, where I is an ideal of L and $U \subset X$ is open. Thus $\mathscr{I}$ can be expressed as the sum (in $\mathscr{I}$, not direct) of subsheaves of the form I_U, and by (b), it suffices to show that every finite sum of sheaves of the form I_U is in $\mathfrak{S}$.

Let $U_1, \dots, U_n$ be open sets and $I_1, \dots, I_n$ ideals of L. Let $\mathscr{J} = (I_1)_{U_1} + \cdots + (I_n)_{U_n}$. For $k \le n$, let V_k be the set of points lying in at least k of the sets U_i. Consider the sequence

$$0 \subset \mathscr{J}_{V_n} \subset \mathscr{J}_{V_{n-1}} \subset \cdots \subset \mathscr{J}_{V_1} = \mathscr{J}.$$

We note that each quotient $\mathscr{J}_{V_k}/\mathscr{J}_{V_{k+1}}$ is the *direct* sum of sheaves of the form J_A, where typically J is the ideal $I_{i_1} + \cdots + I_{i_k}$ of L and A is the locally closed set consisting of those points in $U_{i_1}, U_{i_2}, \dots, U_{i_k}$ but in no other U_j.

Thus, by (a), it suffices to show that $J_A \in \mathfrak{S}$ for J an ideal of L and A locally closed. But if $A = U \cap F$ with U open and F closed, then $J_A = J_U/J_{(X-F)\cap U} \in \mathfrak{S}$ by (a) and (c). □

16.13. Lemma. *If X is a locally compact Hausdorff space and L is a ring with unit, then*

$$H_c^n(X; L) = 0 \quad \Rightarrow \quad H_c^n(X; J) = 0$$

for all ideals J of L.

[35] Also see Exercise 35.

Proof. We have $H_c^n(X;\mathbb{Z}) \otimes L = 0 = H_c^{n+1}(X;\mathbb{Z}) * L$ by the universal coefficient theorem 15.3. Therefore, $H_c^n(X;\mathbb{Z}) \otimes J = H_c^n(X;\mathbb{Z}) \otimes L \otimes_L J = 0$ and $H_c^{n+1}(X;\mathbb{Z}) * J = 0$ since the latter group injects into $H_c^{n+1}(X;\mathbb{Z}) * L$. Thus the universal coefficient theorem over $\mathbb{Z}$ implies that $H_c^n(X;J) = 0$. □

16.14. Theorem. *Let X be locally compact Hausdorff and L a ring with unit. Then $\dim_L X \le n \Leftrightarrow H_c^{n+1}(U;L) = 0$ for all open sets $U \subset X$.*[36]

Proof. The $\Rightarrow$ part is trivial. Thus suppose that the condition is satisfied. Then $H_c^{n+1}(X;J_U) \approx H_c^{n+1}(U;J) = 0$ by 16.13, so that 16.2 implies that $H_c^k(X;J_U) = 0$ for all open $U \subset X$, all ideals J of L, and all $k > n$. Now let $\mathfrak{S} = \{\mathscr{L} \mid H_c^k(X;\mathscr{L}) = 0 \text{ for all } k > n\}$. Then $\mathfrak{S}$ clearly satisfies conditions (a) and (c) of 16.12. Condition (b) follows from 14.5. Thus $\mathfrak{S}$ consists of all sheaves of L-modules and this implies that $\dim_L X = \dim_{c,L} X \le n$. □

The following is an immediate consequence of the definition:

16.15. Proposition. *If X is locally paracompact, then $\dim_L X \le \dim_{\mathbb{Z}} X$ for any ring L with unit.* □

16.16. Proposition. *Let X be locally compact Hausdorff with $\dim_L X \le n$. Let $G \subset H_c^n(X;L)$ be a subgroup such that every point $x \in X$ has a neighborhood U with $\operatorname{Im}\{j_{X,U}^n : H_c^n(U;L) \to H_c^n(X;L)\} = G$. Then $G = H_c^n(X;L)$.*

Proof. Let $\mathfrak{S}$ be the collection of all open subsets U of X with $\operatorname{Im} j_{X,U}^n = G$. The Mayer-Vietoris diagram

$$\begin{array}{ccccc} H_c^n(U) \oplus H_c^n(V) & \to & H_c^n(U \cup V) & \to & 0 \\ \downarrow & & \downarrow & & \\ H_c^n(X) \oplus H_c^n(X) & \to & H_c^n(X) & \to & 0 \end{array}$$

shows that the union of two members of $\mathfrak{S}$ is in $\mathfrak{S}$. Thus $\mathfrak{S}$ is directed upwards, and $X = \bigcup\{W \mid W \in \mathfrak{S}\}$ by assumption. But $H_c^n(X;L) = \bigcup \operatorname{Im} j_{X,W}^n$ for W ranging over $\mathfrak{S}$ [since $L = \varinjlim L_W$, whence $H_c^n(X;L) = \varinjlim H_c^n(X;L_W) = \varinjlim H_c^n(W;L)$ by 14.5] and the result follows. □

16.17. Corollary. *Let X be Hausdorff and locally compact with $\dim_L X \le n$. Suppose that for each open set $U \subset X$ and $x \in U$ there is a neighborhood $W \subset U$ of x such that $j_{U,W}^n : H_c^n(W;L) \to H_c^n(U;L)$ is trivial. Then $\dim_L X < n$.*

[36]Also see 16.25, 16.32, and 16.33.

Proof. Applying 16.16 to each open set U with $G = 0$ gives $H_c^n(U; L) = 0$ for all open sets U, whence $\dim_L X < n$ by 16.14. □

16.18. Example. The one-point union X of a countably infinite number of 2-spheres with radii tending to zero has $\dim_{\mathbb{Z}} X = 2$ as follows from Example 10.10 and 16.17. Since X is metrizable, it follows from 16.8 that $H^p(A; \mathbb{Z}) = 0$ for all subspaces $A \subset X$ and all $p > 2$. As remarked in 10.10, however, the singular cohomology of X is nonzero in arbitrarily high degrees. Also, computation of the singular groups is several orders of magnitude more difficult for such spaces than is sheaf-theoretic cohomology. This illustrates a fundamental difference between the two theories. ◇

16.19. Lemma. *If the space K is locally compact, Hausdorff, and totally disconnected, then $\dim_L K = 0$ for any ring L with unit.*

Proof. Let $\mathscr{A}$ be a sheaf on K. We will suppress coefficients in $\mathscr{A}$ from our notation. Let $\gamma \in H_c^1(K)$ and let the compact set B be the support of some cochain representative of γ. For any point $x \in B$ there is a neighborhood A, which can be assumed to be open and compact, of x such that $H_c^1(K) \to H_c^1(A)$ takes γ to zero, by 10.6. Thus B is contained in a disjoint union of such open and compact sets $A_1, \dots, A_n$. Let $A_0 = K - (A_1 \cup \dots \cup A_n)$. Then γ also restricts to 0 in $H_c^1(A_0)$. (Note that $c \cap A_0 = c|A_0$ and $c \cap A_i = c|A_i = cld$ for $i > 0$.) Since $H_c^1(K) \approx H_c^1(A_0) \oplus \dots \oplus H_c^1(A_n)$, we conclude that $\gamma = 0$, whence $\dim_L K = 0$. □

16.20. Proposition. *If X is a connected space with at least two closed points, then $\dim_{cld,L} X > 0$ for any ring L with unit.*

Proof. Let $x \neq y$ be distinct closed points in X and put $U = X - \{x, y\}$. Then the exact sequence of the pair $(X, \{x, y\})$ has the segment

$$\Gamma(L) \to \Gamma(L|\{x, y\}) \to H^1(X; L_U);$$

see 12.3. This has the form $L \xrightarrow{\text{diag}} L \oplus L \to H^1(X; L_U)$, and hence $H^1(X; L_U) \neq 0$. □

16.21. Corollary. *A locally compact Hausdorff space K has $\dim_L K = 0$ $\Leftrightarrow$ K is totally disconnected.*[37]

Proof. If K^+ is the one-point compactification of K, we have

$\dim_L K = 0$	$\Rightarrow \dim_L K^+ = 0$	by Exercise 11
	$\Rightarrow K^+$ is totally disconnected	by 16.9 and 16.20
	$\Rightarrow K$ is totally disconnected	obviously
	$\Rightarrow \dim_L K = 0$	by 16.19. □

[37] Also see 16.35.

16.22. Example. The "Knaster explosion set" K is an infinite subset of the plane that is connected but has a point $x_0 \in K$ such that $K - \{x_0\}$ is totally disconnected.[38] Now $\dim_L K > 0$ by 16.20, so $\dim_L K - \{x_0\} > 0$ by Exercise 11 for any ring L with unit. This shows that the condition in 16.19 and 16.21 that K be locally compact may not be discarded. Since K embeds in the plane and its closure $\overline{K}$ there has $\dim_L \overline{K} = 1$, it follows from 16.8 that $\dim_L K = 1$. (The fact that $\dim_L \overline{K} = 1$ follows from Exercise 11, the fact that $\overline{K} - \{x_0\} \approx C \times (0, 1]$ where C is a Cantor set, and 16.27.) ◇

16.23. Example. According to Exercise 37 there exists a compact, totally disconnected Hausdorff space X containing a locally closed subspace A such that $\Gamma(L|U) \to \Gamma(L|A)$ is not surjective for any open $U \supset A$. Now, $\dim_L X = 0$ by 16.19. Also, $A = U \cap F$ for some open U and closed F, whence A is closed in U. The exact sequence

$$\Gamma(L|U) \to \Gamma(L|A) \to H^1(U, A; L)$$

and 12.3 show that $H^1(U; L_{U-A}) \approx H^1(U, A; L) \neq 0$. Thus $\dim_{cld,L} U > 0$, while $\dim_L U = \dim_{c,L} U = 0$ by 16.8. Of course, *cld* cannot be paracompactifying on U. ◇

16.24. Corollary. (H. Cohen.) *Let X and K be locally compact Hausdorff spaces. If $\dim_L X = n$ and $\dim_L K > 0$, then $\dim_L(X \times K) > n$.*

Proof. By passing to the one-point compactification and using Exercise 11, we can assume that K is compact. By 16.21 and 16.9 we can also assume that K is connected and is not a single point. By passing to an open subset and using 16.14 we can assume that $H_c^n(X; L) \neq 0$. Let $0 \neq \alpha \in H_c^n(X; L)$ and let $k_1 \neq k_2 \in K$. Consider α as lying in $H_c^n(X \times \{k_1\}; L)$ and let $\beta \in H_c^n(X \times \{k_1, k_2\}; L)$ correspond to $\alpha \oplus 0$ in $H_c^n(X \times \{k_1\}; L) \oplus H_c^n(X \times \{k_2\}; L)$. If $\dim_L(X \times K) \leq n$ then $H_c^{n+1}(X \times (K - \{k_1, k_2\}); L) = 0$ by 16.14, and so the exact sequence

$$H_c^n(X \times K; L) \to H_c^n(X \times \{k_1, k_2\}; L) \to H_c^{n+1}(X \times (K - \{k_1, k_2\}); L)$$

shows that β comes from some class $\gamma \in H_c^n(X \times K; L)$. But the composition

$$H_c^n(X \times K; L) \to H_c^n(X \times \{k\}; L) \xrightarrow{\approx} H_c^n(X; L)$$

is independent of $k \in K$ by 11.9, and this provides a contradiction. □

16.25. Lemma. *For a locally compact Hausdorff space X, suppose that $H_c^k(U; L) = 0$ for all $k > n$ and all U in some basis for the open sets of X that is closed under finite intersections. Then $\dim_L X \leq n$.*

[38] If X is the union of line segments in the plane from $x_0 = (0, 1)$ to points in the Cantor set in the unit interval on the x-axis, then K is the set of points in X of rational height on rays from x_0 to end points of complementary intervals of the Cantor set together with the points of irrational height on the other rays; see [49, Example II-16]. The fact that K is connected is an exercise on the Baire category theorem.

Proof. By the Mayer-Vietoris sequence

$$H_c^k(U) \oplus H_c^k(V) \to H_c^k(U \cup V) \to H_c^{k+1}(U \cap V)$$

we can throw finite unions into the basis. Then any open set U is a union $U = \bigcup U_\alpha$ of elements of the basis *directed* upwards. Then $L_U = \varinjlim L_{U_\alpha}$, and so

$$H_c^{n+1}(U; L) \approx H_c^{n+1}(X; L_U) \approx \varinjlim H_c^{n+1}(X; L_{U_\alpha}) \approx \varinjlim H_c^{n+1}(U_\alpha; L) = 0$$

by 14.5, and so $\dim_L X \leq n$ by 16.14. □

Remark: It does not suffice in 16.25 that $H_c^k(U; L) = 0$ for $k = n+1$ rather than for all $k > n$, since $\mathbb{R}^{n+2}$ satisfies that hypothesis. Also, the Hilbert cube $\mathbb{I}^\infty$ satisfies that hypothesis for any given n.

16.26. Corollary. *Let X and Y be locally compact Hausdorff spaces. Then* $\dim_L(X \times Y) \leq \dim_L X + \dim_L Y$ *for any ring L with unit.*[39]

Proof. Let $p = \dim_L X$ and $q = \dim_L Y$. The sets $U \times V$ where $U \subset X$ and $V \subset Y$ are open form a basis for the topology of $X \times Y$ which is closed under finite intersections since $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. By 15.2, $H_c^n(U \times V; L \widehat{\otimes}_{\mathbb{Z}} L) = 0$ whenever $n > p + q$. Now the composition

$$L \approx \mathbb{Z} \widehat{\otimes}_{\mathbb{Z}} L \to L \widehat{\otimes}_{\mathbb{Z}} L \to L \widehat{\otimes}_L L \approx L$$

is the identity, and so $H_c^n(U \times V; L) = 0$ whenever $n > p + q$. Thus the result follows from 16.25. □

Examples have been constructed by Pontryagin of compact spaces X, Y for which $\dim_L(X \times Y) < \dim_L X + \dim_L Y$.

16.27. Corollary. *Let X and K be locally compact Hausdorff spaces. If* $\dim_L K = 1$, *then* $\dim_L(X \times K) = 1 + \dim_L X$.

Proof. We have $\dim_L(X \times K) > \dim_L X$ by 16.24 and $\dim_L(X \times K) \leq 1 + \dim_L X$ by 16.26. □

16.28. Corollary. *If M^n is a topological n-manifold, then* $\dim_L M^n = n$ *for any L.*

Proof. By the cohomology sequence of the pair $(\mathbb{I}, \partial\mathbb{I})$, $H_c^p((0,1); L) \approx L$ for $p = 1$ and is zero otherwise. Since an open subset of $\mathbb{R}$ is a disjoint union of open intervals, it follows from 16.14 that $\dim_L \mathbb{R} = 1$. By 16.27, $\dim_L \mathbb{R}^n = n$. The result now follows from 16.8. □

[39] Also see IV-8.5.

16.29. Corollary. *Let M be a connected topological n-manifold and K any constant coefficient group. Then:*

(a) *If $F \subsetneq M$ is a proper closed subspace, then $H_c^n(F;K) = 0$.*

(b) *If $\varnothing \neq U \subset M$ is open, then $j_{M,U}^n : H_c^n(U;K) \to H_c^n(M;K)$ is onto.*

(c) *$H_c^n(M;K) \approx K$ or $K/2K$.*

Proof. By 15.3 it suffices to prove this in the case $K = \mathbb{Z}$. First we prove (a) for $M = \mathbb{S}^n$. Let $U \subset \mathbb{S}^n - F$ be an open metric disk. Now $H^n(\mathbb{S}^n) \to H^n(F)$ is surjective since $H_c^{n+1}(\mathbb{S}^n - F) = 0$ by the fact that $\dim \mathbb{S}^n = n$. However, this map factors through $H^n(\mathbb{S}^n - U) = 0$ (since $\mathbb{S}^n - U$ is contractible), proving the contention. By adding a point at infinity, (a) follows for the case $M = \mathbb{R}^n$.

From the exact sequence of the pair $(M, M - U)$ we see that (b) holds when $M = \mathbb{R}^n$.

Suppose now that $U \subset M$ and $U \approx \mathbb{R}^n$. Then for any open $V \subset U$, $j_{U,V}^n : H_c^n(V) \to H_c^n(U)$ is onto by the case of (b) just proved. It follows that $\operatorname{Im} j_{M,V}^n = \operatorname{Im} j_{M,U}^n$.

If U and V are open sets each homeomorphic to $\mathbb{R}^n$ and if $U \cap V \neq \varnothing$, then it follows that $\operatorname{Im} j_{M,V}^n = \operatorname{Im} j_{M,U\cap V}^n = \operatorname{Im} j_{M,U}^n$. Since M is connected, we deduce that $G = \operatorname{Im} j_{M,U}^n$ is independent of U for $U \approx \mathbb{R}^n$. By 16.16 we have $G = H_c^n(M)$, proving (b) in general. The exact sequence of the pair (M, F) proves (a) in general.

Suppose that $\{U_\alpha\}$ is an upward-directed family of connected open sets such that each $H_c^n(U_\alpha)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2$. Then continuity 14.5 (applied to $\mathbb{Z}_{U_\alpha}$) shows that the same is true for $U = \bigcup U_\alpha$. It follows that there is a *maximal* connected open set U satisfying part (c) (for U in place of M). If $U \neq M$, then let $V \approx \mathbb{R}^n$ be an open neighborhood of a point in the boundary of U. Then $U \cup V$ is connected, $U \cap V \neq \varnothing$, and the Mayer-Vietoris sequence

$$H_c^n(U \cap V) \to H_c^n(U) \oplus H_c^n(V) \to H_c^n(U \cup V) \to 0$$

shows that (c) is true for $U \cup V$ since $H_c^n(U \cap V)$ is the direct sum of the cohomology of the components U_β of $U \cap V$ and each $H_c^n(U_\beta) \to H_c^n(U) \oplus H_c^n(V)$ has an image that is the diagonal $\{(\lambda, \lambda)\} \subset \mathbb{Z} \oplus \mathbb{Z} \approx H_c^n(U) \oplus H_c^n(V)$ or the antidiagonal $\{(\lambda, -\lambda)\}$, and similarly for the case in which $H_c^n(U) \approx \mathbb{Z}_2$. This contradicts the maximality of U, and so $U = M$. □

Now we wish to extend 16.14 to general paracompactifying families of supports on general spaces. The main tool is the following extension of 14.3:

16.30. Theorem. (Kuz'minov, Liseĭkin [55].) *Let Φ be a paracompactifying family of supports on X and let $\{\mathscr{L}_\lambda\,;\ \lambda \in \Lambda\}$ be a direct system of Φ-soft sheaves on X. Then $\mathscr{L} = \varinjlim \mathscr{L}_\lambda$ is also Φ-soft.*

Proof. By 9.3 it suffices to treat the case in which X is paracompact and $\Phi = cld$. Let $K \subset X$ be closed and let $s \in \mathscr{L}(K)$. Then we can find a locally finite open covering $\{U_\alpha \mid \alpha \in A\}$ of X and sections $s_\alpha \in \mathscr{L}_{\lambda(\alpha)}(U_\alpha)$ such that $k_{\lambda(\alpha)}(s_\alpha)|K \cap U_\alpha = s|K \cap U_\alpha$, where the $k_\lambda : \mathscr{L}_\lambda \to \mathscr{L}$ are the canonical homomorphisms.

Any finite number of the s_α coincide in the direct limit at a point of K, and hence in a neighborhood of the point. Therefore, by passing to a refinement, it can also be assumed that for any finite subset $F \subset A$ there is an index $\lambda \in \Lambda$ such that $\lambda \geq \lambda(\alpha)$ for all $\alpha \in F$ and the $k_{\lambda,\lambda(\alpha)}(s_\alpha)$ all coincide on $K \cap \bigcap_{\alpha \in F} U_\alpha$.

Let $\{V_\alpha \mid \alpha \in A\}$ be a shrinking of $\{U_\alpha\}$; i.e., $\overline{V}_\alpha \subset U_\alpha$ for all α. For $F \subset A$ finite, put

$$C_F = \bigcap_{\alpha \in F} \overline{V}_\alpha - \bigcup_{\beta \notin F} V_\beta,$$

and note that

$$C_{F_1} \cap C_{F_2} \subset C_{F_1 \cap F_2}. \tag{35}$$

For a given integer $n \geq 0$ and for all $F \subset A$ with fewer than n elements, suppose that we have defined an index $\lambda(F) \in \Lambda$ and sections $s_F \in \mathscr{L}_{\lambda(F)}(C_F)$ such that $\lambda(F) \geq \lambda(G)$ and $s_F = k_{\lambda(F),\lambda(G)}(s_G)$ on $C_F \cap C_G$ for all proper subsets $G \subsetneq F$ and such that $s_F = k_{\lambda(F),\lambda(\alpha)}(s_\alpha)$ on $K \cap C_F$ for all $\alpha \in F$.

For a given subset $F \subset A$ with n elements, there is an index $\lambda(F)$ such that $\lambda(F) \geq \lambda(G)$ for all $G \subsetneq F$ and such that the $k_{\lambda(F),\lambda(\alpha)}(s_\alpha)$ all coincide on $K \cap C_F$ for $\alpha \in F$. Then the sections $k_{\lambda(F),\lambda(G)}(s_G)$, for $G \subsetneq F$, and the $k_{\lambda(F),\lambda(\alpha)}(s_\alpha)|K \cap C_F$ fit together to give a section of $\mathscr{L}_{\lambda(F)}$ over $\bigcup\{C_G \cap C_F \mid G \subsetneq F\} \cup (K \cap C_F)$, which is closed in C_F. Since $\mathscr{L}_{\lambda(\alpha)}$ is soft, this extends to a section $s_F \in \mathscr{L}_{\lambda(F)}(C_F)$.

This completes the inductive construction of the s_F. Because of (35), the projections $k_{\lambda(F)}(s_F) \in \mathscr{L}(C_F)$ fit together to give a section $s' \in \mathscr{L}(X)$; and $s'|K = s$ by construction. □

16.31. Corollary. *Let the base ring L be a principal ideal domain. If Φ is paracompactifying on X and $\mathscr{B}$ is a torsion-free Φ-soft sheaf of L-modules on X, then $\mathscr{A} \otimes_L \mathscr{B}$ is Φ-soft for all sheaves $\mathscr{A}$ of L-modules on X.*

Proof. Let $\mathfrak{S}$ be the collection of all sheaves $\mathscr{A}$ on X such that $\mathscr{A} \otimes \mathscr{B}$ is Φ-soft. Since $L_U \otimes \mathscr{B} \approx \mathscr{B}_U$ is Φ-soft by 9.13, $L_U \in \mathfrak{S}$, and so $\mathfrak{S}$ satisfies condition (c) of 16.12. Condition (b) is satisfied by 16.30. Condition (a) is satisfied by 9.10, since $\mathscr{B}$ is torsion-free, whence $(\bullet) \otimes \mathscr{B}$ is exact. Therefore $\mathfrak{S}$ consists of all sheaves of L-modules by 16.12. □

16.32. Corollary. *If Φ is paracompactifying on X and L is a principal ideal domain, then the following four conditions are equivalent:*

(a) $\dim_{\Phi,L} X \leq n$.

(b) $H^{n+1}_{\Phi|U}(U;L) = 0$ *for all open* $U \subset X$.

(c) $H^{n+1}_{\Phi}(X,F;L) = 0$ *for all closed* $F \subset X$.

(d) $H^n_{\Phi}(X;L) \to H^n_{\Phi}(F;L)$ *is surjective for all closed* $F \subset X$.

Proof. By definition, (a) $\Rightarrow$ (b), since $H^{n+1}_{\Phi|U}(U;L) \approx H^{n+1}_{\Phi}(X;L_U)$ by 10.2. Also, (b) $\Rightarrow$ (c) because $H^{n+1}_{\Phi}(X,F;L) \approx H^{n+1}_{\Phi|X-F}(X-F;L)$ by 10.2 and 12.3. Third, (c) $\Rightarrow$ (d) by the exact sequence of the pair (X,F). To prove the final implication (d) $\Rightarrow$ (a), consider the exact sequences

$$0 \to \mathscr{Z}^{i-1}(X;L)_F \to \mathscr{C}^{i-1}(X;L)_F \to \mathscr{Z}^i(X;L)_F \to 0$$

for $0 < i \le n$ and closed subsets $F \subset X$, where $\mathscr{Z}^0(X;L) = L$. Since $\mathscr{C}^*(X;L)_F$ is Φ-soft by 9.13, the cohomology sequences of these coefficient sequences give the natural isomorphisms

$$H^1_{\Phi}(X;\mathscr{Z}^{n-1}(X;L)_F) \approx \cdots \approx H^n_{\Phi}(X;\mathscr{Z}^0(X;L)_F) = H^n_{\Phi}(X;L_F)$$

and the exact sequences

$$H^0_{\Phi}(X;\mathscr{C}^{n-1}(X;L)_F) \to H^0_{\Phi}(X;\mathscr{Z}^n(X;L)_F) \to H^1_{\Phi}(X;\mathscr{Z}^{n-1}(X;L)_F) \to 0.$$

These combine to give the exact commutative diagram

$$\begin{array}{ccccccc}
\Gamma_{\Phi}(\mathscr{C}^{n-1}(X;L)) & \to & \Gamma_{\Phi}(\mathscr{Z}^n(X;L)) & \to & H^n_{\Phi}(X;L) & \to & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
\Gamma_{\Phi}(\mathscr{C}^{n-1}(X;L)_F) & \to & \Gamma_{\Phi}(\mathscr{Z}^n(X;L)_F) & \to & H^n_{\Phi}(X;L_F) & \to & 0.
\end{array}$$

Now, f is onto by 9.3 since $\Gamma_{\Phi}(\mathscr{A}_F) = \Gamma_{\Phi|F}(\mathscr{A}|F)$ by 10.2 and since $\mathscr{C}^{n-1}(X;L)$ is Φ-soft. Also, h is onto by assumption and 10.2. The 5-lemma implies that g is onto, showing that $\mathscr{Z}^n(X;L)$ is Φ-soft. It is also torsion free, as remarked in Section 2, and so (for tensor products over L)

$$0 \to L \otimes \mathscr{A} \to \mathscr{C}^0(X;L) \otimes \mathscr{A} \to \cdots \to \mathscr{C}^{n-1}(X;L) \otimes \mathscr{A} \to \mathscr{Z}^n(X;L) \otimes \mathscr{A} \to 0$$

is a Φ-soft resolution of $\mathscr{A}$, for any sheaf $\mathscr{A}$ of L-modules. Thus $\dim_{\Phi,L} X \le n$ by 16.4. □

16.33. Corollary. *If X is paracompact, then the following four conditions are equivalent over a principal ideal domain L :*[40]

(a) $\dim_L X \le n$.

(b) $H^{n+1}_{\Phi}(X;L) = 0$ *for all paracompactifying* Φ *on* X.

(c) $H^{n+1}(X,F;L) = 0$ *for all closed* $F \subset X$.

(d) $H^n(X;L) \to H^n(F;L)$ *is surjective for all closed* $F \subset X$.

[40]The equivalence of (c) and (d) is due to E. G. Skljarenko. The equivalence of these with (a) is new at least to the author.

Proof. Items (a), (c), and (d) are just those of 16.32 for $\Phi = cld$. Also, (a) $\Rightarrow$ (b) by 16.5. Given (b), the cases $\Phi = cld|U$ show that $H^{n+1}_{cld|U}(U;L) \approx H^{n+1}_{cld|U}(X;L) = 0$ for all open $U \subset X$ (since $\Gamma_{\Phi|U}(\mathscr{A}|U) = \Gamma_{\Phi|U}(\mathscr{A})$ for all sheaves $\mathscr{A}$ on X). That is just case (b) of 16.32 for $\Phi = cld$, whence $\dim_L X = \dim_{cld,L} X \le n$ by (b) $\Rightarrow$ (a) of 16.32. □

16.34. Corollary. *If Φ is paracompactifying on X, then*

$$\boxed{\dim_{\Phi,L} X \le \sup_{K\in\Phi}\{\operatorname{covdim} K\}}$$

for any ring L.

Proof. By 16.7 it suffices to show that $\dim_L K \le \operatorname{covdim} K$ for $K \in \Phi$. Thus we may assume that $\Phi = cld$ on a paracompact space X. The result is then immediate from (19) on page 29, 9.23, and 16.33(b). □

It is a theorem of Alexandroff [62, p. 243] that if $\operatorname{covdim} X < \infty$, then $\operatorname{covdim} X = n \Leftrightarrow H^k(X;\mathbb{Z}) \to H^k(F;\mathbb{Z})$ is onto for all closed $F \subset X$ and all $k \ge n$. Thus, by 16.33, $\operatorname{covdim} X = \dim_{\mathbb{Z}} X$ as long as $\operatorname{covdim} X < \infty$. There is a recent example of Dranishnikov [33] of a compact metric space X of infinite covering dimension but with $\dim_{\mathbb{Z}} X = 3$. The long line (see Exercise 3) is a nonparacompact (hence of infinite covering dimension) space with $\dim_{\mathbb{Z}} X = 1$. Also see the remarks below 16.39.

We shall now generalize 16.21 to the case of paracompactifying supports on general spaces. To do this we define the strong inductive dimension $\operatorname{Ind}_\Phi X$ with respect to a paracompactifying support family Φ as follows: Put $\operatorname{Ind}_\Phi X = -1$ if $X = \varnothing$. Then we say, inductively, that $\operatorname{Ind}_\Phi X \le n$ if for any two sets $A, C \in \Phi$ with $A \subset \operatorname{int} C$ there exists a set $B \in \Phi$ with $A \subset \operatorname{int} B$ and $B \subset \operatorname{int} C$ and with $\operatorname{Ind}_\Phi \partial B < n$. If $\operatorname{Ind}_\Phi X \le n$ and $\operatorname{Ind}_\Phi X \not\le n-1$, then we say that $\operatorname{Ind}_\Phi X = n$. The case in which $\Phi = cld$ and X is metric is the classical case of the strong inductive dimension, which is known to coincide with covering dimension; see [62].

16.35. Theorem. *If Φ is a paracompactifying family of supports on X and L is a ring with unit, then* $\operatorname{Ind}_\Phi X = 0 \Leftrightarrow \dim_{\Phi,L} X = 0$.

Proof. Let $A, C \in \Phi$ be as in the definition. Let $F = A \cup (X - \operatorname{int} C)$ and let $s \in \Gamma_{\Phi|F}(L|F)$ be 1 on A and 0 on $X - \operatorname{int} C$. If $\dim_{\Phi,L} X = 0$ then L is Φ-soft and so s must extend to some $t \in \Gamma_\Phi(L)$. Let $B = |t|$. Then B is both closed and open since L is a constant sheaf. Thus $\operatorname{Ind}_\Phi X = 0$.

Conversely, suppose that $\operatorname{Ind}_\Phi X = 0$. For a sheaf $\mathscr{A}$ of L-modules, let $c \in C^1_\Phi(X;\mathscr{A})$ be a cocycle with cohomology class γ and put $B = |c| \in \Phi$. Since B has a neighborhood N in Φ, it has one that is both open and closed. As in the proof of 16.21 (and since N is paracompact), there is a locally finite covering $\{U_\alpha\}$ of N consisting of sets both open and closed such that $\gamma|U_\alpha = 0 \in H^1_{\Phi\cap U_\alpha}(U_\alpha;\mathscr{A})$ for all α. Note that $\Phi \cap U_\alpha = \Phi|U_\alpha$ since U_α is

closed as well as open, and so we can just call this family Φ without fear of confusion. Just by passing to finite intersections, we can assume that the U_α are disjoint. Now, $H^1_\Phi(X;\mathscr{A}) \rightarrowtail H^1_\Phi(X-N;\mathscr{A})\oplus\prod H^1_\Phi(U_\alpha;\mathscr{A})$ and γ restricts to 0 in all these sets. Thus $\gamma=0$, so that $H^1_\Phi(X;\mathscr{A})=0$, whence $\dim_{\Phi,L}X=0$. □

Note that it follows that the statement $\dim_{\Phi,L}X=0$ is independent of the base ring L, and also that the statement $\mathrm{Ind}_\Phi X=0$ depends only on the extent $E(\Phi)$ and not on Φ itself. At least the first of these definitely does not extend to higher dimensions; i.e., there are examples for which $\dim_L X$ depends on L.

There is an example, due to E. Pol and R. Pol [67], of a normal, but not paracompact, space X with $\mathrm{Ind}\,X=0$ but $\dim_{cld,L}X>0$ for any L; see [72, 4.2].

16.36. Theorem. *Let Φ be a paracompactifying family of supports on X and let $\{A_\alpha\}$ be a locally finite closed covering of X. Suppose that $\dim_{\Phi,L}A_\alpha\cap A_\beta<n$ for all $\alpha\neq\beta$. Then the restriction maps induce a monomorphism*

$$r^*:H^{n+1}_\Phi(X;\mathscr{A})\rightarrowtail\prod_\alpha H^{n+1}_\Phi(A_\alpha;\mathscr{A})$$

for any sheaf $\mathscr{A}$ of L-modules on X. If, moreover, there is a $K\in\Phi$ with $A_\alpha\subset K$ for all but a finite number of α, then r^* is an isomorphism.

Proof. Let $A=\bigcup_{\alpha\neq\beta}A_\alpha\cap A_\beta$ and note that $\dim_{\Phi,L}A<n$ as follows from 16.8 and the fact that $\dim(B\cup C)=\max\{\dim(B),\dim(C-B)\}$ for closed sets B and C, by Exercise 11. Consider the identification map $f:\coprod_\alpha A_\alpha\to X$. This is closed and finite-to-one. By 11.1 we have

$$H^{n+1}_\Phi(X;ff^*\mathscr{A})\approx H^{n+1}_{f^{-1}\Phi}(\textstyle\coprod_\alpha A_\alpha;f^*\mathscr{A})\rightarrowtail\displaystyle\prod_\alpha H^{n+1}_\Phi(A_\alpha;\mathscr{A}).$$

[If there is a $K\in\Phi$ with $A_\alpha\subset K$ for all but a finite number of α, then this is clearly onto.] There is the canonical monomorphism $\beta:\mathscr{A}\rightarrowtail ff^*\mathscr{A}$. Let $\mathscr{B}=ff^*\mathscr{A}/\beta\mathscr{A}$. Since $\mathscr{B}$ is concentrated on the closed set A and $\dim_{\Phi,L}A<n$, we have that $H^p_\Phi(X;\mathscr{B})=H^p_\Phi(X;\mathscr{B}_A)\approx H^p_\Phi(A;\mathscr{B})=0$ for $p\geq n$. Then the exact cohomology sequence

$$H^n_\Phi(X;\mathscr{B})\to H^{n+1}_\Phi(X;\mathscr{A})\to H^{n+1}_\Phi(X;ff^*\mathscr{A})\to H^{n+1}_\Phi(X;\mathscr{B})$$

shows that $H^{n+1}_\Phi(X;\mathscr{A})\approx H^{n+1}_\Phi(X;ff^*\mathscr{A})\rightarrowtail\prod_\alpha H^{n+1}_\Phi(A_\alpha;\mathscr{A})$. □

16.37. Theorem. (H. Cohen.) *Suppose that X is locally compact Hausdorff and that L is a ring with unit. Let $X=\bigcup_{\alpha\in I}A_\alpha$, where A_α is closed and $\dim_L A_\alpha\leq n$ for each α. Assume that each A_α has an arbitrarily small closed neighborhood B_α with $\dim_L\partial B_\alpha<n$. Then $\dim_L X\leq n$.*[41]

[41] Also see 16.40 and Exercise 60.

Proof. By 16.8 we can assume that X is compact. For a sheaf $\mathscr{A}$ of L-modules let $\gamma \in H^{n+1}(X;\mathscr{A})$. Let $K \in c$ be the support of some cocycle representative of γ. Since $\gamma|A_\alpha = 0$ we have that $\gamma|B_\alpha = 0$ for all sufficiently small closed neighborhoods B_α of A_α, and such sets B_α can be chosen so that $\dim_L \partial B_\alpha < n$. Since the interiors of the B_α cover K, there is a finite subcollection that does so. By passing to the intersections and closures of differences of these B_α, we obtain a finite family $C_1, \ldots, C_r$ of closed sets whose interiors cover K and such that $\gamma|C_i = 0$ and $\dim_L C_i \cap C_j < n$ (since $C_i \cap C_j \subset \partial B_\alpha$ for some α). Throwing in the closure C_0 of the complement of $C_1 \cup \cdots \cup C_r$, we get a collection of sets satisfying the hypotheses of 16.36 and with $\gamma|C_i = 0$ for all i. From 16.36 it follows that $\gamma = 0$. □

Because of the finiteness of the collection $\{B_\alpha\}$ in the foregoing proof, one could use a less sophisticated argument based on Exercise 11 rather than on 16.36.

Recall that the *weak inductive dimension* $\operatorname{ind} X$ is defined by $\operatorname{ind}\varnothing = -1$ and $\operatorname{ind} X \le n$ if each point $x \in X$ has an arbitrarily small neighborhood N with $\operatorname{ind}\partial N \le n-1$. It is well known that $\operatorname{ind} X = \operatorname{Ind} X$ for separable metric spaces but not for general metric spaces. By taking the A_α to be the points of X in 16.37 we deduce:

16.38. Corollary. *If X is locally compact Hausdorff and L is any ring with unit, then* $\dim_L X \le \operatorname{ind} X$. □

According to [49, p. 65] a separable metric space X can be embedded in a compact separable metric space $\overline{X}$ of the same inductive dimension. Thus the corollary and 16.8 imply that $\dim_L X \le \operatorname{ind} X$ for all separable metric X. The following result is somewhat more general.

16.39. Theorem. *For a paracompactifying family Φ of supports on X and any ring L with unit, we have* $\dim_{\Phi,L} X \le \operatorname{Ind}_\Phi X$.

Proof. The proof is very similar to that of 16.37, so that we will just indicate the necessary modifications. Assume that $\operatorname{Ind}_\Phi X = n$ and that the result holds for smaller inductive dimensions. Then, given $\gamma \in H^{n+1}_\Phi(X;\mathscr{A})$, there is a locally finite closed covering $\{B_\alpha\}$ of X such that $\gamma|B_\alpha = 0$ and $\operatorname{Ind}_\Phi \partial B_\alpha < n$. By the inductive hypothesis, $\dim_{\Phi,L}\partial B_\alpha < n$. Construct the C_α, now just *locally* finite, and proceed as before. □

Note that if X is paracompact, then $\operatorname{Ind} X$ is defined, and it is clear that $\operatorname{Ind}_\Phi X \le \operatorname{Ind} X$. (In fact, it follows from the sum theorem for $\operatorname{Ind} X$ on paracompact spaces that $\operatorname{Ind} X = \operatorname{Ind}_\Phi X$ if $E(\Phi) = X$; see [62, p. 193].)

It is known that $\operatorname{covdim} X \le \operatorname{Ind} X$ for X normal; see [62, p. 197]. There is a compact (but not metrizable) space X with $\operatorname{covdim} X = 1$ (whence $\dim_{\mathbb{Z}} X = 1$) but with $\operatorname{Ind} X = 2 = \operatorname{ind} X$; see [62, p. 198]. Moreover, this space is the union $X = F_1 \cup F_2$ of two closed subspaces with $\operatorname{Ind} F_i = 1$.

There is also an example due to P. Roy [70] of a metric space (not separable) X for which $\operatorname{ind} X < \operatorname{Ind} X$.

16.40. Theorem. (The sum theorem.) *Let X be locally paracompact and let L be any ring with unit. Suppose that $X = A_1 \cup A_2 \cup \cdots$, where the A_i are closed and $\dim_L A_i \le n$. Then $\dim_L X \le n$.*

Proof. Let Φ be a paracompactifying support family on X and put $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$. Let $U_0 = X$ and $B_k = A_1 \cup \cdots \cup A_k$. By Exercise 11, $\dim_L B_k \le n$. Let $\alpha \in H^{n+1}_\Phi(X;\mathscr{A}) = H^{n+1}(\Gamma_\Phi(\mathscr{A}^*))$ and suppose that $a_0 \in \Gamma_\Phi(\mathscr{A}^*)$ is a cocycle representative of α. The exact sequence of the pair (X, B_1) has the form

$$H^{n+1}(\Gamma_{\Phi|X-B_1}(\mathscr{A}^*)) \to H^{n+1}(\Gamma_\Phi(\mathscr{A}^*)) \to H^{n+1}(\Gamma_{\Phi\cap B_1}(\mathscr{A}^*|B_1)) = 0,$$

since $H^{n+1}(\Gamma_{\Phi\cap B_1}(\mathscr{A}^*|B_1)) \approx H^{n+1}_{\Phi\cap B_1}(B_1;\mathscr{A}|B_1)$. Therefore there exist elements $a_1 \in \Gamma_{\Phi|X-B_1}(\mathscr{A}^*)$ and $b_0 \in \Gamma_{\Phi|U_0}(\mathscr{A}^*)$ with $a_1 = a_0 - db_0$. Since Φ is paracompactifying, there is an open set U_1 with $|a_1| \subset U_1$ and $\overline{U}_1 \subset X - B_1 = U_0 - B_1$. Thus $a_1 \in \Gamma_{\Phi|U_1}(\mathscr{A}^*)$. An inductive argument along these lines (replacing $\Phi = \Phi|U_0$ with $\Phi|U_1$, etc.) gives a sequence of open sets U_k with $\overline{U}_k \subset U_{k-1} - B_k$ and elements

$$a_k, b_k \in \Gamma_{\Phi|U_k}(\mathscr{A}^*)$$

with

$$a_{k+1} = a_k - db_k.$$

Now, $\bigcap \overline{U}_k = \bigcap U_k = \varnothing$, and so any given point $x \in X$ has a neighborhood meeting only finitely many $\overline{U}_k$. Since $|b_k| \subset U_k$ it follows that it makes sense to define

$$b = b_0 + b_1 + \cdots \in \Gamma_\Phi(\mathscr{A}^*).$$

On $X - U_k$, b coincides with $b_0 + \cdots + b_{k-1}$, and so db coincides with

$$d(b_0 + \cdots + b_{k-1}) = a_0 - a_k,$$

which coincides with a_0 outside U_k. Consequently, $a_0 = db$ everywhere. Thus $\alpha = [a_0] = 0$, whence $H^{n+1}_\Phi(X;\mathscr{A}) = 0$, and so $\dim_L X \le n$. □

16.41. Example. There are examples of countable connected Hausdorff spaces X. See [34, V-problem I-10] for one such example with a countable basis. For such a space, $\dim_{cld,L} X > 0$ by 16.20. Thus a result such as 16.40 (without local paracompactness) cannot hold for X. ◇

16.42. Example. Let X be the set of points in Hilbert space with all coordinates rational. Then X is totally disconnected, but $\operatorname{Ind} X = 1$ as shown in [49]. Thus $\dim_L X = 1$ by 16.35 and 16.39 for any principal ideal domain L. Note that $X \times X \approx X$, so that $1 = \dim_L X \times X < 2 \dim_L X = 2$. Also note that there must be an open set $U \subset X$ with $H^1_\Phi(U;\mathbb{Z}) \ne 0$ by

16.32, where $\Phi = cld|U$. Now $H^1_\Phi(U;\mathbb{Z})$ is torsion-free by Exercise 28. Thus the cross product $H^1_\Phi(U;\mathbb{Z}) \otimes H^1_\Phi(U;\mathbb{Z}) \to H^2_{\Phi\times\Phi}(U \times U;\mathbb{Z}) = 0$ is not monomorphic in this example. ◇

For further results concerning dimension, see Exercises 11 and 22-25 as well as IV-Section 8.

17 Local connectivity

In this section we take cohomology with coefficients in the constant sheaf L, where L is a principal ideal domain. There is the canonical inclusion $L \hookrightarrow \Gamma(L) = H^0(X;L)$, and the cokernel is called the *reduced* cohomology group $\tilde{H}^0(X;L) = H^0(X;L)/L$. For $p \neq 0$ we let $\tilde{H}^p(X;L) = H^p(X;L)$.

17.1. Definition. *The space X is said to be n-clc_L (cohomology locally connected) if for each point $x \in X$ and neighborhood N of x, there is a neighborhood $M \subset N$ of x such that the restriction homomorphism*

$$r^n_{M,N} : \tilde{H}^n(N;L) \to \tilde{H}^n(M;L)$$

is zero.

X is said to be clc^n_L if it is k-clc_L for all $k \leq n$ and clc^∞_L if it is k-clc_L for all k.

X is said to be clc_L if given x and N as above, M can be chosen, independently of n, so that $r^n_{M,N} = 0$ for all n.

Of course, the definition is not affected if we require M and N to be taken from a given neighborhood basis. It is clear that X is 0-clc_L if every point $x \in X$ has arbitrarily small connected neighborhoods (i.e., X is "locally connected"), since $\tilde{H}^0(U;L) = \Gamma(L|U)/L = 0 \Leftrightarrow U$ is connected. Conversely, if X is 0-clc_L, then given an open set N and a point $x \in N$, there is an open neighborhood M of x such that for any separation $N = U \cup V$ into disjoint open sets with $x \in U$ we have $M \subset U$. This means that the quasi-component of N containing x is open. But if all quasi-components of an open set are open then these quasi-components must be connected, and that implies that X is locally connected.[42] Thus we have shown:

X is 0-clc_L $\Leftrightarrow$ X is locally connected.

17.2. Proposition. *The space X is n-clc_L $\Leftrightarrow$ given $x \in X$ and N as in* 17.1, *there is a neighborhood $M \subset N$ of x such that* $\operatorname{Im} r^n_{M,N}$ *is finitely generated.*

[42] Also see [85, pp. 40 ff.].

Proof. This follows from the fact that $\varinjlim \tilde{H}^n(N) = \tilde{H}^n(x) = 0$, since a point is always taut. □

17.3. Lemma. *Consider a commutative diagram of L-modules of the form*

$$\begin{array}{ccccc} & & A_2 & \xrightarrow{s} & A_3 \\ & & \downarrow f & & \downarrow k \\ B_1 & \xrightarrow{i} & B_2 & \xrightarrow{j} & B_3 \\ \downarrow h & & \downarrow g & & \\ C_1 & \xrightarrow{t} & C_2 & & \end{array}$$

in which the middle row is exact. Let "small" mean "zero" or "finitely generated." In both of these cases, if $\operatorname{Im} th$ *and* $\operatorname{Im} ks$ *are small, then* $\operatorname{Im} gf$ *is also small.*

Proof. Put $K = \operatorname{Ker} jf$. Then $gf(K) \subset \operatorname{Im} th$ and hence is small. Also, jf induces a monomorphism $A_2/K \rightarrowtail B_3$, and its image $jf(A_2) = ks(A_2)$ is small, whence A_2/K is small. Thus there is a small submodule $S \subset A_2$ such that $A_2 = K + S$. Then $gf(A_2) = gf(K) + gf(S)$, which is small. □

17.4. Theorem. (Wilder.) *Let X be locally compact Hausdorff. Consider the following four conditions:*

(r^n) *If* K *and* M *are compact subspaces of* X *with* $K \subset \operatorname{int} M$, *then* $\operatorname{Im}(r^n_{K,M} : H^n(M; L) \to H^n(K; L))$ *is finitely generated.*

$(r^n_\bullet)$ *If* M *is a neighborhood of* $x \in X$, *then there is a neighborhood* $K \subset M$ *of* x *with* $\operatorname{Im} r^n_{K,M}$ *finitely generated; i.e.,* X *is* n-clc$_L$.

(j^n) *If* U *and* V *are open, relatively compact subspaces of* X *with* $\overline{V} \subset U$, *then* $\operatorname{Im}(j^n_{U,V} : H^n_c(V; L) \to H^n_c(U; L))$ *is finitely generated.*

$(j^n_\bullet)$ *If* U *is an open neighborhood of* $x \in X$, *then there is an open neighborhood* $V \subset U$ *of* x *with* $\operatorname{Im} j^n_{U,V}$ *finitely generated.*

Then the following implications are true:

$$\begin{array}{rcl} (r^n) & \Rightarrow & (r^n_\bullet), \\ (j^n) & \Rightarrow & (j^n_\bullet), \\ (r^n) \text{ and } (r^{n-1}) & \Rightarrow & (j^n), \\ (r^n_\bullet) \text{ and } (r^{n-1}) & \Rightarrow & (r^n), \\ (j^n) \text{ and } (j^{n-1}) & \Rightarrow & (r^{n-1}), \\ (j^n_\bullet) \text{ and } (j^{n+1}) & \Rightarrow & (j^n). \end{array}$$

Proof. The first two implications are trivial.

Suppose that (r^n) and (r^{n-1}) hold. Given relatively compact open sets U, W with $\overline{U} \subset W$, construct an open set V and compact sets K, L, and M such that $U \subset V \subset W \subset K \subset L \subset M$, with the closure of each contained in the interior of the next. Then 17.3 applied to the diagram

$$\begin{array}{ccccc} & & H_c^n(U) & \to & H^n(M) \\ & & \downarrow & & \downarrow \\ H^{n-1}(L-V) & \to & H_c^n(V) & \to & H^n(L) \\ \downarrow & & \downarrow & & \\ H^{n-1}(K-W) & \to & H_c^n(W) & & \end{array}$$

implies that condition (j^n) holds.

Now assume that X satisfies $(r_\bullet^n)$ and (r^{n-1}). Let M be fixed and let $\mathfrak{S}$ be the collection of compact subsets K of $\operatorname{int} M$ such that K has a neighborhood K' in $\operatorname{int} M$ with $\operatorname{Im} r^n_{K',M}$ finitely generated. Then $\mathfrak{S}$ contains a neighborhood of each point of $\operatorname{int} M$ by $(r_\bullet^n)$. Let $K_1, K_2 \in \mathfrak{S}$ and let K_i' be a compact neighborhood of K_i $(i = 1, 2)$ such that $\operatorname{Im} r^n_{K_i',M}$ is finitely generated and let K_i'' be a compact neighborhood of K_i with $K_i'' \subset \operatorname{int} K_i'$. The Mayer-Vietoris diagram

$$\begin{array}{ccccc} & & H^n(M) & \to & H^n(M) \oplus H^n(M) \\ & & \downarrow & & \downarrow \\ H^{n-1}(K_1' \cap K_2') & \to & H^n(K_1' \cup K_2') & \to & H^n(K_1') \oplus H^n(K_2') \\ \downarrow & & \downarrow & & \\ H^{n-1}(K_1'' \cap K_2'') & \to & H^n(K_1'' \cup K_2'') & & \end{array}$$

together with 17.3 shows that $K_1 \cup K_2 \in \mathfrak{S}$ and consequently that $\mathfrak{S}$ contains all compact subsets of $\operatorname{int} M$, proving (r^n).

Now suppose that (j^n) and (j^{n-1}) hold. Given K and M, construct L compact and U, V, and W open and relatively compact, with $K \subset L \subset M \subset U \subset V \subset W$, the closure of each being in the interior of the next. Condition (r^{n-1}) then follows from the diagram

$$\begin{array}{ccccc} & & H^{n-1}(M) & \to & H_c^n(U-M) \\ & & \downarrow & & \downarrow \\ H_c^{n-1}(V) & \to & H^{n-1}(L) & \to & H_c^n(V-L) \\ \downarrow & & \downarrow & & \\ H_c^{n-1}(W) & \to & H^{n-1}(K). & & \end{array}$$

Finally assume that X satisfies $(j_\bullet^n)$ and (j^{n+1}). Let U be a fixed open and relatively compact subset of X and let $\mathfrak{S}$ be the collection of open subsets V of U such that $\overline{V}$ has an open neighborhood V' with $\overline{V'} \subset U$ and with $\operatorname{Im} j^n_{U,V'}$ finitely generated. Then $\mathfrak{S}$ contains a neighborhood of each point in U by $(j_\bullet^n)$. Let $V_1, V_2 \in \mathfrak{S}$ and let V_i' be an open neighborhood of $\overline{V_i}$ $(i = 1, 2)$ with $\operatorname{Im} j^n_{U,V_i'}$ finitely generated. Also, let V_i'' be an open

neighborhood of $\overline{V_i}$ with $\overline{V_i''} \subset V_i'$. Then the Mayer-Vietoris diagram

$$\begin{array}{ccccc} & & H_c^n(V_1'' \cup V_2'') & \to & H_c^{n+1}(V_1'' \cap V_2'') \\ & & \downarrow & & \downarrow \\ H_c^n(V_1') \oplus H_c^n(V_2') & \to & H_c^n(V_1' \cup V_2') & \to & H_c^{n+1}(V_1' \cap V_2') \\ \downarrow & & \downarrow & & \\ H_c^n(U) \oplus H_c^n(U) & \to & H_c^n(U) & & \end{array}$$

together with 17.3 shows that $V_1 \cup V_2 \in \mathfrak{S}$ and consequently that $\mathfrak{S}$ contains all open sets V with $\overline{V} \subset U$, proving (j^n). □

17.5. Corollary. *If $(r^{\leqslant k})$ stands for "(r^n) for all $n \leq k$," etc., then the following implications are true:*

$$\begin{array}{ccccccc} clc_L^k = (r_\bullet^{\leqslant k}) & \Leftrightarrow & (r^{\leqslant k}) & \Rightarrow & (j^{\leqslant k}) & \Rightarrow & (j_\bullet^{\leqslant k}) \\ & & & & \Downarrow & & \\ & & & & (r_\bullet^{\leqslant k-1}) = clc_L^{k-1}, & & \end{array}$$

$$(\dim_L X < \infty \text{ and } (j_\bullet^{<\infty})) \Rightarrow (j^{<\infty}) \Leftrightarrow clc_L^\infty.$$

In particular $(r_\bullet^{<\infty}) = clc_L^\infty$, $(r^{<\infty})$ and $(j^{<\infty})$ are equivalent, and they are equivalent to $(j_\bullet^{<\infty})$ provided that $\dim_L X < \infty$.

Proof. The proofs of all but the last implication are by an obvious induction on $n \leq k$. The proof of the last implication is a similar downwards induction from $\dim_L X$. □

17.6. Theorem. *If X is compact Hausdorff and clc_L^{n-1}, then $H^n(X; L)$ is finitely generated $\Leftrightarrow$ X satisfies condition (j^n).*

Proof. To see the implication $\Leftarrow$, take $U = V = X$ in (j^n). For the implication $\Rightarrow$, let U and V be open sets in X with $\overline{U} \subset V$. In the diagram

$$\begin{array}{ccccc} H^{n-1}(X-U) & \to & H_c^n(U) & \to & H^n(X) \\ \downarrow{\scriptstyle r^{n-1}} & & \downarrow{\scriptstyle j^n} & & \downarrow{\scriptstyle =} \\ H^{n-1}(X-V) & \to & H_c^n(V) & \to & H^n(X) \end{array}$$

$\operatorname{Im} r^{n-1}$ is finitely generated since $clc_L^{n-1} \Rightarrow (r^{n-1})$. It follows that $\operatorname{Im} j^n$ is finitely generated, which is condition (j^n). □

17.7. Corollary. *If X is compact Hausdorff and clc_L^n, then $H^p(X; L)$ is finitely generated for $0 \leq p \leq n$.* □

17.8. Proposition. *A closed subspace F of an n-manifold M^n satisfies property (r^n). If $M - F$ has only finitely many components, then F also satisfies property $(r_\bullet^{n-1})$.*

Proof. The first part is a trivial consequence of 16.29(a). For the second part, let B be a compact neighborhood in M of a point $x \in F$, so small that it does not contain any component of $U = M - F$. Let $A \subset \operatorname{int} B$ be another compact neighborhood of x. Then $H^n_c(U \cap B) = 0 = H^n_c(U \cap A)$ by 16.29(a). Since M is clc^∞_L, the restriction $H^{n-1}(B) \to H^{n-1}(A)$ has finitely generated image. Then the exact commutative diagram

$$\begin{array}{ccccc} H^{n-1}(B) & \to & H^{n-1}(B \cap F) & \to & H^n_c(B \cap U) = 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-1}(A) & \to & H^{n-1}(A \cap F) & \to & H^n_c(A \cap U) = 0 \end{array}$$

shows that $H^{n-1}(B \cap F) \to H^{n-1}(A \cap F)$ has finitely generated image. □

The following is an immediate consequence of the Mayer-Vietoris sequences:

17.9. Proposition. *Let $X = A \cup B$ be locally compact Hausdorff. If A and B are closed, then*

$$(r^k) \text{ for } A \text{ and } B,\ (r^{k+1}) \text{ for } X \Rightarrow (r^k) \text{ for } A \cap B,$$

and similarly for $(r_\bullet)$ in place of (r). If A and B are open, then

$$(j^k) \text{ for } A \text{ and } B,\ (j^{k-1}) \text{ for } X \Rightarrow (j^k) \text{ for } A \cap B.$$

□

17.10. Example. This example shows that the finite dimensionality assumption is essential for the implication $(j^{<\infty}_\bullet) \Rightarrow (j^{<\infty})$ in 17.5. Let $X = Y \times \mathbb{I}^\infty$ for some locally compact Hausdorff space Y. Then every neighborhood of a point $x \in X$ contains an open neighborhood homeomorphic to $V \times (0,1]$ for some locally compact subspace V. But $H^p_c(V \times (0,1]) = 0$ for all p by the Künneth theorem since $H^*_c((0,1]) \approx H^*([0,1],\{0\}) = 0$. Therefore X satisfies $(j^{<\infty}_\bullet)$. On the other hand, if Y is not locally connected, then neither is X.

Similarly, the infinite product $X = \prod_{i=1}^\infty \mathbb{S}^1$ satisfies $(j^{<\infty}_\bullet)$ because every point has a neighborhood homeomorphic to $\mathbb{R}^k \times X$ for arbitrarily large k. However, $X = \varprojlim (\mathbb{S}^1)^n$, and so $H^1(X;\mathbb{Z}) \approx \bigoplus_{i=1}^\infty \mathbb{Z}$, by continuity, which is not finitely generated. Thus X does not satisfy 17.7. ◇

17.11. Example. This example shows that the implication $(j^{\leqslant k}) \Rightarrow clc^k_L$ is false in general. Let K be the union of k-spheres with radii $1/n$, $n \geq 1$ integral, and with a single point in common. Let X be the cone on K. Then X is compact, contractible, and clc^{k-1}_L, but not clc^k_L. By 17.6, X satisfies condition $(j^{\leqslant k})$. ◇

17.12. Example. The compactified long line[43] is an example of a space that is $clc_{\mathbb{Z}}$ but is not *HLC*, since it is not locally arcwise connected at the point Ω. ◇

[43] See Exercise 3.

17.13. Example. Let Y be the interval $[0, \Omega]$ in the compactified long line. Let $W = Y \times \mathbb{S}^1$, let $z \in \mathbb{S}^1$, and let $A = \{\Omega\} \times \mathbb{S}^1 \cup [1, \Omega] \times \{z\} \subset W$. Let $X = W/A$. Then X is $clc_{\mathbb{Z}}$ (indeed it is a cohomology manifold with boundary; see V-16). Also, X is locally arcwise connected (which is the reason for the inclusion of the interval $[1, \Omega] \times \{z\}$ in the set to be collapsed). However, X is not HLC in dimension 1, for the circle $\{0\} \times \mathbb{S}^1$ cannot bound a singular 2-chain. [If it did, then X would be the image of a finite polyhedron. This would imply that X is metrizable, and it is not.] ◇

17.14. Example. If a space is locally connected, then each point has arbitrarily small neighborhoods that are connected. It is reasonable to ask whether this holds for higher connectivities such as clc_L^k, $k \geq 1$. This example shows that this is false. It is a slightly modified and generalized version of an example due to Wilder [85, p. 198], which we shall call *Wilder's necklace.* First we describe a bead of the necklace. Let S_i be a k-sphere with base point for $i \in \mathbb{Z}_n$, and let $f_i : S_i \twoheadrightarrow S_{i+1}$ be the base point preserving *surjection* of degree zero obtained by mapping S_i to a k-disk by identifying the upper and lower hemispheres and then collapsing the boundary of the k-disk to a point. Let B_i be the mapping cylinder of f_i. That is one bead. The k-sphere S_i is its "top" and S_{i+1} is its "bottom." Let $Y_n = B_0 \cup \cdots \cup B_{n-1}$. (Note that $S_n = S_0$.) This is a "strand" of the necklace. The n line segments made up of the generators I_i of the cylinders B_i between the base points form an n-gon T_n called the "thread" of the strand Y_n. Let X_n be the quotient space of the topological sum $Y_2 + \cdots + Y_n$ obtained by identification of the threads. Let the circle T be the common thread. Now, there is an obvious retraction $\rho_i : B_i \to I_i$, and these provide a retraction of the strand Y_n onto its thread T_n. This gives retractions $\pi_n : X_n \twoheadrightarrow X_{n-1}$ ($n > 2$) forming an inverse system of spaces. We let $X = \varprojlim X_n$, which is the whole necklace. We think of X as the union of the Y_n along T with a topology making Y_n "thin" as $n \to \infty$. It can be seen that X embeds in $\mathbb{R}^{k+2}$, but we do not need that. Note that $\dim_L X = k + 1$ by 16.40.

Now let C_i be the mapping cylinder of the map $S_i \to S_{i+1}$ taking all of S_i to the base point of S_{i+1}. (This is the one-point union, at the vertex, of a cone with a k-sphere.) Let D be the necklace, analogous to X, made up of the beads C_i rather than B_i. (D might be called a *dunce necklace.*) We will still use T to stand for the thread of D. Then there is a homotopy equivalence $\varphi : X \to D$ (restricting to $\varphi_i : B_i \to C_i$) with homotopy inverse $\psi : D \to X$, both being the identity map on the S_i and on the thread T; see [19, p. 48]. But there is an obvious strong deformation retraction $\theta : D \times \mathbb{I} \to D$ of D onto T. Then the composition

$$X \times \mathbb{I} \xrightarrow{\varphi \times 1} D \times \mathbb{I} \xrightarrow{\theta} D \xrightarrow{\psi} X$$

is a strong deformation retraction of X onto T. It follows that X is locally contractible,[44] and a fortiori, it is HLC and clc_L. (The argument just given

[44] This means that for each point $x \in X$ and neighborhood U of x, there is a neigh-

proves local contractibility at points of T. Local contractibility at points outside T is obvious.)

Suppose now that N is some neighborhood in X of a point $x \in T$. We assume that N is small enough so that it omits a neighborhood in X of some other point of the thread T. Then for n sufficiently large, N contains some bead, but not all beads, of the strand Y_n. Clearly there must be some bead B_{i_n} of Y_n such that the top S_{i_n} of B_{i_n} is completely inside N but the entire bead B_{i_n} is not. (This is where the surjectivity of f_i is used.) Let $y_n \in B_{i_n} - N$ and let T'_n be a variant of T_n in Y_n that misses y_n. There is a map $Y_n \to T'_n \cup B_{i_n} \cup C_{i_n+1}$ that is φ_{i_n+1} on B_{i_n+1}, the identity on B_{i_n}, and ρ_j on B_j for $j \neq i_n, i_n+1$. Consideration of several cases shows that $T'_n \cup S_{i_n}$ is a retract of $T'_n \cup B_{i_n} \cup C_{i_n+1} - \{y_n\}$, whence of $Y_n - \{y_n\}$. Putting $N_n = N \cap Y_n$, there are the restriction maps

$$H^k(Y_n - \{y_n\}; L) \to H^k(N_n; L) \to H^k(S_{i_n}; L) \approx L$$

whose composition is surjective. Now, $H^k(N \cap T; L) = 0$.[45] Thus, for $m > n$, a Mayer-Vietoris argument shows that

$$H^k(N) \to H^k(N_2 \cup \cdots \cup N_{n-1}) \oplus H^k(N_n) \oplus \cdots \oplus H^k(N_m) \oplus H^k(N_{m+1} \cup \cdots)$$

is surjective, and it follows that the map $H^k(N; L) \to \bigoplus_{j=n}^{m} H^k(S_{i_j}; L)$ is surjective. Consequently, $H^k(N; L)$ is not even finitely generated.

By taking a union along the threads T of Wilder's necklaces for $k = 1, \ldots, n$ and adding a 2-disk spanning T, one obtains a compact, contractible, and locally contractible space of dimension $n + 1 \geq 2$ that has points x such that any sufficiently small neighborhood N of x has $H^k(N; L)$ not finitely generated for any $1 \leq k \leq n$. ◇

17.15. Theorem. (F. Raymond [69].) *Let X be compact Hausdorff and let $F \subset X$ be closed and totally disconnected. Suppose that $X - F$ is clc_L^{n+1}. Then X is $clc_L^n \Leftrightarrow H^p(X; L)$ is finitely generated for each $p \leq n$.*

Proof. Let $x \in F$. Then any neighborhood of x contains an open neighborhood U such that $U \cap F$ is open and closed in F. Then there is a compact neighborhood $B_1 \subset U$ containing $U \cap F$ and there is a compact neighborhood $A_1 \subset \operatorname{int} B_1$ also containing $U \cap F$. Similarly, using $X - B_1$ as U was used, there are compact sets A_2, B_2 with $F - U \subset \operatorname{int} A_2$, $A_2 \subset \operatorname{int} B_2$, and $B_1 \cap B_2 = \varnothing$. Let $B = B_1 \cup B_2$ and $A = A_1 \cup A_2$. In the commutative diagram

$$\begin{array}{ccccc} H^p(X) & \to & H^p(B) & \to & H_c^{p+1}(X - B) \\ \downarrow{=} & & \downarrow{r^*} & & \downarrow{j^*} \\ H^p(X) & \to & H^p(A) & \to & H_c^{p+1}(X - A) \end{array}$$

borhood $V \subset U$ such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map.

[45] For $k = 1$ this follows from 10.11.

we have that $\operatorname{Im} j^*$ is finitely generated for $p \le n$ since $X - K$ satisfies (j^{p+1}) by 17.5. It follows that $\operatorname{Im} r^*$ is finitely generated for $p \le n$. Since B is the disjoint union of B_1 and B_2, it follows that $H^p(B_1) \to H^p(A_1)$ has a finitely generated image for $p \le n$, showing that X is clc_L^n. The converse follows from 17.7. □

17.16. Corollary. *If X is locally compact Hausdorff and clc_L^{n+1}, then the one-point compactification of X is clc_L^n $\Leftrightarrow$ $H_c^p(X;L)$ is finitely generated for each $p \le n$.*

Proof. This follows from the fact that $\tilde{H}^p(X^+;L) \approx H_c^p(X;L)$. □

We conclude this section by giving one application of 17.5 and 17.8. It is a result of Wilder [85, p. 325] in modern dress and slightly generalized.[46] First we need a lemma.

17.17. Lemma. *Let X be locally compact Hausdorff and $F \subset X$ closed. Let U be a union of some of the components of $X - F$ and put $G = X - U$. Suppose that X is clc_L^{m+1} and that F is clc_L^m. Then G is also clc_L^m.*

Proof. Let P, Q be compact subsets of X with $Q \subset \operatorname{int} P$. Let $k \le m$. By 17.5, $H^i(P) \to H^i(Q)$ has finitely generated image for $i = k, k+1$ and $H^k(P \cap F) \to H^k(Q \cap F)$ has finitely generated image. Then the exact sequences of $(P, P \cap F)$ and $(Q, Q \cap F)$, and an argument using 17.3 (and a set between P and Q), show that the restriction $H_c^{k+1}(P-F) \to H_c^{k+1}(Q-F)$ has finitely generated image. Now, $X - F = U + V = (X - G) + V$ for some open set V, and it follows that $H_c^{k+1}(Q-G) \to H_c^{k+1}(Q-F)$ is a monomorphism (onto a direct summand). From the commutative diagram

$$\begin{array}{ccc} H_c^{k+1}(P-G) & \to & H_c^{k+1}(P-F) \\ \downarrow & & \downarrow \\ H_c^{k+1}(Q-G) & \rightarrowtail & H_c^{k+1}(Q-F) \end{array}$$

it follows that $H_c^{k+1}(P-G) \to H_c^{k+1}(Q-G)$ has a finitely generated image. Then the exact sequences of these pairs show that $H^k(P \cap G) \to H^k(Q \cap G)$ has finitely generated image. □

17.18. Theorem. (Wilder.) *Let M be an n-manifold. Suppose that $F \subset M$ is closed and $clc_{\mathbb{Z}_2}^{n-2}$. Let $U \subset M$ be the union of a finite number of the components of $M - F$. Then $\overline{U}$ and ∂U are locally connected.*

[46] Wilder states his result for "generalized manifolds," but the present proof applies to that case. Wilder also assumes that M has the cohomology of a sphere, but that is not needed here.

Proof. Coefficients will be in $\mathbb{Z}_2$ throughout. As in 17.17, let $G = M - U$. Then G is $clc^\infty_{\mathbb{Z}_2}$ by 17.17 and 17.8. Let W and W' be connected open neighborhoods of a given point $x \in \overline{U}$ with $\overline{W'} \subset W$. Then in the commutative diagram

$$\begin{array}{ccccc} H^{n-1}_c(W'\cap G) & \to & H^n_c(U\cap W') & \to & H^n_c(W') \approx \mathbb{Z}_2 \\ \downarrow f^* & & \downarrow j^* & & \downarrow \\ H^{n-1}_c(W\cap G) & \to & H^n_c(U\cap W) & \to & H^n_c(W) \approx \mathbb{Z}_2, \end{array}$$

$\operatorname{Im} f^*$ is finite by property (j^{n-1}) for G. It follows that $\operatorname{Im} j^*$ is also finite. By 16.29, this implies that only a finite number of components, say $V_1, \dots, V_k$, of $U \cap W$ meet W'. If $x \in \overline{V}_i$ for $i \le k_0$ and $x \notin \overline{V}_i$ for $i > k_0$, then $\overline{V}_1 \cup \cdots \cup \overline{V}_{k_0}$ is a connected neighborhood of x in $\overline{U}$. Thus $\overline{U}$ is locally connected. Since $\partial U = G \cap \overline{U}$, it is also locally connected by 17.9. □

17.19. Corollary. (Torhorst.[47]) *If M is a 2-manifold and if $F \subset M$ is a locally connected closed subset, then for any finite union U of the components of $M - F$, $\overline{U}$ and ∂U are also locally connected.* □

17.20. Example. Let U be the union of disjoint open disks in $X = \mathbb{S}^2$ converging to a point $x \in X$. Then $F = X - U$ is locally connected, but $\overline{U}$ and ∂U are not locally connected at x. This shows that the last result is false without the restriction to finite unions. Similar examples in higher dimensions apply to Wilder's theorem. ◇

18 Change of supports; local cohomology groups

This section will not be used in other parts of this book. It is partially based on Raymond [68].

Let Φ and Ψ be families of supports on X with $\Phi \subset \Psi$. Define the groups

$$\boxed{I^p_{\Psi,\Phi}(X;\mathscr{A}) = H^p(C^*_\Psi(X;\mathscr{A})/C^*_\Phi(X;\mathscr{A})).}$$

The exact sequence

$$0 \to C^*_\Phi(X;\mathscr{A}) \to C^*_\Psi(X;\mathscr{A}) \to C^*_\Psi(X;\mathscr{A})/C^*_\Phi(X;\mathscr{A}) \to 0$$

of cochain complexes yields the exact cohomology sequence

$$\boxed{\cdots \to H^p_\Phi(X;\mathscr{A}) \to H^p_\Psi(X;\mathscr{A}) \to I^p_{\Psi,\Phi}(X;\mathscr{A}) \to H^{p+1}_\Phi(X;\mathscr{A}) \to \cdots} \quad (36)$$

If $\mathscr{L}$ is a flabby sheaf, such as $\mathscr{C}^*(X;\mathscr{A})$, then for $K \in \Phi$, we see that the sequence

$$0 \to \Gamma_{\Psi|K}(\mathscr{L}) \to \Gamma_\Psi(\mathscr{L}) \to \Gamma_{\Psi\cap(X-K)}(\mathscr{L}|X-K) \to 0$$

[47] Torhorst proved this in the case $M = \mathbb{S}^2$.

is exact. But $\Gamma_\Phi(\mathscr{L}) = \varinjlim_{K\in\Phi} \Gamma_{\Psi|K}(\mathscr{L})$, and hence

$$\Gamma_\Psi(\mathscr{L})/\Gamma_\Phi(\mathscr{L}) \approx \varinjlim_{K\in\Phi} \Gamma_{\Psi\cap(X-K)}(\mathscr{L}|X-K)$$

naturally. Applying this to $\mathscr{L} = \mathscr{C}^*(X;\mathscr{A})$ we have

$$\begin{aligned} I^*_{\Psi,\Phi}(X;\mathscr{A}) &\approx H^*(\varinjlim_{K\in\Phi} C^*_{\Psi\cap(X-K)}(X-K;\mathscr{A}|X-K)) \\ &\approx \varinjlim_{K\in\Phi} H^*_{\Psi\cap(X-K)}(X-K;\mathscr{A}|X-K). \end{aligned} \tag{37}$$

In case $\Psi = cld$, we put

$$\boxed{I^*_\Phi(X;\mathscr{A}) = I^*_{cld,\Phi}(X;\mathscr{A}).}$$

If X is locally compact, then in [68], $I^*_c(X)$ is denoted by $I^*(X)$ and is called the *cohomology of the ideal boundary.*

18.1. Let $F \subset X$ be closed. We define

$$\boxed{I^p_{F,\Phi}(X;\mathscr{A}) = I^p_{\Phi\cap(X-F),\Phi|(X-F)}(X-F;\mathscr{A}).}$$

Assume now that F is Φ-taut (e.g., Φ paracompactifying) and let $U = X - F$. For a flabby sheaf $\mathscr{L}$ on X we have the exact sequence

$$0 \to \Gamma_{\Phi|F}(\mathscr{L}) \to \Gamma_\Phi(\mathscr{L}) \to \Gamma_{\Phi\cap U}(\mathscr{L}|U) \to 0$$

(since U is always Φ-taut). Moreover, the subgroup $\Gamma_{\Phi|U}(\mathscr{L})$ of $\Gamma_\Phi(\mathscr{L})$ maps isomorphically onto $\Gamma_{\Phi|U}(\mathscr{L}|U) \subset \Gamma_{\Phi\cap U}(\mathscr{L}|U)$. Thus the sequence

$$0 \to \Gamma_{\Phi|F}(\mathscr{L}) \to \frac{\Gamma_\Phi(\mathscr{L})}{\Gamma_{\Phi|U}(\mathscr{L})} \to \frac{\Gamma_{\Phi\cap U}(\mathscr{L}|U)}{\Gamma_{\Phi|U}(\mathscr{L}|U)} \to 0$$

is exact. Since F is Φ-taut, the middle group is naturally isomorphic to $\Gamma_{\Phi\cap F}(\mathscr{L}|F)$. Replacing $\mathscr{L}$ by $\mathscr{C}^*(X;\mathscr{A})$, where $\mathscr{A}$ is a sheaf on X, and passing to cohomology, we obtain the exact sequence (coefficients in $\mathscr{A}$)

$$\boxed{\cdots \to H^p_\Phi(X,U) \to H^p_{\Phi\cap F}(F) \to I^p_{F,\Phi}(X) \to H^{p+1}_\Phi(X,U) \to \cdots} \tag{38}$$

Note that when $\Phi = cld$, then by (37) it follows that

$$\boxed{I^p_F(X;\mathscr{A}) \approx \varinjlim H^p(W-F;\mathscr{A}),} \tag{39}$$

W ranging over the neighborhoods of F.

18.2. Perhaps the most interesting case is that of the groups $I_c^*(U;\mathscr{A})$, where U is a dense open subspace of a compact space X. Then (36) becomes

$$\boxed{\cdots\to H_c^p(U;\mathscr{A})\to H^p(U;\mathscr{A})\to I_c^p(U;\mathscr{A})\to H_c^{p+1}(U;\mathscr{A})\to\cdots}$$

Letting $F = X - U$, we have $I_F^*(X;\mathscr{A}) = I_c^*(U;\mathscr{A})$, so that (38) becomes

$$\boxed{\cdots\to H^p(X,U;\mathscr{A})\to H^p(F;\mathscr{A})\to I_c^p(U;\mathscr{A})\to H^{p+1}(X,U;\mathscr{A})\to\cdots}$$

The latter sequence shows, for example, that if F is totally disconnected, then $I_c^p(U;\mathscr{A})\approx H^{p+1}(X,U;\mathscr{A})$ for $p>0$. Finally, (39) becomes

$$\boxed{I_c^p(U;\mathscr{A})\approx \varinjlim H^p(W-F;\mathscr{A}),}$$

W ranging over a neighborhood basis of F in X.

18.3. When F in (39) consists of a single point, we obtain a type of local cohomology group.[48] We now consider a different, but closely related, notion of local cohomology.

The *local cohomology group at the point* $x\in X$ with coefficients in $\mathscr{A}$ is defined to be $H_x^*(X;\mathscr{A})$ where the subscript x denotes the family of supports consisting only of $\{x\}$ and the empty set. For any family Φ containing $\{x\}$ we have, by 12.1 (provided that $\{x\}$ is closed),

$$\boxed{H_x^*(X;\mathscr{A}) = H_\Phi^*(X,X-\{x\};\mathscr{A}).}$$

If $\Phi = cld$ and if $\{x\}$ is closed, then $\{x\}$ is Φ-taut, so that (38) and (39) yield the exact sequence

$$\boxed{0\to H_x^0(X;\mathscr{A})\to \mathscr{A}_x\to \varinjlim \mathscr{A}(W-\{x\})\to H_x^1(X;\mathscr{A})\to 0} \tag{40}$$

and the isomorphisms

$$\boxed{H_x^p(X;\mathscr{A})\approx \varinjlim H^{p-1}(W-\{x\};\mathscr{A}),\quad \text{for } p>1} \tag{41}$$

(where W ranges over the neighborhoods of x). [This can, of course, be obtained directly by passing to the limit of the cohomology sequences of the pairs $(W,W-\{x\})$.]

Assume now that $\mathscr{A} = L$ is constant. Then if x is not isolated, we have $H_x^0(X;L)=\Gamma_x(L)=0$. If, furthermore, $U-\{x\}$ is connected for a fundamental system of neighborhoods U of x, then (40) implies that $H_x^1(X;L)=0$ also.

If X is an n-manifold with boundary B, then it can be seen that

$$H_x^p(X;L)\approx\begin{cases}0, & \text{for } p\neq n \quad\text{or}\quad x\in B,\\ L, & \text{for } p=n \quad\text{and}\quad x\notin B.\end{cases}$$

[48] For an n-manifold and constant coefficients it is the cohomology of the $(n-1)$-sphere.

Also, if X is a compact n-manifold with boundary B, then $I_c^p(X - B; L) \approx H^p(B; L)$, whence the term "ideal boundary." To see this, note that $I_B^p(X; L) = I_c^p(X - B; L)$ and apply (38).

The local cohomology groups introduced here have some pleasant properties, mainly because the limit in (41) is a direct limit. However, the problem of "comparing" the local groups at different points is quite difficult. See Raymond [68] in this regard. The situation in homology is considerably simpler because of the fact that the local *homology* groups form a *sheaf* (see Chapter V).

19 The transfer homomorphism and the Smith sequences

In this section let X be any topological space and let G be a finite group of transformations of X. Let X/G be the orbit space and $\pi : X \to X/G$ the orbit map [that is, X/G is the set of orbits $\{G(x) \mid x \in X\}$ given the quotient topology via the canonical map $\pi : x \mapsto G(x)$]. We shall *assume* that each orbit is relatively Hausdorff in X (e.g., X Hausdorff). Note that the map π is both open and closed.

19.1. Theorem. *Let Ψ be any family of supports on X/G and let $\Phi = \pi^{-1}\Psi$. Let $\mathscr{B}$ be any sheaf on X/G and let $\mathscr{A}$ be the sheaf $\pi^*\mathscr{B}$ on X. Then there is a canonical action of G as a group of automorphisms of $\pi\mathscr{A}$. The induced action of G on $H_\Psi^p(X/G; \pi\mathscr{A})$ coincides with that on $H_\Phi^p(X; \mathscr{A})$ via the isomorphism*

$$\boxed{\pi^\dagger : H_\Psi^p(X/G; \pi\mathscr{A}) \xrightarrow{\approx} H_\Phi^p(X; \mathscr{A})}$$

of 11.1. *Moreover, the canonical homomorphism $\beta : \mathscr{B} \to \pi\mathscr{A} = \pi\pi^*\mathscr{B}$ of* I-4 *is a monomorphism onto the subsheaf $(\pi\mathscr{A})^G$ of G-invariant elements, whence*

$$(\pi\mathscr{A})^G = (\pi\pi^*\mathscr{B})^G \approx \mathscr{B}, \tag{42}$$

so that there is the canonical isomorphism

$$\boxed{\bar{\beta}^* : H_\Psi^p(X/G; \mathscr{B}) \xrightarrow{\approx} H_\Psi^p(X/G; (\pi\mathscr{A})^G)}$$

with

$$\boxed{\beta^* = \iota^*\bar{\beta}^* \quad \text{and} \quad \pi^* = \pi^\dagger\beta^*,}$$

where $\iota : (\pi\mathscr{A})^G \hookrightarrow \pi\mathscr{A}$.

Proof. The equation $\beta^* = \iota^*\bar{\beta}^*$ is by definition. The equation $\pi^* = \pi^\dagger\beta^*$ has already been noted in Section 8.

For $g \in G$, regarded as a map $g : X \to X$, we have $\pi g = \pi$. Thus $g^*\mathscr{A} = g^*\pi^*\mathscr{B} = (\pi g)^*\mathscr{B} = \mathscr{A}$. Therefore we have the g-cohomomorphism

$$g^* : \mathscr{A} \rightsquigarrow g^*\mathscr{A} = \mathscr{A}$$

such that the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ g^*\ } & \mathscr{A} \\ & {\scriptstyle\pi^*}\nwarrow \quad \nearrow{\scriptstyle\pi^*} & \\ & \mathscr{B} & \end{array} \tag{43}$$

of cohomomorphisms commutes. On stalks, (43) is the commutative diagram

$$\begin{array}{ccc} \mathscr{A}_{g(x)} & \xrightarrow{\ g_x^*\ } & \mathscr{A}_x \\ & {\scriptstyle\pi_{g(x)}^*}\nwarrow \quad \nearrow{\scriptstyle\pi_x^*} & \\ & \mathscr{B}_{\pi(x)} & \end{array}$$

of isomorphisms. If $U \subset X/G$ is open, then we obtain the induced commutative diagram

$$\begin{array}{ccc} (\pi\mathscr{A})(U) & \xrightarrow{\ g_U^*\ } & (\pi\mathscr{A})(U) \\ = \downarrow & & \downarrow = \\ \mathscr{A}(\pi^{-1}U) & \xrightarrow{\ g_{\pi^{-1}U}^*\ } & \mathscr{A}(\pi^{-1}U) \\ & {\scriptstyle\pi_U}\nwarrow \quad \nearrow{\scriptstyle\pi_U} & \\ & \mathscr{B}(U) & \end{array}$$

where g_U^* is defined by commutativity.

Thus we have the commutative diagram

$$\begin{array}{ccc} \pi\mathscr{A} & \xrightarrow{\ g^*\ } & \pi\mathscr{A} \\ & {\scriptstyle\beta}\nwarrow \quad \nearrow{\scriptstyle\beta} & \\ & \mathscr{B} & \end{array}$$

of homomorphisms of sheaves on X/G and the commutative diagram

$$\begin{array}{ccc} \pi\mathscr{A} & \xrightarrow{g^*} & \pi\mathscr{A} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathscr{A} & \xrightarrow{g^*} & \mathscr{A} \end{array} \tag{44}$$

of cohomomorphisms. The diagram (44) induces the commutative diagram

$$\begin{array}{ccc} H_\Psi^*(X/G; \pi\mathscr{A}) & \xrightarrow{g^*} & H_\Psi^*(X/G; \pi\mathscr{A}) \\ {\scriptstyle\pi^\dagger}\downarrow{\scriptstyle\approx} & & {\scriptstyle\pi^\dagger}\downarrow{\scriptstyle\approx} \\ H_\Phi^*(X; \mathscr{A}) & \xrightarrow{g^*} & H_\Phi^*(X; \mathscr{A}) \end{array}$$

in which the vertical maps are isomorphisms by 11.1 and the horizontal maps are also isomorphisms since each $g : X \to X$ is a homeomorphism.

For $y \in X/G$, we have

$$(\pi\mathscr{A})_y = \varinjlim_{y \in U} \mathscr{A}(\pi^{-1}U) = \mathscr{A}(\pi^{-1}(y)) \approx \underbrace{\mathscr{B}_y \oplus \cdots \oplus \mathscr{B}_y}_{n \text{ times}},$$

where $n = \#(\pi^{-1}(y))$. The homomorphism $\beta_y : \mathscr{B}_y \to (\pi\mathscr{A})_y$ is the diagonal map, and $g_y^* : (\pi\mathscr{A})_y \to (\pi\mathscr{A})_y$ permutes the factors $\mathscr{B}_y$ as $g^{-1} : \pi^{-1}(y) \to \pi^{-1}(y)$ permutes the points of $\pi^{-1}(y)$. Thus we see that $\beta : \mathscr{B} \to \pi\mathscr{A}$ is a monomorphism onto $(\pi\mathscr{A})^G$. □

Let $\sigma : \pi\mathscr{A} \to \pi\mathscr{A}$ be the endomorphism of $\pi\mathscr{A}$ defined by $\sigma = \sum_{g \in G} g^*$. Since the image of σ is invariant under G, σ induces a homomorphism $\mu : \pi\mathscr{A} \to (\pi\mathscr{A})^G \approx \mathscr{B}$. Therefore, we have the homomorphisms

$$\mathscr{B} \underset{\mu}{\overset{\beta}{\rightleftarrows}} \pi\mathscr{A} \tag{45}$$

with $\beta\mu = \sigma = \sum_{g \in G} g^*$ and $\mu\beta = \operatorname{ord} G$ (i.e., multiplication by the integer $\operatorname{ord} G$).

Since $\pi^\dagger\beta^* = \pi^*$ we see that together with the isomorphism $\pi^\dagger$, the homomorphisms (45) induce the homomorphisms

$$\boxed{H_\Psi^*(X/G; \mathscr{B}) \underset{\mu^*}{\overset{\pi^*}{\rightleftarrows}} H_{\pi^{-1}\Psi}^*(X; \pi^*\mathscr{B})} \tag{46}$$

with

$$\boxed{\mu^*\pi^*(b) = \operatorname{ord}(G)b \quad \text{and} \quad \pi^*\mu^*(a) = \sum_{g \in G} g^*(a).} \tag{47}$$

The map μ^* is called the *transfer* homomorphism.

Let $H_{\pi^{-1}\Psi}^*(X; \pi^*\mathscr{B})^G$ denote the subgroup of invariant elements. The image of π^* clearly consists of invariant elements. The following result is immediate from (47):

19.2. Theorem. *Let $\mathscr{B}$ be a sheaf of L-modules, where L is a field of characteristic relatively prime to* $\operatorname{ord} G$. *Then*

$$\pi^* : H_\Psi^*(X/G; \mathscr{B}) \to H_{\pi^{-1}\Psi}^*(X; \pi^*\mathscr{B})^G$$

is an isomorphism. □

19.3. Theorem. *If X is Hausdorff and n-clc$_L$, where L is a field of characteristic relatively prime to* $\operatorname{ord} G$, *then X/G is also n-clc$_L$.*[49]

[49] Also see 19.14.

Proof. The case $n = 0$ is clear, so assume that $n > 0$. Let $x \in X$ and let $\breve{x} \in X/G$ be its image. Let $U \subset X/G$ be a given open neighborhood of $\breve{x}$. Let $S \subset G$ be a set of representatives of the left cosets of G_x in G, so that $\{g(x) \mid g \in S\}$ is an enumeration of the orbit $G(x)$. For any $g \in S$ there is an open neighborhood $W_g \subset \pi^{-1}(U)$ of $g(x)$ so small that $H^n(W_g; L) \to H^n(\pi^{-1}(U); L)$ is zero by the definition of n-clc_L. Since X is Hausdorff, we can assume that $W_g \cap W_h = \varnothing$ if $g \neq h$. Put $V = \bigcap \pi(W_g)$ and $V_g = \pi^{-1}(V) \cap W_g$. (Recall that π is an open map.) Then $V_g \subset W_g$, and $\pi^{-1}(V)$ is the disjoint union of the V_g. Therefore the restriction $r^* : H^n(\pi^{-1}(U); L) \to H^n(\pi^{-1}(V); L)$ is zero. From the diagram

$$\begin{array}{ccc} H^n(U;L) & \xrightarrow{s^*} & H^n(V);L) \\ {\scriptstyle \pi_U^*}\big\downarrow\big\uparrow{\scriptstyle \mu_U^*} & & {\scriptstyle \pi_V^*}\big\downarrow\big\uparrow{\scriptstyle \mu_V^*} \\ H^n(\pi^{-1}(U);L) & \xrightarrow{r^*} & H^n(\pi^{-1}(V);L) \end{array}$$

we have $\operatorname{ord}(G)s^* = \mu_V^* \pi_V^* s^* = \mu_V^* r^* \pi_U^* = 0$, whence $s^* = 0$ since the characteristic of L is relatively prime to $\operatorname{ord}(G)$. □

We wish to generalize the transfer to the case of the map $X/H \to X/G$, where H is a subgroup of G. First we need a lemma:

19.4. Lemma. *Let $\alpha : \mathscr{A} \to \mathscr{A}$ be an endomorphism of a sheaf $\mathscr{A}$ on any space X and put $\mathscr{A}^\alpha = \{s \in \mathscr{A} \mid \alpha(s) = s\}$. If $\pi : X \to Y$, then α induces an endomorphism $\pi(\alpha) : \pi\mathscr{A} \to \pi\mathscr{A}$ and there is a canonical isomorphism*

$$\pi(\mathscr{A}^\alpha) \approx (\pi\mathscr{A})^{\pi(\alpha)}.$$

Proof. The composition $(\pi\mathscr{A})(U) = \mathscr{A}(\pi^{-1}U) \xrightarrow{\alpha} \mathscr{A}(\pi^{-1}U) = (\pi\mathscr{A})(U)$ gives the endomorphism $\pi(\alpha)$. Also, $\mathscr{A}^\alpha = \operatorname{Ker}(1-\alpha)$, so that the sequence

$$0 \to \mathscr{A}^\alpha \to \mathscr{A} \xrightarrow{1-\alpha} \mathscr{A}$$

is exact. Applying the left exact functor π to this gives the exact sequence

$$0 \to \pi(\mathscr{A}^\alpha) \to \pi\mathscr{A} \xrightarrow{1-\pi(\alpha)} \pi\mathscr{A},$$

which shows that $\pi(\mathscr{A}^\alpha) \approx (\pi\mathscr{A})^{\pi(\alpha)}$. □

Returning to the main discussion, let $\pi = \pi_G$ factor as

$$X \xrightarrow{\pi_H} X/H \xrightarrow{\pi_{G/H}} X/G.$$

Then the lemma and (42) provide the isomorphism

$$\begin{aligned} (\pi\mathscr{A})^H &= (\pi_{G/H}\pi_H\pi_H^*\pi_{G/H}^*\mathscr{B})^H \\ &\approx \pi_{G/H}((\pi_H\pi_H^*\pi_{G/H}^*\mathscr{B})^H \\ &\approx \pi_{G/H}(\pi_{G/H}^*\mathscr{B}), \end{aligned} \tag{48}$$

whence

$$H^p_\Psi(X/G;(\pi\mathscr{A})^H) \approx H^p_\Psi(X/G;\pi_{G/H}(\pi^*_{G/H}\mathscr{B})) \approx H^p_\Psi(X/H;\pi^*_{G/H}\mathscr{B})$$

by 11.1. There is the homomorphism $\mu_{G/H} = \sum_i g_i^* : (\pi\mathscr{A})^H \to (\pi\mathscr{A})^G$, where $\{g_i\} \subset G$ is a set of representatives of the right cosets of H in G.[50] This induces a map $H^p_\Psi(X/G;(\pi\mathscr{A})^H) \to H^p_\Psi(X/G;(\pi\mathscr{A})^G)$, giving the transfer homomorphism

$$\boxed{\mu^*_{G/H} : H^p_{\pi^{-1}_{G/H}\Psi}(X/H;\pi^*_{G/H}\mathscr{B}) \to H^p_\Psi(X/G;\mathscr{B})}$$

such that

$$\boxed{\mu^*_{G/H}\pi^*_{G/H} = \mathrm{ord}(G/H).}$$

Since $\sum_{g\in G} g^* = \sum_i \sum_{h\in H}(hg_i)^* = \sum_i g_i^* \sum_{h\in H} h^*$, we have that $\mu_G = \mu_{G/H}\mu_H$, giving the relationship

$$\boxed{\mu^*_G = \mu^*_{G/H}\mu^*_H.}$$

19.5. We shall now restrict our attention to the case in which G is cyclic of prime order p. Let L be a field of characteristic p and let $\mathscr{B}$ be a sheaf of L-modules on X/G. Let $F \subset X$ be the fixed-point set of G on X and note that F is closed. We shall also consider F to be a subset of X/G. Let $g \in G$ be a generator, chosen once and for all. Let τ and σ denote the elements

$$\tau = 1 - g,$$
$$\sigma = 1 + g + g^2 + \cdots + g^{p-1}$$

of the group ring $L(G)$, and note that $\sigma = \tau^{p-1}$ since $\mathrm{char}(L) = p$.

Since $\mathscr{B}$, and hence $\mathscr{A} = \pi^*\mathscr{B}$ and $\pi\mathscr{A}$, are sheaves of L-modules, $L(G)$ operates on $\pi\mathscr{A}$.

If ρ denotes either σ or τ, let $\bar{\rho}$ denote the other. Let $\rho(\pi\mathscr{A})$ denote the image of $\pi\mathscr{A}$ under

$$\rho : \pi\mathscr{A} \to \pi\mathscr{A}.$$

Consider the sequence

$$\boxed{0 \to \bar{\rho}(\pi\mathscr{A}) \xrightarrow{i} \pi\mathscr{A} \xrightarrow{\rho\oplus j} \rho(\pi\mathscr{A}) \oplus \mathscr{B}_F \to 0,} \qquad (49)$$

where i is the inclusion and j is the canonical map $\pi\mathscr{A} \to (\pi\mathscr{A})_F \approx \mathscr{B}_F$.

We claim that the sequence (49) is exact. On the stalk at $y \in F$ this is clear since $\rho(\pi\mathscr{A})_y = 0$ for both $\rho = \sigma, \tau$. For $y \notin F$, $(\pi\mathscr{A})_y \approx \mathscr{B}_y \otimes L(G)$, where the operation of $L(G)$ is via the regular representation on the factor $L(G)$. Thus it suffices to show that

$$0 \to \bar{\rho}L(G) \to L(G) \to \rho L(G) \to 0$$

[50]It is immediate that $\mu_{G/H}$ is independent of the choice of the representatives g_i, whence its image is G-invariant.

is exact. Since it has order two (because $\sigma\tau = \tau\sigma = 1 - g^p = 0$) it suffices to show that $\dim L(G) = \dim \rho L(G) + \dim \bar\rho L(G)$ as vector spaces over L. Now $\operatorname{Ker}\{\tau : L(G) \to L(G)\}$ consists of the invariant elements and hence has dimension one. Consider the homomorphisms

$$L(G) \xrightarrow{\tau} \tau L(G) \xrightarrow{\tau} \tau^2 L(G) \xrightarrow{\tau} \cdots \xrightarrow{\tau} \tau^p L(G) = 0.$$

The kernel of each of these homomorphisms has dimension at most one, but the composition τ^p has kernel $L(G)$ of dimension p. It follows that $\dim \tau^i L(G) = p - i$, whence $\dim \sigma L(G) + \dim \tau L(G) = 1 + (p-1) = p = \dim L(G)$ as claimed.

This discussion also shows that $\tau^i(\pi\mathscr{A})/\tau^{i+1}(\pi\mathscr{A}) \approx \operatorname{Ker}\{\tau : \pi\mathscr{A} \to \pi\mathscr{A}\}$ on $(X-F)/G$ and is zero on F, for $i = 1, 2, ..., p-1$. That is,

$$\boxed{\tau^i(\pi\mathscr{A})/\tau^{i+1}(\pi\mathscr{A}) \approx \sigma(\pi\mathscr{A}) \approx \mathscr{B}_{(X-F)/G} \quad \text{for } 1 \le i < p.} \tag{50}$$

We define

$$\boxed{{}_\rho H^s_\Phi(X;\mathscr{A}) = H^s_\Psi(X/G; \rho(\pi\mathscr{A})),}$$

called the "Smith special cohomology group." More generally, put

$${}_{\tau^k} H^s_\Phi(X;\mathscr{A}) = H^s_\Psi(X/G; \tau^k(\pi\mathscr{A})).$$

Note that by (50) and 12.3, we have

$$\boxed{{}_\sigma H^s_\Phi(X;\mathscr{A}) \approx H^s_\Psi(X/G, F; \mathscr{B}).} \tag{51}$$

By (49), 10.2, and 11.1, we have the exact *Smith sequences*

$$\boxed{\begin{aligned}\cdots \to {}_{\bar\rho}H^s_\Phi(X;\mathscr{A}) \xrightarrow{i^*} H^s_\Phi(X;\mathscr{A}) \xrightarrow{\rho^*\oplus j^*} {}_\rho H^s_\Phi(X;\mathscr{A}) \oplus H^s_{\Phi|F}(F;\mathscr{A})\\ \xrightarrow{\delta^*} {}_{\bar\rho}H^{s+1}_\Phi(X;\mathscr{A}) \to \cdots\end{aligned}} \tag{52}$$

(where $\mathscr{A} = \pi^*\mathscr{B}$ and $\Phi = \pi^{-1}\Psi$). It is easy to check that j^* is the usual restriction map. Similarly, using (50) we derive the exact sequences

$$\begin{aligned}\cdots \to {}_{\tau^{k+1}}H^s_\Phi(X;\mathscr{A}) \xrightarrow{i^*} {}_{\tau^k}H^s_\Phi(X;\mathscr{A}) \xrightarrow{j^*} {}_\sigma H^s_\Phi(X;\mathscr{A})\\ \xrightarrow{\delta^*} {}_{\tau^{k+1}}H^{s+1}_\Phi(X;\mathscr{A}) \to \cdots\end{aligned} \tag{53}$$

Note that there is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{B}_{(X-F)/G} & \longrightarrow & \mathscr{B} & \longrightarrow & \mathscr{B}_F & \longrightarrow & 0 \\ & & \downarrow{\approx} & & \downarrow{\beta} & & \downarrow{0\oplus 1} & & \\ 0 & \longrightarrow & \sigma(\pi\mathscr{A}) & \xrightarrow{i} & \pi\mathscr{A} & \xrightarrow{\tau\oplus j} & \tau(\pi\mathscr{A}) \oplus \mathscr{B}_F & \longrightarrow & 0, \end{array}$$

which induces the diagram

$$\begin{array}{ccccccc} \cdots \to H^i_\Psi(X/G,F;\mathscr{B}) & \to & H^i_\Psi(X/G;\mathscr{B}) & \longrightarrow & H^i_{\Psi|F}(F;\mathscr{B}) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\pi^*} & & \downarrow{\scriptstyle 0\oplus 1} & & \\ \cdots \to {}_\sigma H^i_\Phi(X;\mathscr{A}) & \to & H^i_\Phi(X;\mathscr{A}) & \to & {}_\tau H^i_\Phi(X;\mathscr{A}) \oplus H^i_\Phi(F;\mathscr{A}) & \to & \cdots \end{array} \tag{54}$$

Similarly, using the fact that $\sigma = \tau\tau^{p-2}$, we have the following commutative diagram for $p > 2$,

$$\begin{array}{ccccccccc} 0 & \to & \sigma(\pi_*\mathscr{A}) & \to & \pi_*\mathscr{A} & \to & \tau(\pi_*\mathscr{A}) \oplus \mathscr{B}_F & \to & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tau^{p-2}\oplus 1} & & \\ 0 & \to & \tau(\pi_*\mathscr{A}) & \to & \pi_*\mathscr{A} & \to & \sigma(\pi_*\mathscr{A}) \oplus \mathscr{B}_F & \to & 0 \\ & & \downarrow{\scriptstyle \tau^{p-2}} & & \downarrow{\scriptstyle \tau^{p-2}} & & \downarrow{\scriptstyle 1\oplus 0} & & \\ 0 & \to & \sigma(\pi_*\mathscr{A}) & \to & \pi_*\mathscr{A} & \to & \tau(\pi_*\mathscr{A}) \oplus \mathscr{B}_F & \to & 0, \end{array} \tag{55}$$

which yields a relationship between the sequences (52) for $\rho = \sigma, \tau$.

Here are some applications of our general considerations:

19.6. Theorem. *Let X be a Hausdorff space that has a connected double covering space $\tilde{X}$. Then $H^1(X;\mathbb{Z}_2) \neq 0$.*

Proof. Since $\tau = \sigma$ for $p = 2$, the exact sequence (52) has the segment

$$0 \to H^0(X;\mathbb{Z}_2) \xrightarrow{\approx} H^0(\tilde{X};\mathbb{Z}_2) \to H^0(X;\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$$

by (51), and the result follows. □

19.7. Theorem. (P. A. Smith and E. E. Floyd.) *Let X be a Hausdorff space with an action of $G = \mathbb{Z}_p$ with fixed point set F. Let $L = \mathbb{Z}_p$ and let $\mathscr{B}$ be a sheaf of L-modules on X/G, and put $\mathscr{A} = \pi^*\mathscr{B}$. Let Ψ be a paracompactifying family of supports on X/G and put $\Phi = \pi^{-1}\Psi$, which is also paracompactifying. Assume that* $\dim_{\Phi,L} X = n < \infty$. *Then* $\dim_{\Phi,L} F \leq n$ *and* $\dim_{\Psi,L} X/G \leq n$. *Also, for each r,*

$$\boxed{\dim_L H^r_\Psi(X/G,F;\mathscr{B}) + \sum_{i=r}^{\infty} \dim_L H^i_{\Phi|F}(F;\mathscr{A}) \leq \sum_{i=r}^{\infty} \dim_L H^i_\Phi(X;\mathscr{A}).}$$

Moreover, if the Euler characteristic $\chi(X) = \sum(-1)^i \dim_L H^i_\Phi(X;\mathscr{A})$ of X is defined, then so are those of F and of X/G, and

$$\boxed{\chi(X) = \chi(F) + p\chi(X/G,F).}$$

Proof. The statements on dimension of F and X/G follow immediately from Exercise 11 and the local nature of dimension from 16.8. Let us use the shorthand notation $H^i(X) = H^i_\Phi(X;\mathscr{A})$ and $H^i(\tau^k) = {}_{\tau^k}H^i_\Phi(X;\mathscr{A})$, etc. Then, from (52) and (53) we derive the inequalities (with $\rho = \tau$ or $\rho = \sigma$)

$$\begin{aligned}
\dim H^r(F) + \dim H^r(\rho) &\le \dim H^r(X) + \dim H^{r+1}(\bar\rho),\\
\dim H^{r+1}(F) + \dim H^{r+1}(\bar\rho) &\le \dim H^{r+1}(X) + \dim H^{r+2}(\rho),\\
&\cdots
\end{aligned}$$

These eventually become totally zero by the dimension assumption, and a downwards induction shows that all terms are finite if those for X are finite. Adding these inequalities gives

$$\dim H^r(\rho) + \sum_{i=r}^{\infty} \dim H^i(F) \le \sum_{i=r}^{\infty} \dim H^i(X),$$

which gives the inequality of the theorem upon application of (51). Also, the exact sequences (52) and (53) give the following equations about Euler characteristics (with the obvious notation):

$$\begin{aligned}
\chi(X) &= \chi(F) + \chi(\tau) + \chi(\sigma),\\
\chi(\tau) &= \chi(\tau^2) + \chi(\sigma),\\
\chi(\tau^2) &= \chi(\tau^3) + \chi(\sigma),\\
&\cdots\\
\chi(\tau^{p-2}) &= \chi(\tau^{p-1}) + \chi(\sigma).
\end{aligned}$$

Since $\tau^{p-1} = \sigma$ in the present situation, adding these and cancelling gives the desired result. □

19.8. Corollary. *Under the hypotheses of* 19.7, *suppose that X is acyclic. Then so are F and X/G. In particular, $F \neq \varnothing$.* □

19.9. Corollary. *Under the hypotheses of* 19.7, *suppose that $H^*_\Phi(X;\mathscr{A}) \approx H^*(\mathbb{S}^m;\mathbb{Z}_p)$. Then $H^*_{\Phi|F}(F;\mathscr{A}|F) \approx H^*(\mathbb{S}^r;\mathbb{Z}_p)$ for some $-1 \le r \le m$. Moreover, $m - r$ is even if p is odd.* □

19.10. Corollary. *Let X be a paracompact space with* $\dim_{\mathbb{Z}_p} X < \infty$ *and with $H^*(X;\mathbb{Z}_p) \approx H^*(\mathbb{S}^n;\mathbb{Z}_p)$. Suppose we are given a free action of $G = \mathbb{Z}_p$ on X. If $p = 2$ then*

$$H^*(X/G;\mathbb{Z}_2) \approx \mathbb{Z}_2[u]/(u^{n+1}),$$

where $\deg u = 1$. *If p is an odd prime, then n must be odd and*

$$H^*(X/G;\mathbb{Z}_p) \approx \Lambda(u) \otimes \mathbb{Z}_p[v]/(v^{(n+1)/2}),$$

where $\deg u = 1$ *and* $\deg v = 2$.

Proof. Coefficients will be in $\mathbb{Z}_p$ unless otherwise indicated. Since $\mathbb{Z}_p$ has no automorphisms of period p, $\rho^* : H^*(X) \to H^*(X)$ is zero for both $\rho^* = \tau^* = 1 - g^*$ and $\rho^* = \sigma^* = (\tau^*)^{p-1}$. The exact sequences

$$0 \to {}_{\bar\rho}H^0(X) \xrightarrow{i^*} H^0(X) \xrightarrow{\rho^*} {}_\rho H^0(X)$$

then show that ${}_\rho H^0(X) \approx \mathbb{Z}_p$ and that $\rho^* : H^0(X) \to {}_\rho H^0(X)$ is zero, because the composition $H^0(X) \xrightarrow{\rho^*} {}_\rho H^0(X) \xrightarrow{i^*} H^0(X)$ is zero.

By 19.7, $H^k(X/G) = 0$ for $k > n$. Similar considerations then show that $\rho^* : H^n(X) \to {}_\rho H^n(X)$ is an isomorphism and that the maps

$${}_\rho H^0(X) \xrightarrow{\delta^*} {}_{\bar\rho}H^1(X) \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} {}_\eta H^n(X)$$

are all isomorphisms, where $\eta = \rho$ or $\eta = \bar\rho$, depending on the parity of n.

Let $L = \mathbb{Z}_p$ as a sheaf on X. Consider the case $p = 2$, so that $\rho = \sigma = \tau = \bar\rho$. We have the exact coefficient sequence $0 \to \sigma(\pi L) \to \pi L \to \sigma(\pi L) \to 0$ of (49). If $1 \in H^0(X/G) = H^0(X/G; \sigma(\pi L))$ and $a \in H^k(X/G; L)$, then for the connecting homomorphism $\delta^* : H^k(X/G; \sigma(\pi L)) \to H^{k+1}(X/G; \sigma(\pi L))$, we have $\delta^*(1 \cup a) = \delta^*(1) \cup a = u \cup a$ by 7.1(b), where $u = \delta^*(1) \in H^1(X/G; \sigma(\pi L)) = H^1(X/G)$. Since

$$\delta^* : H^k(X/G) \approx {}_\sigma H^k(X) \to {}_\sigma H^{k+1}(X) \approx H^{k+1}(X/G)$$

is an isomorphism for $0 \le k < n$, the result follows.

Now let p be an odd prime. Then n is odd by 19.9. The proof proceeds as in the case $p = 2$ except that σ and τ now alternate. Let $\delta_1^* : {}_\sigma H^k(X) \to {}_\tau H^{k+1}(X)$ and $\delta_2^* : {}_\tau H^k(X) \to {}_\sigma H^{k+1}(X)$ be the connecting homomorphisms for the exact coefficient sequences $0 \to \tau(\pi L) \to \pi L \to \sigma(\pi L) \to 0$ and $0 \to \sigma(\pi L) \to \pi L \to \tau(\pi L) \to 0$ on X/G, respectively, which we know to be isomorphisms for $0 \le k < n$. Then for $1 \in H^0(X/G) = {}_\sigma H^k(X)$ we have the elements

$$w = \delta_1^*(1) \in {}_\tau H^1(X) \text{ and } v = \delta_2^*(w) \in {}_\sigma H^2(X) = H^2(X/G).$$

Still, for $a \in H^k(X/G) = {}_\sigma H^k(X)$, we have $\delta_2^*\delta_1^*(1 \cup a) = \delta_2^*(w \cup a) = v \cup a$ by 7.1(b). It follows that v^i generates $H^{2i}(X/G)$, $1 \le 2i < n$. We know that $H^1(X/G) = {}_\sigma H^1(X) \approx \mathbb{Z}_p$, so let $u \in H^1(X/G)$ be any generator. Then the preceding remarks also show that uv^i generates $H^{2i+1}(X/G)$, $1 \le 2i + 1 \le n$. Now $u^2 = 0$ for the usual reason: $u^2 = u \cup u = -u \cup u$. The result follows. □

When $\mathbb{Z}_p$ acts on a space with the integral cohomology of $\mathbb{S}^n$, then the fixed-point set F is a cohomology r-sphere over $\mathbb{Z}_p$ by 19.9, but it may well have other torsion. (The first such example, given by the author [13], was a differentiable action of $\mathbb{Z}_2$ on $\mathbb{S}^5$ with F a lens space.) It is now known, due mainly to work of Jones and Oliver, that this torsion is virtually arbitrary. Thus it may be somewhat surprising that the *integral* cohomology of the *orbit space* is completely determined by n and r, as the following result shows.

19.11. Corollary. [17] *Let X be a paracompact space with* $\dim_{\mathbb{Z}_p} X < \infty$. *Let L be either $\mathbb{Z}$ or $\mathbb{Z}_p$, p prime, and assume that $H^*(X;L) \approx H^*(\mathbb{S}^n;L)$. Suppose that $G = \mathbb{Z}_p$ acts on X with $H^*(F;\mathbb{Z}_p) \approx H^*(\mathbb{S}^r;\mathbb{Z}_p)$. Then in both cases,*

$$\tilde{H}^k(X/G;\mathbb{Z}_p) \approx \begin{cases} \mathbb{Z}_p, & \text{for } r+2 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

If $L = \mathbb{Z}$, then also

$$\tilde{H}^k(X/G;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}_p, & \text{for } r+3 \le k \le n,\ k-r \text{ odd}, \\ \mathbb{Z}, & \text{for } k = n \text{ if } n-r \text{ is even}, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $n - r$ is even $\Leftrightarrow g^* = 1$ on $H^n(X;\mathbb{Z})$.

Proof. By suspending X twice, it can be assumed that $r \ge 1$. The Smith sequence shows that the maps

$$H^r(F;\mathbb{Z}_p) \to {}_\rho H^{r+1}(X;\mathbb{Z}_p) \to {}_{\bar\rho} H^{r+2}(X;\mathbb{Z}_p) \to \cdots \to {}_\eta H^n(X;\mathbb{Z}_p)$$

are all isomorphisms, as is

$$H^n(X;\mathbb{Z}_p) \to {}_\rho H^n(X;\mathbb{Z}_p).$$

Then it follows from (54) that in the exact sequence of the pair $(X/G, F)$,

$$H^r(F;\mathbb{Z}_p) \to H^{r+1}(X/G,F;\mathbb{Z}_p) \text{ and } H^{n-1}(F;\mathbb{Z}_p) \to H^n(X/G,F;\mathbb{Z}_p)$$

are isomorphisms. The cohomology with $\mathbb{Z}_p$ coefficients then follows.

For integer coefficients, the composition

$$H^k(X/G;\mathbb{Z}) \xrightarrow{\pi^*} H^k(X;\mathbb{Z}) \xrightarrow{\sigma^*} H^k(X/G;\mathbb{Z}) \tag{56}$$

is multiplication by p by (47). Therefore, $\tilde{H}^k(X/G;\mathbb{Z})$ is all p-torsion for $k \ne n$. The exact coefficient sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ shows that there is the exact sequence

$$0 \to \tilde{H}^k(X/G;\mathbb{Z}) \to \tilde{H}^k(X/G;\mathbb{Z}_p) \to H^{k+1}(X/G;\mathbb{Z}) \to 0$$

for $0 \le k < n-1$ and for $k > n$, and this sequence is left exact for $k = n-1$. An induction then shows that $\tilde{H}^k(X/G;\mathbb{Z})$ is as stated for $k \le n-1$ and for $k > n$.

Now suppose that $n - r$ is even. Then $H^{n-1}(X/G;\mathbb{Z}) \approx \mathbb{Z}_p$ maps monomorphically into $H^{n-1}(X/G;\mathbb{Z}_p) \approx \mathbb{Z}_p$, and so there is the exact sequence

$$0 \to H^n(X/G;\mathbb{Z}) \xrightarrow{p} H^n(X/G;\mathbb{Z}) \to H^n(X/G;\mathbb{Z}_p) \to 0,$$

where the marked map is the composition $\sigma^*\pi^*$ of (56). This implies that $\pi^* : H^n(X/G;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ is monomorphic, whence $H^n(X/G;\mathbb{Z}) \approx \mathbb{Z}$. Moreover, $\sigma^* \neq 0$, and so $1 + g^* = \pi^*\sigma^* \neq 0$, whence $g^* = 1$.

Now suppose that $n - r$ is odd. Then $\tilde{H}^{n-1}(X/G;\mathbb{Z}) = 0$, and so there is the exact sequence

$$0 \to H^{n-1}(X/G;\mathbb{Z}_p) \xrightarrow{\delta} H^n(X/G;\mathbb{Z}) \xrightarrow{p} H^n(X/G;\mathbb{Z}) \to H^n(X/G;\mathbb{Z}_p) \to 0,$$

in which the middle map is the composition (56). It follows that $K = \operatorname{Coker}\delta$ is a subgroup of $\mathbb{Z}$, whence $K = 0$, or $K \approx \mathbb{Z}$ and $H^n(X/G;\mathbb{Z}) \approx \mathbb{Z}_p \oplus K$. If $K \approx \mathbb{Z}$ then the sequence implies that $H^n(X/G;\mathbb{Z}_p) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$, and so $K = 0$. Thus $H^n(X/G;\mathbb{Z}) \approx \mathbb{Z}_p$. The composition

$$\mathbb{Z} \approx H^n(X;\mathbb{Z}) \xrightarrow{\sigma^*} H^n(X/G;\mathbb{Z}) \xrightarrow{\pi^*} H^n(X;\mathbb{Z}) \approx \mathbb{Z}$$

is $1 + g^*$. Since the middle group is all torsion, this must be zero and so $g^* = -1$ and this can only happen when $p = 2$. (The reader should verify the somewhat exceptional case $p = 2$ and $r = n - 1$.) □

19.12. Corollary. *Suppose that X is paracompact with $\dim_{\mathbb{Z}_p} X < \infty$ (p prime) and $H^*(X;\mathbb{Z}) \approx H^*(\mathbb{S}^n;\mathbb{Z})$. Suppose that $G = \mathbb{Z}_p$ acts on X with $H^*(F;\mathbb{Z}_p) \approx H^*(\mathbb{S}^r;\mathbb{Z}_p)$. If $r = n - 2$ then $H^*(X/G;\mathbb{Z}) \approx H^*(\mathbb{S}^n;\mathbb{Z})$. If $r = n - 1$, then $\tilde{H}^*(X/G;\mathbb{Z}) = 0$.* □

19.13. Theorem. (E. E. Floyd.) *Let X be a paracompact space with $\dim_{\mathbb{Z}} X < \infty$. Suppose that G is a finite group acting on X. If X is $\mathbb{Z}$-acyclic then so is X/G.*

Proof. If $H \subset G$ is cyclic of prime order p, then $H^k(X/H;\mathbb{Z}_p) = 0$ for $k > 0$ by 19.8. Since a p-group has nonzero center, an induction proves the same thing for H being the p-Sylow subgroup of G. Now, $\mu^*_{G/H}\pi^*_{G/H} = \operatorname{ord}(G) : H^k(X/G;\mathbb{Z}_p) \to H^k(X;\mathbb{Z}_p) \to H^k(X/G;\mathbb{Z}_p)$ is an isomorphism factoring through zero, so that $H^k(X/G;\mathbb{Z}_p) = 0$ for $k > 0$ and all primes p. An induction, using the cohomology sequences of the coefficient sequences

$$0 \to \mathbb{Z}_i \to \mathbb{Z}_{ij} \to \mathbb{Z}_j \to 0,$$

shows that $H^k(X/G;\mathbb{Z}_m) = 0$ for all m. Taking $m = \operatorname{ord} G$, we have that $\mu^*_G\pi^*_G = m$, whence $H^k(X/G;\mathbb{Z}) \xrightarrow{m} H^k(X/G;\mathbb{Z})$ is zero. The cohomology sequence of the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$ then shows that $H^k(X/G;\mathbb{Z}) = 0$ for $k > 0$. □

Remark: One can use Exercise 53 to replace the finite-dimensionality hypothesis in 19.7 and its corollaries, as well as 19.13, by an assumption that X is compact Hausdorff. This is done by showing, as in the proof of 19.10, that the connecting homomorphisms in the Smith sequences are cup products with fixed elements of $H^1(X/G;\tau^k(\pi\mathbb{Z}_p))$, $0 < k < p$.

19.14. Theorem. (E. E. Floyd.) *Let X be a locally paracompact Hausdorff space with $\dim_{\mathbb{Z}} X < \infty$. Suppose that G is a finite group acting on X. Let L be $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p. If X is clc^∞_L, then so is X/G.*

Proof. Let K be a closed paracompact neighborhood of $y \in X/G$. Since X is clc_L, there is a closed neighborhood $M \subset K$ of y such that the map $H^k(K^\bullet; L) \to H^k(M^\bullet; L)$ is zero for $k > 0$, where $K^\bullet$ denotes $\pi^{-1}(K)$. Assume for the moment that G is cyclic of prime order p. Then by a downwards induction using 17.3 and the Smith sequences of the form (coefficients in $\mathbb{Z}_p$)

$$\tilde{H}^k(K^\bullet) \to {}_\rho H^k(K^\bullet) \oplus \tilde{H}^k(F \cap K^\bullet) \to {}_{\bar\rho}H^{k+1}(K^\bullet),$$

M can be chosen to be so small that ${}_\rho H^k(K^\bullet) \to {}_\rho H^k(M^\bullet)$ and $\tilde{H}^k(F \cap K^\bullet) \to \tilde{H}^k(F \cap M^\bullet)$ are both zero for all k and for both $\rho = \tau$ and $\rho = \sigma$. From the exact sequence of the pair $(X/G, F)$, we see that M can be taken so small that $\tilde{H}^k(K; \mathbb{Z}_p) \to \tilde{H}^k(M; \mathbb{Z}_p)$ is zero, i.e., that X/G is $clc^\infty_{\mathbb{Z}_p}$. By an induction, as in the proof of 19.13, this also follows for G being any p-group. For general G and for H being the p-Sylow subgroup of G, $\mu^*_{G/H}\pi^*_{G/H} = \operatorname{ord} G/H$ is an isomorphism for $\mathbb{Z}_p$ coefficients, so that X/H being $clc^\infty_{\mathbb{Z}_p}$ implies that X/G is $clc^\infty_{\mathbb{Z}_p}$, proving the case $L = \mathbb{Z}_p$ for any p. For any given integer m, an induction shows that M can be taken to be so small that $r^*_{M,K} : \tilde{H}^k(K; \mathbb{Z}_m) \to \tilde{H}^k(M; \mathbb{Z}_m)$ is zero for all k. For a closed neighborhood $N \subset M$ of y and with $m = \operatorname{ord} G$, we have that $m r^*_{N,M} = \mu^*_G \pi^*_G r^*_{N,M} = \mu^*_G r^*_{N^\bullet,M^\bullet} \pi^*_G = 0$ for N sufficiently small. Then 17.3, applied to the diagram

$$\begin{array}{ccccc}
 & & \tilde{H}^k(K;\mathbb{Z}) & \longrightarrow & \tilde{H}^k(K;\mathbb{Z}_m) \\
 & & \downarrow & & \downarrow{\scriptstyle 0} \\
\tilde{H}^k(M;\mathbb{Z}) & \xrightarrow{m} & \tilde{H}^k(M;\mathbb{Z}) & \longrightarrow & \tilde{H}^k(M;\mathbb{Z}_m) \\
\downarrow{\scriptstyle r^*_{N,M}} & & \downarrow & & \\
\tilde{H}^k(N;\mathbb{Z}) & \xrightarrow{m} & \tilde{H}^k(N;\mathbb{Z}), & &
\end{array}$$

gives the result for $L = \mathbb{Z}$. □

The subjects of the transfer map and of Smith theory will be taken up again in V-19 and V-20.

20 Steenrod's cyclic reduced powers

In this section and the next we shall construct the Steenrod cohomology operations (the reduced squares and pth powers) in the context of sheaf theory and shall derive several of their properties. These two sections are not used elsewhere in this book and may be skipped. Several details of a straightforward computational nature are omitted.

Throughout these two sections p will denote a prime number, although much of what we do goes through for general integers p. The case $p = 2$ differs in several details from the case of odd p, but it is basically much simpler. To avoid undue repetition, we shall concentrate on the case of odd p and merely note modifications that are necessary for the case $p = 2$.

20.1. If $\mathscr{A}$ is a sheaf on X, we let

$$T_p(\mathscr{A}) = \overbrace{\mathscr{A} \otimes \cdots \otimes \mathscr{A}}^{p \text{ times}}.$$

The symmetric group S_p on p elements acts on $T_p(\mathscr{A})$ as a group of automorphisms in the obvious way. Let $\mathscr{A}$ and $\mathscr{B}$ be given sheaves on X and assume that we are given a homomorphism

$$h : T_p(\mathscr{A}) \to \mathscr{B}$$

that is *symmetric* (i.e., such that $h\gamma = h$ for any $\gamma \in S_p$). The skew-symmetric case can also be treated, and most of what we shall do applies to both, but it is not as important as the symmetric case and will be omitted for the sake of brevity.

For a differential sheaf $\mathscr{A}^*$, $T_p(\mathscr{A}^*)$ is also a differential sheaf with the usual total degree and differential. In this case, we assume that the action of S_p includes the usual sign conventions (i.e., so that a transposition of adjacent terms of degrees r and s is given the sign $(-1)^{rs}$). Then the action of S_p commutes with the differential.

Let Φ be a family of supports on X and let $\mathscr{A}^*$ be any Φ-acyclic resolution of $\mathscr{A}$ for which $T_p(\mathscr{A}^*)$ is a resolution of $T_p(\mathscr{A})$. For example, the canonical resolution $\mathscr{C}^*(X, \mathscr{A})$, or any pointwise homotopically trivial resolution, may be used. Let $\mathscr{B}^*$ be any *injective* resolution of $\mathscr{B}$.

Let $\alpha \in S_p$ denote any cyclic permutation of the factors of $T_p(\mathscr{A}^*)$ and of $T_p(\mathscr{A})$. For notational convenience, and for nonambiguity of later definitions, we shall take α to be that permutation taking the ith factor to the $(i-1)$st place. We also introduce the notation

$$\begin{aligned} \tau &= 1 - \alpha, \\ \sigma &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}. \end{aligned}$$

These are endomorphisms of $T_p(\mathscr{A}^*)$ commuting with differentials. Note that

$$\tau\sigma = 0 = \sigma\tau.$$

20.2. Let $h_0 : T_p(\mathscr{A}^*) \to \mathscr{B}^*$ be a homomorphism of resolutions extending h. Recall that h_0 induces the cup product

$$H_\Phi^{n_1}(X; \mathscr{A}) \otimes \cdots \otimes H_\Phi^{n_p}(X; \mathscr{A}) \to H_\Phi^n(X; T_p(\mathscr{A})) \xrightarrow{h^*} H_\Phi^n(X; \mathscr{B}),$$

where $n = \sum n_i$. (We have not actually shown this, but it will follow from later developments in this section.)

The composition $h_0\tau$ extends $h\tau = h - h\alpha = 0 : T_p(\mathscr{A}) \to \mathscr{B}$, and it follows that $h_0\tau$ is homotopically trivial, since $\mathscr{B}^*$ is injective. Thus there exists a homomorphism $h_1 : T_p(\mathscr{A}^*) \to \mathscr{B}^*$, of degree -1, such that

$$h_0\tau = h_1 d + d h_1. \tag{57}$$

Applying σ to the right of (57), we see that $h_1\sigma$ anticommutes with d. Preceding $h_1\sigma$ by the sign $(-1)^{\deg}$, one obtains a chain map $T_p(\mathscr{A}^*) \to \mathscr{B}^*$ (of degree -1), and since this must extend $T_p(\mathscr{A}) \to 0$, it must be homotopically trivial. In terms of $h_1\sigma$ itself, this means that there is a homomorphism $h_2 : T_p(\mathscr{A}^*) \to \mathscr{B}^*$ of degree -2 such that

$$h_1\sigma = h_2 d - dh_2. \tag{58}$$

Similarly, applying τ to the right of (58) shows that $h_2\tau$ is a chain map and must be homotopically trivial. Continuing inductively one obtains homomorphisms $h_i : T_p(\mathscr{A}^*) \to \mathscr{B}^*$ of degree $-i$ such that

$$\begin{cases} h_{2n}\tau &= h_{2n+1}d + dh_{2n+1}, \\ h_{2n-1}\sigma &= h_{2n}d - dh_{2n}. \end{cases} \tag{59}$$

20.3. In this subsection we shall assume that $p\mathscr{B} = 0$ (i.e., that $\mathscr{B}$ is a sheaf of $\mathbb{Z}_p$-modules). Then $h\sigma = ph = 0$, so that taking $k_0 = h_0$, we can modify the above constructions to find homomorphisms $k_i : T_p(\mathscr{A}^*) \to \mathscr{B}^*$ such that

$$\begin{cases} k_{2n}\sigma &= k_{2n+1}d + dk_{2n+1}, \\ k_{2n-1}\tau &= k_{2n}d - dk_{2n}. \end{cases} \tag{60}$$

In the present situation, of course, we can factor $h : T_p(\mathscr{A}) \to \mathscr{B}$ through $T_p(\mathscr{A}/p\mathscr{A})$. Thus it is actually of no loss of generality to assume that $\mathscr{A}$ and $\mathscr{B}$ are both sheaves of $\mathbb{Z}_p$-modules. With this assumption, $\mathscr{A}^*$ and $\mathscr{B}^*$ can be taken to be sheaves of $\mathbb{Z}_p$-modules, with $\mathscr{B}^*$ now $\mathbb{Z}_p$-injective. (Note that in fact $\mathscr{C}^*(X;\mathscr{B})$ would be $\mathbb{Z}_p$-injective because of formula (6) on page 41.)

For $\mathbb{Z}_p$-modules we have that $\sigma = \tau^{p-1}$, and if we define

$$\begin{cases} k_{2n} &= h_{2n}, \\ k_{2n-1} &= h_{2n-1}\tau, \end{cases} \tag{61}$$

we see immediately that the k_i will satisfy (60) when the h_i satisfy (59).

20.4. By applying Γ_Φ and using the canonical map $T_p(\Gamma_\Phi(\mathscr{A}^*)) \to \Gamma_\Phi(T_p(\mathscr{A}^*))$, we obtain homomorphisms $h_i : T_p(\Gamma_\Phi(\mathscr{A}^*)) \to \Gamma_\Phi(\mathscr{B}^*)$, which still satisfy (59).

In this subsection we shall suppose, generally, that we are given chain complexes A^* and B^* or differential sheaves, with differentials of degree $+1$, and homomorphisms

$$h_i : T_p(A^*) \to B^*$$

of degree $-i$ such that the formulas (59) hold. For $a \in A^*$ we put

$$\Delta a = a \otimes \cdots \otimes a \in T_p(A^*). \tag{62}$$

Since the case $p = 2$ differs from the case of odd p, we shall first assume that p is odd and shall then give the modifications necessary for $p = 2$. If p

is odd then $\tau(\Delta a) = 0$, and it follows from (59) that $h_{2n-1}(\Delta a)$ is a cycle when a is a cycle. We wish to show that $h_{2n-1}(\Delta a)$ is a boundary when a is a boundary and, moreover, to obtain a formula for $h_{2n-1}(\Delta(db))$. We claim that there is a natural formula of the following type:

$$h_{2n-1}(\Delta(db)) = \sum_{i=1}^{p}(-1)^i dh_{2n-i}(\gamma_i) \qquad (p \text{ odd}) \tag{63}$$

where γ_i is a natural integral linear combination of terms of the form $a_1 \otimes \cdots \otimes a_p$ in which i of the a_j are equal to b and the rest are equal to db. Moreover, we claim that γ_p, which must be an integral multiple of Δb, is given by

$$\gamma_p = (-1)^{m(q+1)} m!(\Delta b), \tag{64}$$

where

$$m = (p-1)/2 \quad \text{and} \quad q = \deg b.$$

To obtain these formulas, we shall consider the chain complex L^*, where L^q is the free abelian group generated by the symbol b, L^{q+1} is the free abelian group generated by the symbol db, and the remainder of the L^i are zero. The differential is defined to take $b \mapsto db$. Let $\lambda : L^n \to L^{n-1}$ be that homomorphism taking $db \mapsto b$. Let $\Lambda : T_p(L^*) \to T_p(L^*)$ be $\lambda \otimes 1 \otimes 1 \otimes \cdots \otimes 1$. Then

$$\Lambda d + d\Lambda = 1. \tag{65}$$

Note that $\tau\Delta(db) = 0$. Let

$$\gamma_1 = \Lambda(\Delta(db))$$

and note that $\Delta(db) = d\gamma_1$. We define, inductively,

$$\begin{cases} \gamma_{2i} &= \Lambda\tau\gamma_{2i-1}, \\ \gamma_{2i+1} &= \Lambda\sigma\gamma_{2i}. \end{cases}$$

Using (65), it follows by an easy inductive argument that for $i \geq 1$,

$$\begin{cases} d\gamma_{2i} &= \tau\gamma_{2i-1}, \\ d\gamma_{2i+1} &= \sigma\gamma_{2i}. \end{cases} \tag{66}$$

These definitions make sense in $T_p(A^*)$, and the equations (66) remain valid, since we have a natural chain map $L^* \to A^*$. [In $T_p(A^*)$, Λ can be thought of as a formal operator that tells us how to write down γ_{i+1} from the expression for γ_i.] Using (59) and (66) inductively leads immediately to equation (63). (Note that γ_{p+1} is necessarily zero.) It remains for us to prove formula (64).

For the proof of (64) let us define, in $T_p(L^*)$, the operator Λ_i given by

$$\Lambda_i(a_1 \otimes \cdots \otimes a_p) = (-1)^{q_1+\cdots+q_{i-1}} a_1 \otimes \cdots \otimes a_{i-1} \otimes \lambda a_i \otimes a_{i+1} \otimes \cdots \otimes a_p,$$

where $q_j = \deg a_j$. Thus $\Lambda = \Lambda_1$, and it is easily computed that

$$\Lambda_i \alpha^j = \alpha^j \Lambda_{i+j}.$$

In particular, we have

$$\begin{cases} \Lambda_i \tau = \Lambda_i - \alpha\Lambda_{i+1}, \\ \Lambda_i \sigma = \sum_j \alpha^j \Lambda_{i+j}. \end{cases} \tag{67}$$

Moreover,

$$\begin{cases} \Lambda_i \Lambda_j = -\Lambda_j \Lambda_i, \\ \Lambda_i \Lambda_i = 0. \end{cases} \tag{68}$$

Now let $M_i = \Lambda_{i+1}\Lambda_i$. Using (67) and (68) to move the operators Λ_i to the right, we see that $\gamma_2 = -\alpha M_1(\Delta db)$, $\gamma_3 = -\sum \alpha^{i-1}\Lambda_i M_1(\Delta db)$, and, by an easy induction, that

$$\begin{cases} \gamma_{2r} = (-1)^r \sum \alpha^j M_j \cdots M_i M_1(\Delta db), \\ \gamma_{2r+1} = (-1)^r \sum \alpha^{k-1} \Lambda_k M_j \cdots M_i M_1(\Delta db), \end{cases} \tag{69}$$

where the number of M's occurring in each equation is r and the summations run over all free indices.

Now, in the computation of γ_p note that the term $\Lambda_k M_j \cdots M_i M_1(\Delta db)$, when it is nonzero, must be Δb up to sign. Each M_i contributes the sign $(-1)^q$, where $q = \deg b$, and Λ_k contributes no sign (since k is necessarily odd and the first $k-1$ terms in the tensor product it operates on are of the same degree). Thus, when $r = (p-1)/2 = m$, we have $\Lambda_k M_j \cdots M_i M_1(\Delta db) = (-1)^{mq}(\Delta b)$. It follows that γ_p is $(-1)^m$ times $(-1)^{mq}$ times the number of permutations of m objects [the M's and Λ_k in (69)] times Δb. This yields formula (64).

Now let $a, b \in A^q$ be cycles, and consider $\Delta(a+b) - \Delta a - \Delta b \in T_p(A^q)$. This is a sum of monomials in a and b. These monomials are permuted by α, and none of them is left fixed by α. Thus the cyclic group generated by α, being of prime order, acts freely on this set of monomials. Select one of these monomials out of each orbit of this group action and let c be the sum of these. Then $dc = 0$ and $\sigma c = \Delta(a+b) - \Delta a - \Delta b$. It follows from (59) that

$$h_{2n-1}(\Delta(a+b)) - h_{2n-1}(\Delta a) - h_{2n-1}(\Delta b) = -dh_{2n}(c). \tag{70}$$

By (59) the mapping $a \mapsto h_{2n-1}(\Delta a)$ takes cycles into cycles, and by (63) it takes boundaries into boundaries. Thus, by (70), it yields homomorphisms

$$St_{2n-1} : H^q(A^*) \to H^{pq-(2n-1)}(B^*) \quad \text{for } p \text{ odd.} \tag{71}$$

For $p = 2$ we have that $\sigma(\Delta a) = 0$ when $q = \deg a$ is odd, and $\tau(\Delta a) = 0$ when q is even. It follows easily from (59) that

$$h_{n+1}(da \otimes da) = (-1)^{n+1} dh_{n+1}(a \otimes da) + (-1)^{q-n} dh_n(a \otimes a) \quad \text{for } q-n \text{ odd.} \tag{72}$$

As before, it is easily seen that $a \mapsto h_n(a \otimes a)$ induces a homomorphism

$$St_n : H^q(A^*) \to H^{2q-n}(B^*) \quad \text{for } p = 2 \text{ and } q - n \text{ odd.} \tag{73}$$

It is convenient to introduce the notation $St^j = St_n$ on $H^q(A^*)$, where $j = q(p-1) - n$. Thus (71) and (73) become, for any p,

$$St^j : H^q(A^*) \to H^{q+j}(B^*) \quad \text{for } j \text{ odd.}$$

Finally, we claim that the image of St^j consists of elements of order p. In fact, for p odd, we have that $p(\Delta a) = \sigma(\Delta a)$, and (59) shows that $ph_{2n-1}(\Delta a) = h_{2n-1}(\sigma(\Delta a))$ is a boundary when a is a cycle, as claimed. This also follows when $p = 2$ in a similar manner.

20.5. Now suppose that for chain complexes A^* and B^* we are given homomorphisms

$$k_i : T_p(A^*) \to B^*$$

(of degree $-i$) satisfying the equations (60). Then as in 20.4, we can derive the formula

$$k_{2n}(\Delta(db)) = \sum_{i=0}^{p-1} (-1)^i dk_{2n-i}(\gamma_{i+1}) \quad \text{for } p \text{ odd,} \tag{74}$$

with γ_i as in 20.4 and, in particular, with γ_p given by formula (64). Similarly to (70), we also have, for $da = 0 = db$,

$$k_{2n}(\Delta(a+b)) - k_{2n}(\Delta a) - k_{2n}(\Delta b) = dk_{2n+1}(c). \tag{75}$$

For $p = 2$ we have formula (72) with k_i in place of h_i and valid for $q - n$ even ($q = \deg a$), as well as an analogue of (74).

Thus, $a \mapsto k_{2n}(\Delta a)$ [or $a \mapsto k_n(a \otimes a)$ for $p = 2$] induces homomorphisms

$$\begin{cases} St_{2n} : H^q(A^*) \to H^{pq-2n}(B^*) & \text{for } p \text{ odd,} \\ St_n : H^q(A^*) \to H^{2q-n}(B^*) & \text{for } p = 2 \text{ and } q - n \text{ even.} \end{cases}$$

That is, we obtain homomorphisms

$$St^j : H^q(A^*) \to H^{q+j}(B^*) \quad \text{for } j \text{ even.} \tag{76}$$

20.6. We may apply 20.4 to the case in which $A^* = \Gamma_\Phi(\mathscr{A}^*)$ and $B^* = \Gamma_\Phi(\mathscr{B}^*)$ with the h_i induced from the h_i defined in 20.2. Thus we obtain homomorphisms

$$St^j : H^q_\Phi(X; \mathscr{A}) \to H^{q+j}_\Phi(X; \mathscr{B}) \quad \text{for } j \text{ odd.}$$

Now suppose that $p\mathscr{B} = 0$. Then the k_i constructed in 20.3 induce

$$St^j : H^q_\Phi(X; \mathscr{A}) \to H^{q+j}_\Phi(X; \mathscr{B}) \quad \text{for } j \text{ even.} \tag{77}$$

If, as we may as well assume, $\mathscr{A}$ and $\mathscr{B}$ are both sheaves of $\mathbb{Z}_p$-modules, then (for p odd) $St^{q(p-1)-2n} = St_{2n}$ of (77) is induced by $a \mapsto h_{2n}(\Delta a)$ because of (61). A similar remark holds for $p = 2$.

If $j = (p-1)q$, then $St^j = St_0$ is induced by $a \mapsto h_0(a \otimes \cdots \otimes a)$ and hence it is the p-fold cup product followed by h^*.

Also note that if $j > (p-1)q$, then $St^j = St_{(p-1)q-j} = 0$, since $h_i = 0$ for $i < 0$ by definition.

20.7. We wish to show that the homomorphisms St^j constructed in 20.6 are independent of the choices involved. In particular, we wish to show that the definitions do not depend on the choice of the h_i in 20.2.

Suppose that $\{h_i'\}$ is another system of homomorphisms $T_p(\mathscr{A}^*) \to \mathscr{B}^*$ satisfying (59) and such that h_0' extends the given map h. Then h_0 is chain homotopic to h_0', so that there is a homomorphism D_1 with

$$h_0 - h_0' = D_1 d + dD_1.$$

Applying τ to the right of this equation, we obtain

$$(D_1\tau)d + d(D_1\tau) = h_0\tau - h_0'\tau = (h_1 d + dh_1) - (h_1' d + dh_1').$$

Rearranging terms, we have

$$(h_1 - h_1' - D_1\tau)d + d(h_1 - h_1' - D_1\tau) = 0.$$

As in (57), this implies the existence of a homomorphism D_2 with

$$h_1 - h_1' - D_1\tau = D_2 d - dD_2.$$

Apply σ to the right of this equation and proceed as before. An easy inductive argument provides the existence of homomorphisms D_i with

$$\begin{cases} h_{2n-1} - h_{2n-1}' - D_{2n-1}\tau = D_{2n}d - dD_{2n}, \\ h_{2n} - h_{2n}' - D_{2n}\sigma = D_{2n+1}d + dD_{2n+1}. \end{cases} \tag{78}$$

[Note that the construction of the D_n (and of the h_i themselves) does not use the assumption that $\mathscr{B}^*$ is a resolution of $\mathscr{B}$ but only that it is an injective differential sheaf.]

Applying Γ_Φ to these equations, we see that for p odd and $a \in \Gamma_\Phi(\mathscr{A}^*)$ with $da = 0$, we have

$$h_{2n-1}(\Delta a) - h_{2n-1}'(\Delta a) = -dD_{2n}(\Delta a)$$

since $\tau\Delta a = 0$ and $d(\Delta a) = 0$. This proves our contention for odd p, and the case $p = 2$ is similar.

The analogous result for the k_i (when $p\mathscr{B} = 0$) is proved in exactly the same manner.

We shall also show in 20.8 that the definitions of the St^j are independent of the choices of the resolutions $\mathscr{A}^*$ and $\mathscr{B}^*$ (for the latter this should be clear). It is shown in Section 21 that the St^j are also independent, up to sign, of the choice of the particular cyclic permutation α.

20.8. Suppose that $f : X \to Y$ is a map and let $h' : T_p(\mathscr{L}) \to \mathscr{M}$ be a symmetric homomorphism of sheaves on Y. Let Ψ be a family of supports on Y with $f^{-1}\Psi \subset \Phi$ and suppose that we are given Ψ-acyclic resolutions $\mathscr{L}^*$ and $\mathscr{M}^*$ of $\mathscr{L}$ and $\mathscr{M}$ such that $T_p(\mathscr{L}^*)$ is a resolution of $T_p(\mathscr{L})$. Also assume that we are given homomorphisms $h'_i : T_p(\mathscr{L}^*) \to \mathscr{M}^*$ satisfying (59) [or k'_i satisfying (60)] and extending h'. Suppose further that we are given f-cohomomorphisms $k : \mathscr{L} \rightsquigarrow \mathscr{A}$ and $g : \mathscr{M} \rightsquigarrow \mathscr{B}$ such that the diagram

$$\begin{array}{ccc} T_p(\mathscr{L}) & \xrightarrow{h'} & \mathscr{M} \\ {\scriptstyle \Delta k}\downarrow & & \downarrow{\scriptstyle g} \\ T_p(\mathscr{A}) & \xrightarrow{h} & \mathscr{B} \end{array}$$

commutes (where $\Delta k = k \otimes \cdots \otimes k$). Also suppose that k and g extend to f-cohomomorphisms $k^* : \mathscr{L}^* \rightsquigarrow \mathscr{A}^*$ and $g^* : \mathscr{M}^* \rightsquigarrow \mathscr{B}^*$ of resolutions. Consider the (not necessarily commutative) diagram

$$\begin{array}{ccc} T_p(\mathscr{L}^*) & \xrightarrow{h'_n} & \mathscr{M}^* \\ {\scriptstyle \Delta k^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ f(T_p(\mathscr{A}^*)) & \xrightarrow{f(h_n)} & f(\mathscr{B}^*). \end{array} \tag{79}$$

The maps $f(h_n)\Delta(k^*)$ and $g^*h'_n$ of $T_p(\mathscr{L}^*) \to f(\mathscr{B}^*)$ both satisfy (59) and extend $f(h)\Delta k = gh'$. Since $f(\mathscr{B}^*)$ is injective, it follows that homomorphisms $D_n : T_p(\mathscr{L}^*) \to f(\mathscr{B}^*)$ can be constructed, as in (78), such that

$$f(h_n)\Delta k^* - g^*h'_n - D_n\eta_n = D_{n+1}d + (-1)^n dD_{n+1},$$

where $\eta_n = \tau$ for n odd and $\eta_n = \sigma$ for n even. This formula remains valid upon passing to sections with supports in Ψ [of (79)], and it follows immediately that the diagram

$$\begin{array}{ccc} H^q_\Psi(Y;\mathscr{L}) & \xrightarrow{St^j} & H^{q+j}_\Psi(Y;\mathscr{M}) \\ \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle g^*} \\ H^q_\Phi(X;\mathscr{A}) & \xrightarrow{St^j} & H^{q+j}_\Phi(X;\mathscr{B}) \end{array} \tag{80}$$

commutes, where j is odd. This may also be shown for even j when $p\mathscr{B} = 0 = p\mathscr{M}$ and St^j is defined as in (76).

When f is the identity, these results show that the St^j are independent of the choice of the resolution $\mathscr{A}^*$ (the independence of the choice of $\mathscr{B}^*$ is clear). In fact, any $\mathscr{A}^*$ such that $T_p(\mathscr{A}^*)$ is a resolution of $T_p(\mathscr{A})$ can be compared with $\mathscr{C}^*(X;\mathscr{A})$ by means of the maps

$$\mathscr{A}^* \to \mathscr{C}^*(X;\mathscr{A}^*) \leftarrow \mathscr{C}^*(X;\mathscr{A}),$$

and the middle term satisfies our hypothesis that the tensor product of the resolutions be a resolution of the tensor product. (This follows from Exercise 48, which asserts that the first map is a pointwise homotopy equivalence.)

20.9. Suppose that we are given exact sequences $0 \to \mathscr{A}'^* \to \mathscr{A}^* \to \mathscr{A}''^* \to 0$ and $0 \to \mathscr{B}'^* \to \mathscr{B}^* \to \mathscr{B}''^* \to 0$ of chain complexes (with differentials of degree +1). Suppose, moreover, that we have *commutative* diagrams

$$\begin{array}{ccccc} T_p(\mathscr{A}'^*) & \xrightarrow{i} & T_p(\mathscr{A}^*) & \xrightarrow{j} & T_p(\mathscr{A}''^*) \\ \downarrow{\scriptstyle h'_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h''_n} \\ B'^* & \xrightarrow{f} & B^* & \xrightarrow{g} & B''^*, \end{array} \tag{81}$$

where the h'_n, h_n, and h''_n satisfy (59) [respectively (60)]. Consider the induced diagram

$$\begin{array}{ccccccccc} \cdots \to & H^q(A'^*) & \xrightarrow{i^*} & H^q(A^*) & \xrightarrow{j^*} & H^q(A''^*) & \xrightarrow{\delta} & H^{q+1}(A'^*) & \to \cdots \\ & \downarrow{\scriptstyle St^s} & & \downarrow{\scriptstyle St^s} & & \downarrow{\scriptstyle St^s} & & \downarrow{\scriptstyle St^s} & \\ \cdots \to & H^{q+s}(B'^*) & \xrightarrow{f^*} & H^{q+s}(B^*) & \xrightarrow{g^*} & H^{q+s}(B''^*) & \xrightarrow{\delta} & H^{q+s+1}(B'^*) & \to \cdots \end{array} \tag{82}$$

which is defined for odd s (respectively, for even s). This diagram is clearly commutative except for the square containing connecting homomorphisms. For this square, let $a \in A^q$ represent a *cycle* $j(a) \in A''^*$. Let a' be such that $i(a') = da$. Then in the case of odd s and odd p, for example, we have

$$\begin{aligned} fh'_{2n-1}(\Delta a') &= h_{2n-1}(\Delta(da)) \\ &= d\left(-r_p h_{2n-p}(\Delta a) + \sum_{i=1}^{p-1}(-1)^i h_{2n-i}(\gamma_i)\right) \end{aligned} \tag{83}$$

by (63) and (64), where $r_p = (-1)^{m(q+1)}m!$ and $m = (p-1)/2$.

Now, $gh_{2n-i}(\gamma_i) = h''_{2n-i}(j(\gamma_i)) = 0$ for $i < p$ (since γ_i, for $i < p$, is a sum of monomials each of which contains a factor $da = i(a')$). Thus we also have that

$$\begin{aligned} -r_p h''_{2n-p}(\Delta j(a)) &= g(-r_p h_{2n-p}(\Delta a)) \\ &= g\left(-r_p h_{2n-p}(\Delta a) + \sum_{i=1}^{p-1}(-1)^i h_{2n-i}(\gamma_i)\right). \end{aligned} \tag{84}$$

From (83) and (84) we see that $-r_p\delta St_{2n-p}[j(a)]$ is represented by $h'_{2n-1}(\Delta a')$. But the latter also represents $St_{2n-1}\delta[j(a)]$. In terms of St^s this means that

$$St^s\delta = (-1)^s(-1)^{m(q+1)}(m!)\delta St^s \quad \text{for } p \text{ odd}. \tag{85}$$

We have proved this for odd s. The case of even s follows in the same way from (74).

In the case $p = 2$ it can be seen, in the same way, that (82) is commutative when it is defined (since in this case, the image of St^s consists of elements of order 2).

We wish to apply these remarks to spaces. Let A be a Φ-taut subspace of X and assume that $\mathscr{A}^*$ is flabby. The maps $h_n : T_p(\mathscr{A}^*) \to \mathscr{B}^*$ restrict to maps $T_p(\mathscr{A}^*|A) \to \mathscr{B}^*|A$, and the diagrams

$$\begin{array}{ccccc} T_p(\Gamma_{\Phi|X-A}(\mathscr{A}^*)) & \to & T_p(\Gamma_\Phi(\mathscr{A}^*)) & \to & T_p(\Gamma_{\Phi\cap A}(\mathscr{A}^*|A)) \\ \downarrow h_n & & \downarrow h_n & & \downarrow h_n \\ \Gamma_{\Phi|X-A}(\mathscr{B}^*) & \to & \Gamma_\Phi(\mathscr{B}^*) & \to & \Gamma_{\Phi\cap A}(\mathscr{B}^*|A) \end{array}$$

clearly commute. Thus, if we define the St^s on the relative cohomology of (X, A), for A Φ-taut, via the isomorphism 12.1, it follows that the diagram

$$\begin{array}{ccccccccc} \cdots\to & H^q_\Phi(X,A;\mathscr{A}) & \to & H^q_\Phi(X;\mathscr{A}) & \to & H^q_{\Phi\cap A}(A;\mathscr{A}) & \to & H^{q+1}_\Phi(X,A;\mathscr{A}) & \to\cdots \\ & \downarrow St^s & & \downarrow St^s & & \downarrow St^s & & \downarrow St^s & \\ \cdots\to & H^{q+s}_\Phi(X,A;\mathscr{A}) & \to & H^{q+s}_\Phi(X;\mathscr{A}) & \to & H^{q+s}_{\Phi\cap A}(A;\mathscr{A}) & \to & H^{q+s+1}_\Phi(X,A;\mathscr{A}) & \to\cdots \end{array}$$

commutes when defined, except for the square involving connecting homomorphisms. This square commutes when $n = 2$ and satisfies (85) when p is odd.

We remark that this result can be extended to nontaut A by using the naturality of the particular set of h_n constructed in Section 21.

20.10. We shall now consider the case in which $\mathscr{A}$ and $\mathscr{B}$ are torsion-free. We put $\mathscr{A}_p = \mathscr{A}/p\mathscr{A}$ and note that the given map $h : T_p(\mathscr{A}) \to \mathscr{B}$ induces a symmetric homomorphism $T_p(\mathscr{A}_p) \to \mathscr{B}_p$.

Now $St^j : H^q_\Phi(X;\mathscr{A}) \to H^{q+j}_\Phi(X;\mathscr{B})$ is defined for odd j, while $St^j : H^q_\Phi(X;\mathscr{A}_p) \to H^{q+j}_\Phi(X;\mathscr{B}_p)$ (or from coefficients in $\mathscr{A}$) is defined for all j. We wish to find relationships between these operations.

If we use the fact, proved in Section 21, that the St^j can be defined using *torsion-free* resolutions of $\mathscr{A}$ *and* $\mathscr{B}$ then we could reduce these resolutions mod p to obtain resolutions of $\mathscr{A}_p$ and $\mathscr{B}_p$ and could use these for our comparison. However, we prefer to use another method, which, although longer, does not depend on this fact and has, perhaps, some independent interest.

The method we adopt utilizes the "mapping cone," of multiplication by p, of a differential sheaf. The mapping cone of $p : \mathscr{L}^* \to \mathscr{L}^*$ is the differential sheaf $M_p(\mathscr{L}^*)$, where

$$\boxed{M_p(\mathscr{L}^*)^q = \mathscr{L}^{q+1} \oplus \mathscr{L}^q}$$

and $d : M_p(\mathscr{L}^*)^q \to M_p(\mathscr{L}^*)^{q+1}$ is given by

$$\boxed{d(a, b) = (-da, pa + db).}$$

We have the exact sequences

$$0 \to \mathscr{L}^q \xrightarrow{\rho} M_p(\mathscr{L}^*)^q \xrightarrow{\beta'} \mathscr{L}^{q+1} \to 0, \tag{86}$$

where $\rho(b) = (0, b)$ and $\beta'(a, b) = a$. If $\mathscr{L}^*$ is a Φ-acyclic resolution of a torsion-free sheaf $\mathscr{L}$, then according to Exercise 50, there is a natural isomorphism

$$H^q(\Gamma_\Phi(M_p(\mathscr{L}^*))) \approx H^q_\Phi(X; \mathscr{L}_p),$$

and the homology sequence induced by Γ_Φ of (86) can be identified with the cohomology sequence of the exact coefficient sequence $0 \to \mathscr{L} \xrightarrow{p} \mathscr{L} \to \mathscr{L}_p \to 0$. [Note that ρ becomes reduction mod p and $\beta = -\beta'$ becomes the connecting homomorphism, which is called the "Bockstein" homomorphism in this case. We retain the notation ρ and β for these induced cohomology homomorphisms.]

Remark: The homology sequence of (86) shows that if $\mathscr{L}^*$ is a resolution of a torsion-free sheaf $\mathscr{L}$, then

$$\mathscr{H}^q(M_p(\mathscr{L}^*)) \approx \begin{cases} \mathscr{L}_p, & \text{for } q = 0, \\ 0, & \text{for } q \neq 0. \end{cases}$$

However, $M_p(\mathscr{L}^*)$ is not a resolution of $\mathscr{L}_p$ since $M_p(\mathscr{L}^*)^{-1} \approx \mathscr{L}^0 \neq 0$. The minor alteration of dividing $M_p(\mathscr{L}^*)^0$ by $d(M_p(\mathscr{L}^*)^{-1})$ provides a resolution of $\mathscr{L}_p$, but we need not use this fact. The augmentation is induced by the canonical inclusion $b \mapsto (0, b)$ of $\mathscr{L}$ in $\mathscr{L}^1 \oplus \mathscr{L}^0$.

Note that $p - \sigma$, as a polynomial in α, is divisible by τ. We let ω denote the quotient, so that

$$p - \sigma = \omega\tau.$$

Now suppose that $h_n : T_p(\mathscr{L}^*) \to \mathscr{M}^*$ are homomorphisms that satisfy (59). We claim that the following two statements hold:

(i) Let $h'_n : T_p(\mathscr{L}^*) \to M_p(\mathscr{M}^*)$ be defined by $h'_n = (0, h_n)$. Then the h'_n satisfy (59).

(ii) Let $h''_n : T_p(\mathscr{L}^*) \to M_p(\mathscr{M}^*)$ be defined by $h''_{2n} = (h_{2n-1}, h_{2n})$ and $h''_{2n+1} = (h_{2n}, -h_{2n+1}\omega)$. Then the h''_n satisfy (60).

These facts are easily obtained by straightforward computations, which will be omitted.

We shall digress for a moment to consider a construction to be used below. Define, for any differential sheaf $\mathscr{A}^*$ (or chain complex), the homomorphisms θ and ψ of $T_p(M_p(\mathscr{A}^*)) \to T_p(\mathscr{A}^*)$ as follows: If $x = (a_1, b_1) \otimes \cdots \otimes (a_p, b_p) \in T_p(M_p(\mathscr{L}^*))$ and if $q_i = \deg b_i$, then we put

$$\begin{aligned} \theta(x) &= b_1 \otimes \cdots \otimes b_p, \\ \psi(x) &= a_1 \otimes b_2 \otimes \cdots \otimes b_p + (-1)^{q_1} b_1 \otimes a_2 \otimes b_3 \otimes \cdots \otimes b_p \\ &\quad + (-1)^{q_1+q_2} b_1 \otimes b_2 \otimes a_3 \otimes b_4 \otimes \cdots \otimes b_p + \cdots \\ &\quad + (-1)^{q_1+\cdots+q_{p-1}} b_1 \otimes \cdots \otimes b_{p-1} \otimes a_p. \end{aligned}$$

It is easy to check that

$$\begin{cases} \theta d = p\psi + d\theta, \\ \psi d = -d\psi. \end{cases}$$

Also note that

$$\begin{cases} \theta\alpha = \alpha\theta, \\ \psi\alpha = \alpha\psi. \end{cases}$$

Now define the homomorphisms

$$\lambda_n : T_p(M_p(\mathscr{A}^*)) \to \mathscr{B}^*$$

by the equations

$$\begin{cases} \lambda_{2n} = h_{2n}\theta + h_{2n+1}\omega\psi, \\ \lambda_{2n-1} = h_{2n-1}\theta + h_{2n}\psi. \end{cases}$$

We compute

$$\begin{aligned} \lambda_{2n}d - d\lambda_{2n} &= (h_{2n}d - dh_{2n})\theta + h_{2n}p\psi - (h_{2n+1}d + dh_{2n+1})\omega\psi \\ &= h_{2n-1}\sigma\theta + h_{2n}(p - \tau\omega)\psi \\ &= (h_{2n-1}\theta + h_{2n}\psi)\sigma = \lambda_{2n-1}\sigma. \end{aligned} \tag{87}$$

Similarly we have

$$\begin{aligned} \lambda_{2n-1}d + d\lambda_{2n-1} &= (h_{2n-1}d + dh_{2n-1})\theta + h_{2n-1}p\psi - (h_{2n}d - dh_{2n})\psi \\ &= h_{2n-2}\tau\theta + h_{2n-1}(p - \sigma)\psi \\ &= (h_{2n-2}\theta + h_{2n-1}\omega\psi)\tau = \lambda_{2n-2}\tau. \end{aligned} \tag{88}$$

Thus, by (87) and (88), the λ_n satisfy (59).

Now we let $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$ and let $\mathscr{B}^*$ be any injective resolution of $\mathscr{B}$ as usual. By statement (i) above, we see that the homomorphisms

$$\mu_n = (0, \lambda_n) : T_p(M_p(\mathscr{A}^*)) \to M_p(\mathscr{B}^*)$$

satisfy (59), and by (ii), we see that the homomorphisms ν_n defined by

$$\begin{cases} \nu_{2n} = (\lambda_{2n-1}, \lambda_{2n}), \\ \nu_{2n+1} = (\lambda_{2n}, -\lambda_{2n+1}\omega) \end{cases}$$

satisfy (60).

Since, in degree zero, $\lambda_{2n}((0,b_1)\otimes\cdots\otimes(0,b_p)) = h_{2n}(b_1\otimes\cdots\otimes b_p)$, it follows that μ_0 and ν_0 both extend the homomorphism $h : T_p(\mathscr{A}_p) \to \mathscr{B}_p$. We claim that the homomorphism $St^j : H^q_\Phi(X;\mathscr{A}_p) \to H^{q+j}_\Phi(X;\mathscr{B}_p)$ is induced by $(a,b) \mapsto \mu_n(\Delta(a,b))$ for j odd and by $(a,b) \mapsto \nu_n(\Delta(a,b))$ for j even [where, of course, $n = q(p-1) - j$]. To see this, we note that there is a canonical homomorphism

$$M_p(\mathscr{A}^*) \to \mathscr{C}^*(X;\mathscr{A}_p) = \mathscr{C}^*(X;\mathscr{A})/p\mathscr{C}^*(X;\mathscr{A}),$$

which takes $(a,b) \mapsto b \pmod p$. This homomorphism clearly induces an isomorphism of derived sheaves, and in fact, it is just the map that identifies $H^q(\Gamma_\Phi(M_p(\mathscr{A}^*)))$ with $H^q_\Phi(X;\mathscr{A}_p)$ (see Exercise 49). The diagram

$$\begin{array}{ccc} T_p(M_p(\mathscr{A}^*)) & \to & M_p(\mathscr{B}^*) \\ \downarrow & & \downarrow \\ T_p(\mathscr{C}^*(X;\mathscr{A}_p)) & \to & (\mathscr{B}_p)^* \end{array}$$

[where $(\mathscr{B}_p)^*$ is any $\mathbb{Z}$-injective resolution of $\mathscr{B}_p$] can then be treated as in 20.8, and our contention follows immediately. [Note that without modification, $T_p(M_p(\mathscr{A}^*))$ is not a resolution of $T_p(\mathscr{A}_p)$, and in fact, some of the derived sheaves in negative degrees are nonzero. However, the derived sheaves in positive degrees are all zero by the algebraic Künneth formula applied to the stalks, and this is sufficient for the construction of the homotopies of 20.7.]

Assume for the moment that p is odd. Taking sections, a q-cycle of $\Gamma_\Phi(M_p(\mathscr{A}^*))$ is a pair (a,b) with $\deg b = q$, $da = 0$, and $db = -pa$. Thus, for coefficients in $\mathscr{A}_p$ and $\mathscr{B}_p$, St_{2n-1} is induced by

$$(a,b) \mapsto \mu_{2n-1}(\Delta(a,b)) = (0, h_{2n-1}(\theta\Delta(a,b)) + h_{2n}(\psi\Delta(a,b))),$$

while St_{2n} is induced by

$$\begin{aligned}(a,b) \mapsto \nu_{2n}(\Delta(a,b)) = (&h_{2n-1}(\theta\Delta(a,b)) + h_{2n}(\psi\Delta(a,b)),\\ &h_{2n}(\theta\Delta(a,b)) + h_{2n+1}(\omega\psi\theta\Delta(a,b))).\end{aligned}$$

Note that $\theta\Delta(0,b) = \Delta b$ and $\psi\Delta(0,b) = 0$.

Recall that reduction modulo p is induced by $b \mapsto (0,b)$ and that the Bockstein is induced by $(a,b) \mapsto -a$. It follows immediately that in the diagram

$$\begin{array}{ccccc}
H^q_\Phi(X,\mathscr{A}) & & \xrightarrow{St^{2j+1}} & & H^{q+2j+1}_\Phi(X,\mathscr{B}) \\
& & & \nearrow{\scriptstyle\beta} & \\
\downarrow{\scriptstyle\rho} & & H^{q+2j}_\Phi(X,\mathscr{B}_p) & & \downarrow{\scriptstyle\rho} \\
& \nearrow{\scriptstyle St^{2j}} & & & \\
H^q_\Phi(X,\mathscr{A}_p) & & \xrightarrow[St^{2j+1}]{} & & H^{q+2j+1}_\Phi(X,\mathscr{B}_p)
\end{array} \tag{89}$$

both of the triangles *anticommute* (and hence the outside square commutes). This also holds for $p = 2$ by the same arguments.

Remark: If in (59) and in similar equations we had chosen to use $dh \pm hd$ rather than $hd \pm dh$ (which would have been more logical in some ways), we would have obtained strict commutativity in (89). Clearly, such a change merely appends the sign $(-1)^n$ to h_{2n} and h_{2n+1}. We have adopted the present conventions because they provide slightly simpler formulas in several places, such as (63).

20.11. Let $\mathscr{A}^*$, $\mathscr{B}^*$, $\mathscr{L}^*$, and $\mathscr{M}^*$ be differential sheaves (or chain complexes) and assume that we are given homomorphisms

$$\begin{cases} h_n : T_p(\mathscr{A}^*) \to \mathscr{B}^*, \\ k_n : T_p(\mathscr{L}^*) \to \mathscr{M}^*, \end{cases} \tag{90}$$

both of which satisfy the system (59) of equations. Let

$$\lambda : T_p(\mathscr{A}^* \otimes \mathscr{L}^*) \to T_p(\mathscr{A}^*) \otimes T_p(\mathscr{L}^*)$$

be the obvious "unshuffling" isomorphism (with the usual sign convention). It commutes with differentials. On $T_p(\mathscr{A}^*) \otimes T_p(\mathscr{L}^*)$ we use the notation $\Delta\tau = 1\otimes 1 - \alpha\otimes\alpha$ and $\Delta\sigma = 1\otimes 1 + \alpha\otimes\alpha + \cdots \alpha^{p-1}\otimes\alpha^{p-1}$ [so that $\lambda^{-1}(\Delta\tau)\lambda$ and $\lambda^{-1}(\Delta\sigma)\lambda$ are the operators "τ" and "σ" on $T_p(\mathscr{A}^* \otimes \mathscr{L}^*)$]. Also, let $\Omega = \sum \alpha^i \otimes \alpha^j$, where the summation is over the range $0 \le i < j < p$. The following identities are easy to verify:

$$\left\{\begin{aligned} \sigma\otimes 1 + (1\otimes\tau)\Omega &= (1\otimes\alpha)(\Delta\sigma), \\ 1\otimes\sigma - (\tau\otimes 1)\Omega &= \Delta\sigma, \\ 1\otimes\tau + \tau\otimes\alpha &= \Delta\tau, \\ 1\otimes\sigma - \sigma\otimes 1 &= \Omega(\Delta\tau). \end{aligned}\right. \tag{91}$$

We define homomorphisms $j_n : T_p(\mathscr{A}^* \otimes \mathscr{L}^*) \to \mathscr{B}^* \otimes \mathscr{M}^*$ by

$$\left\{\begin{aligned} j_{2n} &= \sum[(h_{2i}\otimes k_{2n-2i}) + (h_{2i+1}\otimes k_{2n-2i-1})\Omega]\lambda, \\ j_{2n+1} &= \sum[(h_{2i}\otimes k_{2n-2i+1}) + (h_{2i+1}\otimes k_{2n-2i})\alpha]\lambda. \end{aligned}\right.$$

We claim that the j_n also satisfy (59). This is easily shown by formal manipulation using the identities (91). Note that $\Omega = 1\otimes\alpha$ if $p = 2$, so that both equations can be combined in that case.

Now suppose that $\mathscr{A}$, $\mathscr{B}$, $\mathscr{L}$, and $\mathscr{M}$ are all sheaves of $\mathbb{Z}_p$-modules and that we are given symmetric homomorphisms

$$h : T_p(\mathscr{A}) \to \mathscr{B} \quad \text{and} \quad k : T_p(\mathscr{L}) \to \mathscr{M}.$$

Let $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$ and similarly for $\mathscr{B}^*$, $\mathscr{L}^*$, and $\mathscr{M}^*$. These are all $\mathbb{Z}_p$-injective resolutions by (6) on page 6 since every (classical) $\mathbb{Z}_p$-module is $\mathbb{Z}_p$-injective. Thus the homomorphisms (90) can be constructed as in 20.2. Defining j_n as above, we see that j_0 extends

$$j = (h\otimes k)\lambda : T_p(\mathscr{A}\otimes\mathscr{L}) \to \mathscr{B}\otimes\mathscr{M}.$$

Also, $\mathscr{B}^* \otimes \mathscr{M}^*$, being a resolution of $\mathscr{B}\otimes\mathscr{M}$, can be mapped into any injective resolution.

Now, for $a \in \mathscr{A}^s$ and $b \in \mathscr{L}^t$ and for p odd, we have

$$j_{2n}(\Delta(a\otimes b)) = (-1)^{p(p-1)st/2} \sum h_{2i}(\Delta a)\otimes k_{2n-2i}(\Delta b) \tag{92}$$

since $\lambda(\Delta(a\otimes b)) = (-1)^{p(p-1)st/2}(\Delta a)\otimes(\Delta b)$ and since $\Omega(\Delta a\otimes\Delta b) = p\left(\frac{p-1}{2}\right)(\Delta a\otimes\Delta b) = 0$. Similarly, if $p = 2$, we have

$$j_n((a\otimes b)\otimes(a\otimes b)) = (-1)^{st}\sum h_i(a\otimes a)\otimes k_{n-i}(b\otimes b) \tag{93}$$

(the sign is, of course, immaterial here).

Taking sections, passing to homology, and converting to upper indices, we get

$$St^{2n}(a \cup b) = (-1)^{p(p-1)st/2} \sum St^{2i}(a) \cup St^{2n-2i}(b) \quad \text{for } p \text{ odd} \tag{94}$$

by (92). These are maps $H^s_\Phi(X;\mathscr{A}) \otimes H^t_\Psi(X;\mathscr{L}) \to H^{s+t+2n}_{\Phi\cap\Psi}(X;\mathscr{B}\otimes\mathscr{M})$ for sheaves of $\mathbb{Z}_p$-modules where Φ and Ψ are arbitrary support families. Similarly, by (93),

$$St^n(a \cup b) = \sum St^i(a) \cup St^{n-i}(b) \quad \text{for } p = 2$$

of $H^s_\Phi(X;\mathscr{A})\otimes H^t_\Psi(X;\mathscr{L}) \to H^{s+t+n}_{\Phi\cap\Psi}(X;\mathscr{B}\otimes\mathscr{M})$ for sheaves of $\mathbb{Z}_2$-modules.
These equations are known as the *Cartan formulas.*

21 The Steenrod operations

In this section, which is a continuation of the last section, we shall find certain values of j for which the operation St^j is trivial. Using this information, we then alter these operations to define operations, the "Steenrod powers," that possess somewhat simpler properties than do the St^j.

21.1. We shall first apply the results of 20.7 to show that up to sign, the St^j are independent of the choice of the particular cyclic permutation α. We shall then use this fact to find, for p odd, various values of j for which the operation St^j is trivial.

Let α denote the particular cyclic permutation that we have been using and let α' be any cyclic permutation of the factors of a p-fold tensor product. Then α and α' are conjugate in the symmetric group S_p. That is, there is an element $\gamma \in S_p$ with $\gamma\alpha' = \alpha\gamma$. If $\tau' = 1 - \alpha'$ and $\sigma' = 1 + \alpha' + \cdots + \alpha'^{p-1}$, then we have

$$\gamma\tau' = \tau\gamma \quad \text{and} \quad \gamma\sigma' = \sigma\gamma.$$

Applying γ to the right side of (59), we see that

$$\begin{cases} (h_{2n}\gamma)\tau' &= (h_{2n+1}\gamma)d + d(h_{2n+1}\gamma), \\ (h_{2n-1}\gamma)\sigma' &= (h_{2n}\gamma)d - d(h_{2n}\gamma). \end{cases}$$

Also, $h_0\gamma$ extends $h\gamma = h$, since h is symmetric. However, if $a \in \Gamma_\Phi(\mathscr{A}^*)$ is a cycle of degree q, then $\gamma(\Delta a) = (\operatorname{sgn}\gamma)^q \Delta a$, so that

$$\begin{cases} (h_{2n-1}\gamma)(\Delta a) &= (\operatorname{sgn}\gamma)^q h_{2n-1}(\Delta a), \\ (h_{2n}\gamma)(\Delta a) &= (\operatorname{sgn}\gamma)^q h_{2n}(\Delta a). \end{cases} \tag{95}$$

Thus St_{2n-1}, defined by means of α', differs from St_{2n-1}, defined by means of α, by the sign $(\operatorname{sgn}\gamma)^q$ in degree q. This is also true, clearly, for St_{2n} when it is defined (i.e., for $p\mathscr{B} = 0$).

Let us now consider a particular choice of $\gamma \in S_p$ and let p be odd. Let γ take the ith term into the (i/k)th place (modulo p), where k is a generator of the multiplicative group of $\mathbb{Z}_p$ (which is cyclic of order $p-1$). Then $\alpha' = \gamma^{-1}\alpha\gamma$ takes the ith term into the $k(i/k-1)$st $= (i-k)$th place. Thus $\alpha' = \alpha^k$. Moreover, γ takes $\langle k^1, k^2, \ldots, k^{p-1} = 1\rangle \mapsto \langle k^{p-1}, k^1, k^2, \ldots\rangle$, and so γ is an *odd* permutation.

Now $\sigma' = \sigma$ for this choice of $\alpha' = \alpha^k$ and $\tau' = 1 - \alpha^k = (1 + \alpha + \cdots + \alpha^{k-1})\tau$. Let us define

$$\begin{cases} h'_{2n} &= h_{2n}(1+\alpha+\cdots+\alpha^{k-1})^n, \\ h'_{2n-1} &= h_{2n-1}(1+\alpha+\cdots+\alpha^{k-1})^n. \end{cases}$$

An easy computation shows that the $\{h'_n\}$ satisfy (59) with τ replaced by $\tau' = 1-\alpha^k$. But so do the $\{h_n\gamma\}$. By 20.7 it follows that for $a \in \Gamma_\Phi(\mathscr{A}^*)$ a cycle of degree q, we have that $(h_{2n-1}\gamma)(\Delta a)$ is homologous to $(h'_{2n-1})(\Delta a) = h_{2n-1}((1+\alpha+\cdots+\alpha^{k-1})^n(\Delta a)) = k^n h_{2n-1}(\Delta a)$. From this and (95) it follows that

$$St_{2n-1} : H^q_\Phi(X;\mathscr{A}) \to H^{pq-2n+1}_\Phi(X;\mathscr{B})$$

is zero unless

$$k^n \equiv (-1)^q \pmod{p}. \tag{96}$$

(Recall that the image of St_i consists of elements of order p.) If $p\mathscr{B} = 0$, we can prove a similar fact for St_{2n} using the k_n of 20.3. However, we can treat this case directly by using formula (61), which reduces the question to one concerning h_{2n}. Again, it is seen that $(h_{2n}\gamma)(\Delta a)$ is homologous to $(h'_{2n})(\Delta a) = k^n h_{2n}(\Delta a)$, and by (95), it follows that

$$St_{2n} : H^q_\Phi(X;\mathscr{A}) \to H^{pq-2n}_\Phi(X;\mathscr{B})$$

is zero unless formula (96) holds.

Since k generates the multiplicative group of $\mathbb{Z}_p$, the equation $k^n \equiv 1 \pmod{p}$ is equivalent to $n = r(p-1)$ and the equation $k^n \equiv -1 \pmod{p}$ is equivalent to $n = (2r+1)(p-1)/2$ (where r is some integer). In terms of St^j, which is more convenient, a short calculation shows that our criterion (96) becomes

$$St^j = 0 \quad \text{unless } j = 2r(p-1) \text{ or } j = 2r(p-1)+1, \text{ for some } r \in \mathbb{Z}. \tag{97}$$

21.2. We wish to show that $St^j = 0$ for $j < 0$ and to identify St^0. To do this we must, for the first time, give an explicit construction of one particular system of h_i. We shall, in fact, construct such homomorphisms $h_i : T_p(\mathscr{A}^*) \to \mathscr{B}^*$, where $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$ and $\mathscr{B}^* = \mathscr{C}^*(X;\mathscr{B})$. This can then be followed by a map to an injective resolution of $\mathscr{B}$ if so desired.

Recall that $\mathscr{C}^*(X;\mathscr{A})$ is pointwise homotopically trivial. In fact, as in Section 2, let $\eta_x : \mathscr{C}^0(X;\mathscr{A})_x \to \mathscr{A}_x$ be the map assigning to a germ of a

serration $f : X \dashrightarrow \mathscr{A}$ its value $\eta_x(f) = f(x) \in \mathscr{A}_x$ at x. Then η_x provides a splitting

$$\mathscr{A}_x \underset{\eta_x}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}^0(X;\mathscr{A})_x \underset{\nu_x}{\overset{\partial}{\rightleftarrows}} \mathscr{Z}^1(X;\mathscr{A})_x$$

and hence, generally, a splitting

$$\mathscr{Z}^n(X;\mathscr{A})_x \underset{\eta_x}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}^n(X;\mathscr{A})_x \underset{\nu_x}{\overset{\partial}{\rightleftarrows}} \mathscr{Z}^{n+1}(X;\mathscr{A})_x.$$

Then $d = \varepsilon\partial : \mathscr{C}^n(X;\mathscr{A}) \to \mathscr{C}^{n+1}(X;\mathscr{A})$, and letting

$$D_x = \nu_x\eta_x : \mathscr{C}^{n+1}(X;\mathscr{A})_x \to \mathscr{C}^n(X;\mathscr{A})_x,$$

we have that

$$\begin{cases} D_x d + dD_x = 1 : \mathscr{C}^n(X;\mathscr{A})_x \to \mathscr{C}^n(X;\mathscr{A})_x & \text{for } n > 0, \\ D_x d = 1 - \varepsilon\eta_x : \mathscr{C}^0(X;\mathscr{A})_x \to \mathscr{C}^0(X;\mathscr{A})_x. \end{cases}$$

Since $\eta_x\nu_x = 0$, we have

$$D_x^2 = 0.$$

$T_p(\mathscr{A}^*) = T_p(\mathscr{C}^*(X;\mathscr{A}))$ is also pointwise homotopically trivial, with the homotopy provided by the operator Λ_x defined by

$$\Lambda_x(a_1 \otimes \cdots \otimes a_p) = \varepsilon\eta_x(a_1) \otimes \cdots \otimes \varepsilon\eta_x(a_{i-1}) \otimes D_x(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_p,$$

where $a_1, ..., a_{i-1}$ have degree zero and $\deg(a_i) > 0$. It is easily computed that, as claimed,

$$\begin{cases} \Lambda_x d + d\Lambda_x = 1 & \text{in positive degrees,} \\ \Lambda_x d = 1 - \varepsilon\eta_x & \text{in degree zero,} \\ \Lambda_x^2 = 0, \end{cases}$$

where $\varepsilon\eta_x$ stands for $\varepsilon\eta_x \otimes \cdots \otimes \varepsilon\eta_x$ here. [Note that $\varepsilon = \varepsilon \otimes \cdots \otimes \varepsilon : T_p(\mathscr{A}) \to T_p(\mathscr{A}^*)$ is the augmentation.]

We shall often suppress the variable x in Λ_x and in other operators. Define operators Λ_i on $T_p(\mathscr{A}^*)_x$ inductively by

$$\begin{cases} \Lambda_1 &= \Lambda, \\ \Lambda_{2n} &= \Lambda\tau\Lambda_{2n-1}, \\ \Lambda_{2n+1} &= \Lambda\sigma\Lambda_{2n}. \end{cases}$$

Also define M_i on $T_p(\mathscr{A}^*)_x$ by

$$\begin{cases} M_1 &= \Lambda = \Lambda_1, \\ M_{2n} &= \Lambda\sigma M_{2n-1}, \\ M_{2n+1} &= \Lambda\tau M_{2n}. \end{cases}$$

Since $\Lambda^2 = 0$ we have

$$\Lambda_i\Lambda_j = \Lambda_i M_j = M_i\Lambda_j = M_i M_j = 0. \tag{98}$$

Since $D^2 = 0$, it is clear that Λ_n and M_n are linear combinations of terms of the form $K_1 \otimes K_2 \otimes \cdots \otimes K_p$, where n of the K_i are D and the rest are 1 (or possibly $\varepsilon\eta$). It follows that

$$\Lambda_{p+1} = 0 = M_{p+1}. \tag{99}$$

It is easily shown by induction that for $n \geq 1$, we have

$$\begin{cases} \Lambda_{2n}d &= M_{2n-1}\tau - \tau\Lambda_{2n-1} + d\Lambda_{2n}, \\ \Lambda_{2n+1}d &= M_{2n}\tau + \sigma\Lambda_{2n} - d\Lambda_{2n+1}, \end{cases} \tag{100}$$

where the first equation holds in degrees at least $2n$ and the second in degrees at least $2n+1$. However, another induction shows that for $n \geq 1$, these equations hold in degrees $2n-1$ and $2n$, respectively (using the facts that η commutes with τ and σ in degree zero, that $\eta\Lambda = 0$, and that $\Lambda_n = 0$ in degrees less than n). The equations also hold in degrees less than $2n-1$ and $2n$ respectively, since all terms are zero in that case. Thus (100) is valid for $n \geq 1$ in all degrees. Similarly we obtain, for $n \geq 1$,

$$\begin{cases} M_{2n}d &= \Lambda_{2n-1}\sigma - \sigma M_{2n-1} + dM_{2n}, \\ M_{2n+1}d &= \Lambda_{2n}\sigma + \tau M_{2n} - dM_{2n+1}. \end{cases} \tag{101}$$

We shall now define $h_n : T_p(\mathscr{A}^*) \to \mathscr{B}^*$. Note that to define h_n it suffices to define $\eta_x h_n$ for each $x \in X$. The definition proceeds by induction on n and on the total degree in $T_p(\mathscr{A}^*)$. Define h_0 in degree zero by

$$\eta_x h_0 = h\eta_x.$$

[That is, the *serrations* $a_1, \ldots, a_p$ of $\mathscr{A}$ are taken by h_0 into the serration $x \mapsto h(a_1(x) \otimes \cdots \otimes a_p(x))$.] Note that $\eta_x h_0 \varepsilon \eta_x = h\eta_x \varepsilon \eta_x = h\eta_x = \eta_x \varepsilon h \eta_x$ so that

$$h_0 \varepsilon \eta_x = \varepsilon h \eta_x = \varepsilon \eta_x h_0.$$

Define h_0 in positive degrees, inductively, by

$$\eta_x h_0 = \partial h_0 \Lambda.$$

More generally we define, by double induction,

$$\begin{cases} \eta_x h_{2n} &= \sum\limits_{i=0}^{p-1} (-1)^i \partial h_{2n-i} M_{i+1}, \\ \eta_x h_{2n-1} &= \sum\limits_{i=1}^{p} (-1)^i \partial h_{2n-i} \Lambda_i. \end{cases} \tag{102}$$

(We could leave the upper limits of summation open because of (99). The reader should notice the formal similarity with (63) and (74).) Note that if N is either M_j or Λ_j then $\eta_x h_n N = 0$ by (102) and (98). Thus

$$\nu_x \partial h_n N = h_n N \quad \text{for } N = M_j \text{ or } N = \Lambda_j, \tag{103}$$

since $\nu_x\partial = 1 - \varepsilon\eta_x$.

To show that the h_n satisfy (59), it suffices to prove that

$$\eta_x(h_{2n}d - dh_{2n}) = \eta_x h_{2n-1}\sigma \tag{104}$$

and

$$\eta_x(h_{2n-1}d + dh_{2n-1}) = \eta_x h_{2n-2}\tau.$$

We shall only prove (104), which is typical. The proof proceeds by double induction as in the definition (102). In degrees less than $2n-1$ both sides of (104) are zero. In degree $2n-1$ (so that the image has degree zero), the left-hand side of (104) is

$$\eta_x h_{2n}d = \sum_{i=0}(-1)^i\partial h_{2n-i}M_{i+1}d = 0$$

by degree. The right-hand side of (104) is

$$\eta_x h_{2n-1}\sigma = \sum_{i=1}(-1)^i\partial h_{2n-i}\Lambda_i\sigma = 0$$

by (102) and degree.

Now, for degrees greater than $2n-1$, it suffices to show that

$$D_x h_{2n}d - D_x dh_{2n} = D_x h_{2n-1}\sigma,$$

since ν_x is one-to-one and $D_x = \nu_x\eta_x$. Using (102), (103), and (101) we have

$$\begin{aligned} D_x h_{2n}d &= \sum_{i=0}(-1)^i h_{2n-i}M_{i+1}d \\ &= h_{2n} - h_{2n}dM_1 + \sum_{j=1} h_{2n-2j}(\Lambda_{2j}\sigma + \tau M_{2j} - dM_{2j+1}) \\ &\qquad - \sum_{j=1} h_{2n-2j+1}(\Lambda_{2j-1}\sigma - \sigma M_{2j-1} + dM_{2j}). \end{aligned}$$

Also, by (102),

$$D_x dh_{2n} = h_{2n} - dD_x h_{2n} = h_{2n} - \sum_{i=0}(-1)^i dh_{2n-i}M_{i+1}.$$

Subtracting and rearranging, we obtain

$$\begin{aligned} D_x h_{2n}d - D_x dh_{2n} &= -\sum_{j=0}(h_{2n-2j}d - dh_{2n-2j})M_{2j+1} \\ &\quad - \sum_{j=1}(h_{2n-2j+1}d + dh_{2n-2j+1})M_{2j} \\ &\quad + \sum_{j=1} h_{2n-2j}(\Lambda_{2j}\sigma + \tau M_{2j}) \\ &\quad - \sum_{j=0} h_{2n-2j-1}(\Lambda_{2j+1}\sigma - \sigma M_{2j+1}). \end{aligned}$$

Using the inductive assumption on the first two terms yields

$$\begin{aligned} D_x h_{2n} d - D_x d h_{2n} &= -\sum_{j=0} h_{2n-2j-1}\sigma M_{2j+1} - \sum_{j=1} h_{2n-2j}\tau M_{2j} \\ &\quad - \sum_{j=0} h_{2n-2j-1}(\Lambda_{2j+1}\sigma - \sigma M_{2j+1}) \\ &\quad + \sum_{j=1} h_{2n-2j}(\Lambda_{2j}\sigma + \tau M_{2j}) \\ &= -\sum_{j=0} h_{2n-2j-1}\Lambda_{2j+1}\sigma + \sum_{j=1} h_{2n-2j}\Lambda_{2j}\sigma \\ &= (\sum_{i=1} (-1)^i h_{2n-i}\Lambda_i)\sigma = D_x h_{2n-1}\sigma. \end{aligned}$$

as was to be shown. [The last equation is obtained by applying ν_x to (102) and using (103).]

We claim that for the h_n we have just constructed,

$$h_n(a_1 \otimes \cdots \otimes a_p) = 0 \quad \text{if } pn > (p-1)\deg(a_1 \otimes \cdots \otimes a_p). \tag{105}$$

This is easily proved by induction using the definition (102). In particular, we have

$$h_n(\Delta a) = 0 \quad \text{if } n > (p-1)\deg(a). \tag{106}$$

Remark: The construction of this special system of h_n was undertaken for one purpose only, the proof of (105), and from now on this property is all we will use. It might be possible to show directly that the h_n defined by the general procedure of 20.2 can be so chosen to possess this property, but it is not clear how to do this. Of course, the present h_n have the added advantage of being homomorphisms into $\mathscr{C}^*(X;\mathscr{B})$ rather than into an injective resolution, and of being natural. This fact could be used to simplify and shorten the discussion of 20.10, but we prefer the method used there.

21.3. It follows directly from (106) that

$$St^j = 0 \quad \text{for} \quad j < 0. \tag{107}$$

Now assume that $\mathscr{A}$ and $\mathscr{B}$ are sheaves of $\mathbb{Z}_p$-modules. By (61), (64), and (74), equation (105) implies that

$$h_{(p-1)(q+1)}(\Delta(da)) = (-1)^{m(q+1)}(m!)dh_{(p-1)q}(\Delta a),$$

where $q = \deg a$, p is odd, and $m = (p-1)/2$. For $p = 2$ we have, similarly,

$$h_{q+1}(da \otimes da) = dh_q(a \otimes a), \quad \text{where} \quad q = \deg a.$$

If p is odd we define

$$t_q = (-1)^{mq(q+1)/2}(m!)^{-q} \in \mathbb{Z}_p,$$

and if $p = 2$, we put $t_q = 1$. Then it follows that the maps

$$\lambda : a \mapsto t_q h_{(p-1)q}(\Delta a)$$

of $\mathscr{A}^q \to \mathscr{B}^q$ commute with the differential. For $q = 0$ this is just $a \mapsto h_0(\Delta a)$, so that λ extends the map $h\Delta : a \mapsto h(\Delta a)$ of $\mathscr{A} \to \mathscr{B}$. Now $h\Delta$ is a homomorphism since $p\mathscr{B} = 0$. Moreover, by (61), (75), or their analogues for $p = 2$, it follows from (105) that λ is a homomorphism. Thus λ is a homomorphism of resolutions extending $h\Delta$, and it follows that $\lambda^* : H^q_\Phi(X;\mathscr{A}) \to H^q_\Phi(X;\mathscr{B})$ is just the homomorphism $(h\Delta)^*$ induced by the coefficient homomorphism $h\Delta$. Consequently,

$$t_q St^0 = (h\Delta)^* : H^q_\Phi(X;\mathscr{A}) \to H^q_\Phi(X;\mathscr{B}). \tag{108}$$

21.4. For the remainder of this section we shall restrict our attention exclusively to sheaves of $\mathbb{Z}_p$-modules.

For $p = 2$ we define the *Steenrod squares* by

$$\boxed{Sq^j = St^j : H^q_\Phi(X;\mathscr{A}) \to H^{q+j}_\Phi(X;\mathscr{B})} \tag{109}$$

since these homomorphisms already possess the properties we desire.

We shall now restrict the discussion to the case of odd p. Taking cognizance of (97) and (108) as well as (89), we define the *Steenrod* pth *powers*

$$\boxed{\wp^r_p : H^q_\Phi(X;\mathscr{A}) \to H^{q+2r(p-1)}_\Phi(X;\mathscr{B})} \tag{110}$$

by

$$\boxed{\wp^r_p = (-1)^r t_q St^{2r(p-1)} = (-1)^r t_q St_{(p-1)(q-2r)}.} \tag{111}$$

By (108) we have

$$\boxed{\wp^0_p = (h\Delta)^* : H^q_\Phi(X;\mathscr{A}) \to H^q_\Phi(X;\mathscr{B}),}$$

which is the map induced by the coefficient homomorphism $h\Delta : \mathscr{A} \to \mathscr{B}$.

According to (62), $St_0(a) = h^*(a^p)$ (a^p denotes the cup pth power of a), and it follows easily from this and (107) that

$$\boxed{\wp^r_p(a) = \begin{cases} h^*(a^p), & \text{if} \quad \deg a = 2r, \\ 0, & \text{if} \quad \deg a < 2r. \end{cases}} \tag{112}$$

[The verification of this uses the fact that $(m!)^2 \equiv (-1)^{(p+1)/2} \pmod p$, which follows easily from Wilson's theorem. Formula (112) is the reason for using the sign $(-1)^r$ in (111).]

Similarly,

$$\boxed{Sq^r(a) = \begin{cases} h^*(a^2), & \text{if} \quad \deg a = r, \\ 0, & \text{if} \quad \deg a < r. \end{cases}}$$

An easy computation using (94) yields the Cartan formula

$$\boxed{\wp^r_p(a \cup b) = \textstyle\sum_{i=0}^r \wp^i_p(a) \cup \wp^{r-i}_p(b)}$$

in the situation of (94), and similarly for the Sq^r.

Trivially, $\wp_p^r$ commutes with the homomorphisms induced by maps (or, generally, cohomomorphisms of coefficient sheaves) in the situation of 20.8. (See the diagram (80).) It also follows from (85) that $\wp_p^r$ commutes with connecting homomorphisms. That is, for $A \subset X$ Φ-taut, the diagram

$$\begin{array}{ccccccccc} \cdots\to & H^q_\Phi(X,A;\mathscr{A}) & \to & H^q_\Phi(X;\mathscr{A}) & \to & H^q_{\Phi\cap A}(A;\mathscr{A}) & \to & H^{q+1}_\Phi(X,A;\mathscr{A}) & \to\cdots \\ & \downarrow{\scriptstyle \wp_p^r} & & \downarrow{\scriptstyle \wp_p^r} & & \downarrow{\scriptstyle \wp_p^r} & & \downarrow{\scriptstyle \wp_p^r} & \\ \cdots\to & H^{q+s}_\Phi(X,A;\mathscr{B}) & \to & H^{q+s}_\Phi(X;\mathscr{B}) & \to & H^{q+s}_{\Phi\cap A}(A;\mathscr{B}) & \to & H^{q+s+1}_\Phi(X,A;\mathscr{B}) & \to\cdots \end{array}$$

commutes, where $s = 2r(p-1)$, and similarly for Sq^r.

Exercises

1. Ⓢ Show that 10.1 need not hold for A open and Φ *not* paracompactifying. In fact, give an example for which $\Phi = cld|X - A$ and $H^*_\Phi(X;\mathscr{B}^X) \neq 0$, where $\mathscr{B}$ is a constant sheaf on A.

2. Ⓢ Let X be a simply ordered set with the order topology and assume that X is compact. Prove that $H^p(X) = 0$ for $p > 0$ with any constant coefficient sheaf. [*Hint*: Use the minimality principle 10.8 and the Mayer-Vietoris sequence.]

3. Ⓢ Show that the "long line"[51] is acyclic with respect to any constant coefficients. [*Hint*: Use the "long interval," a compactified long ray, as a parameter space to define a contracting "long homotopy" of the long line and apply 11.12 and Exercise 2.]

4. Ⓢ Let X be the real line and let $\{\mathscr{A}_\lambda\}$ be a direct system of sheaves on X with $\mathscr{A} = \varinjlim \mathscr{A}_\lambda$. Show, by examples, that the canonical map $\theta : \varinjlim \mathscr{A}_\lambda(X) \to \mathscr{A}(X)$ need be neither one-to-one nor onto.

5. Let Φ be a family of supports on X, and $\{U_\alpha\}$ an upward-directed family of open sets such that each $K \in \Phi$ is contained in some U_α. Then show that
$$H^*_\Phi(X;\mathscr{A}) \approx \varinjlim H^*_{\Phi|U_\alpha}(U_\alpha;\mathscr{A}).$$

6. Ⓢ If X is not normal then show that X contains a closed, nontaut subspace. Give such an example in which X is Hausdorff and the subspace is paracompact. [Thus it does not suffice for A to be paracompact in 9.5.]

7. If $\mathscr{A}$ is a sheaf of L-modules on X, where L is a ring with unit, show that $\mathscr{C}^*(X,A;\mathscr{A})$ is a $\mathscr{C}^0(X;L)$-module and hence that it is Φ-fine for Φ paracompactifying.

8. Call A *hereditarily Φ-taut* in X if $A \cap U$ is Φ-taut in X for every open $U \subset X$. [Then $A \cap U$ is $(\Phi \cap U)$-taut in U.] Prove that if A is hereditarily Φ-taut and $B \subset A$, then B is Φ-taut in $X \Leftrightarrow B$ is $(\Phi \cap A)$-taut in A.

[51]If Ω is the set of countable ordinal numbers as a well-ordered set, then the *long ray* Y is $\Omega \times [0,1)$ with the dictionary order and the order topology. Then 0×0 is the least element of Y. The *long line* X is two copies of Y with their least elements identified. It is a nonparacompact topological 1-manifold.

9. Ⓢ Define the "one-point paracompactification" X^+ of a space X with a given paracompactifying family of supports Φ for which $E(\Phi) = X$, and show that $H^*_\Phi(X;\mathscr{A}) \approx H^*(X^+;\mathscr{A}^{X^+})$.

10. Ⓢ If $\mathscr{A}$ is a sheaf on X such that $\mathscr{A}|U_x$ is flabby for some open neighborhood U_x of each $x \in X$, then show that $\mathscr{A}$ is flabby.

11. Ⓢ For paracompactifying Φ, $F \subset X$ closed and $U = X - F$, show that

$$\dim_{\Phi,L} X = \max\{\dim_{\Phi|U,L} U,\ \dim_{\Phi|F,L} F\}.$$

12. Ⓢ Let Φ be paracompactifying. Show that a sheaf $\mathscr{A}$ is Φ-fine $\Leftrightarrow$ for every $K \in \Phi$ and neighborhood U of K there is an endomorphism $\mathscr{A} \to \mathscr{A}$ that is 1 on K and 0 outside U.

13. Let Φ be paracompactifying. Show that a sheaf $\mathscr{A}$ is Φ-soft $\Leftrightarrow$ for every $K \in \Phi$ and $s \in \mathscr{A}(K)$ and for every locally finite covering $\{U_\alpha\}$ of K in X there exist elements $s_\alpha \in \mathscr{A}(X)$ with $|s_\alpha| \subset U_\alpha$ and with $s = (\sum s_\alpha)|K$ [i.e., $s(x) = \sum s_\alpha(x)$ for $x \in K$]. Also show that $\mathscr{A}$ is Φ-fine $\Leftrightarrow$ there exists a partition of unity subordinate to any locally finite covering $\{U_\alpha\}$ of X containing (at least) one member of the form $X - K$ where $K \in \Phi$ [i.e., there exist endomorphisms $h_\alpha \in \mathscr{H}om(\mathscr{A},\mathscr{A})$ with $|h_\alpha| \subset U_\alpha$ and $\sum h_\alpha = 1$].

14. Ⓢ (a) Let X and Y be locally compact Hausdorff and let $\mathscr{A}$ and $\mathscr{B}$ be c-fine sheaves on X and Y respectively. Show that $\mathscr{A}\widehat{\otimes}\mathscr{B}$ is also c-fine. [*Hint*: By taking one-point compactifications, reduce this to the compact case. If $K \subset X \times Y$ and W is a neighborhood of K, let $\{U_\alpha\}$ and $\{V_\beta\}$ be finite coverings of X and Y respectively such that $\{U_\alpha \times V_\beta\}$ refines $\{W, X - K\}$. Apply Exercises 12 and 13.]

    (b) If $\mathscr{A}$ and $\mathscr{B}$ are c-soft sheaves on the locally compact Hausdorff spaces X and Y respectively and if $\mathscr{A}\widehat{*}\mathscr{B} = 0$, then show that $\mathscr{A}\widehat{\otimes}\mathscr{B}$ is c-soft. [*Hint*: Consider the collection of all open sets $W \subset X \times Y$ such that $(\mathscr{A}\widehat{\otimes}\mathscr{B})_W$ is c-acyclic, and use the Künneth Theorem 15.2, the Mayer-Vietoris sequence (27), continuity 14.5, and 16.1.]

15. Ⓢ If W is the one-point compactification of the locally compact Hausdorff space X and if $i : X \hookrightarrow W$ is the inclusion, show that $i\mathscr{A}$ is soft for any c-soft sheaf $\mathscr{A}$ on X.

16. If L is a ring with unit and $\mathscr{I}$ is an injective sheaf of L-modules on X, show that $\mathscr{I}(X)$ is an injective L-module. [*Hint*: Map X into a point.]

17. If $\mathscr{I}$ is an injective sheaf on X, show that the sheaf $\mathscr{H}om(\mathscr{A},\mathscr{I})$ is flabby for any sheaf $\mathscr{A}$ on X. (For $\mathscr{A} = \mathscr{I}$, it follows that $\mathscr{I}$ is Φ-fine for any paracompactifying family Φ of supports.)

18. Ⓢ If the derived functors $H^p_{\Phi|X-A}(X;\bullet)$ of $H^0_\Phi(X,A;\bullet) = \Gamma_{\Phi|X-A}(\bullet)$ fit in an exact "cohomology sequence of a pair"

$$\cdots \to H^p_{\Phi|X-A}(X;\mathscr{A}) \to H^p_\Phi(X;\mathscr{A}) \to H^p_{\Phi\cap A}(A;\mathscr{A}|A) \to H^{p+1}_{\Phi|X-A}(X;\mathscr{A}) \to \cdots,$$

then show that A is Φ-taut in X.

19. Let (A, B) be a $\Phi\cap\Psi$-excisive pair of subspaces of X and assume that A is Φ-taut, B is Ψ-taut, and $A \cup B$ is $(\Phi\cap\Psi)$-taut. Show that, through 12.1, the cup product (33) on page 100 coincides with that of 7.1.

20. Let A and B be closed subspaces of X. Show that, through 12.3, the cup product (33) on page 100 coincides with that of 7.1.

21. Ⓢ Show that for a sheaf $\mathscr{L}$ on the space X, the following four conditions are equivalent:

    (a) $\mathscr{L}$ is flabby.

    (b) $\mathscr{L}$ is Φ-acyclic for all families Φ of supports on X.

    (c) $H^1_\Phi(X;\mathscr{L}) = 0$ for all Φ.

    (d) $H^1(X,U;\mathscr{L}) = 0$ for all open sets $U \subset X$.

22. Define $\operatorname{Dim} X$ to be the least integer n (or ∞) such that $H^k_\Phi(X;\mathscr{A}) = 0$ for all $k > n$, all sheaves $\mathscr{A}$ on X, and all Φ; i.e., $\operatorname{Dim} X = \sup_\Phi \dim_{\Phi,\mathbb{Z}} X$. Show that the following three statements are equivalent:

    (a) $\operatorname{Dim} X \leq n$.

    (b) $\mathscr{Z}^n(X;\mathscr{A})$ is flabby for all $\mathscr{A}$.

    (c) Every sheaf $\mathscr{A}$ on X has a flabby resolution of length n.

23. Ⓢ Let X be the subspace of the unit interval $[0,1]$ consisting of the points $\{0\}$ and $\{1/n\}$ for integral $n > 0$. Show that $\dim_{\mathbb{Z}} X = 0$ but $\operatorname{Dim} X = 1$.

24. If every point of X has an open neighborhood U with $\operatorname{Dim} U \leq n$, show that $\operatorname{Dim} X \leq n$. [*Hint*: Use Exercises 10 and 22.]

25. Ⓢ If X is locally hereditarily paracompact, show that $\operatorname{Dim} X \leq \dim_{\mathbb{Z}} X + 1$. By 16.28, a topological n-manifold M^n has $\dim_{\mathbb{Z}} M^n = n$, and thus $\operatorname{Dim} M^n$ is either n or $n+1$. Show that in fact, $\operatorname{Dim} M^n = n+1$ for $n \geq 1$. This latter fact is due to Satya Deo [31].

26. Let $f : X \to Y$ be a closed surjection between locally paracompact spaces. Assume that X is second countable and that each $x \in X$ has a neighborhood N such that $f|N : N \to f(N)$ is a homeomorphism. Then show that $\dim_L Y = \dim_L X$.

27. Ⓢ Define a topology on the set $\mathbb{N}$ of positive integers by taking the sets $U_n = \{1, 2, ..., n\}$ to be the only open sets, together with the whole space and the empty set. Show that every sheaf on $\mathbb{N}$ is equivalent to an inverse system of abelian groups based on the directed set $\mathbb{N}$. If $\mathscr{A}$ is a sheaf on $\mathbb{N}$, show that $C^0(U_n;\mathscr{A})$ is isomorphic to the set of all n-tuples $\langle a_1, ..., a_n\rangle$, where $a_i \in \mathscr{A}_i = \mathscr{A}(U_i)$ and where the restriction to U_m $(m < n)$ takes $\langle a_1, ..., a_n\rangle$ into $\langle a_1, ..., a_m\rangle$. Show that $Z^1(U_n;\mathscr{A}) = \mathscr{Z}^1(\mathbb{N};\mathscr{A})(U_n) \approx C^0(U_{n-1};\mathscr{A})$ with the induced restriction map and such that the differential $C^0(U_n;\mathscr{A}) \to Z^1(U_n;\mathscr{A})$ takes $\langle a_1, ..., a_n\rangle$ into $\langle a_1 - \pi_2 a_2, ..., a_{n-1} - \pi_n a_n\rangle$, where π_n denotes the restriction $\mathscr{A}(U_n) \to \mathscr{A}(U_{n-1})$. Show that $\dim_{cld,\mathbb{Z}} \mathbb{N} = 1$. Show that $H^0(\mathbb{N};\mathscr{A}) = \varprojlim \mathscr{A}_n$, so that $H^1(\mathbb{N};\mathscr{A})$ is the right derived functor $\varprojlim{}^1 \mathscr{A}_n$ of the inverse limit functor. Show that $\mathscr{A}$ is flabby $\Leftrightarrow$ each $\pi_n : \mathscr{A}_n \to \mathscr{A}_{n-1}$ is surjective, and that $\mathscr{A}$ is acyclic $\Leftrightarrow$ for every system $\{a'_i \in \mathscr{A}_i\}$ the system of equations

$$\pi_2 a_2 = a_1 - a'_1,$$
$$\pi_3 a_3 = a_2 - a'_2,$$
$$\cdots$$

has a solution for the a_i.

Also show that an inverse sequence $\mathscr{A} = \{\mathscr{A}_1, \mathscr{A}_2, \dots\}$ is acyclic if it satisfies the "Mittag-Leffler condition" that given i, there is a $j \geq i$ such that $\operatorname{Im}(\mathscr{A}_j \to \mathscr{A}_i) = \operatorname{Im}(\mathscr{A}_k \to \mathscr{A}_i)$ for all $k \geq j$.

28. Ⓢ If M is a torsion-free L-module, where L is an arbitrary ring, show that $H^1_\Phi(X; M)$ is torsion-free over L for any space X and support family Φ. [*Hint*: Use I-Exercise 11.] Give an example of a torsion-free sheaf $\mathscr{A}$ on $X = \mathbb{S}^1$ such that $H^1(X; \mathscr{A})$ is not torsion-free.

29. Ⓢ Give an example of a sequence $\{\mathscr{L}_i\}$ of flabby sheaves on the unit interval such that $\varinjlim \mathscr{L}_i$ is not flabby.

30. A *Zariski space* is a space satisfying the descending chain condition on closed subsets (such as an algebraic variety with the Zariski topology). If $\{\mathscr{L}_\lambda\}$ is a direct system of sheaves on a Zariski space X, show that the natural map
$$\varinjlim H^*(X; \mathscr{L}_\lambda) \to H^*(X; \mathscr{L})$$
is an isomorphism, where $\mathscr{L} = \varinjlim \mathscr{L}_\lambda$. Also show that $\mathscr{L}$ is flabby if each $\mathscr{L}_\lambda$ is flabby.

31. Ⓢ A Zariski space is called *irreducible* if it is not the union of two proper closed subspaces. The *Zariski dimension* $Z\text{-}\dim X$ of a space X is the least integer n (or ∞) such that every chain $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_p \neq \varnothing$ of closed irreducible subspaces of X has "length" $p \leq n$. Show that $\dim_{cld,\mathbb{Z}} X \leq Z\text{-}\dim X$. [*Hint*: Prove an anologue of 16.14 and show that if X is irreducible then all constant sheaves on X are flabby.]

32. A differential sheaf $\mathscr{L}^*$ on X is said to be *homotopically Φ-fine*, where Φ is a paracompactifying family, if for every locally finite covering $\{U_\alpha\}$ with one member of the form $X - K$ for some $K \in \Phi$, there are endomorphisms $h_\alpha \in \operatorname{Hom}(\mathscr{L}^*, \mathscr{L}^*)$ of degree zero with $|h_\alpha| \subset U_\alpha$ and such that $h = \sum h_\alpha$ is chain homotopic to the identity. [That is, there exist homomorphisms $D : \mathscr{L}^* \to \mathscr{L}^*$ of degree -1 with $1 - h = dD + Dd$ (d being the differential on $\mathscr{L}^*$).] If this holds for every locally finite covering of X, we merely say that $\mathscr{L}^*$ is *homotopically fine*.

(a) If $\mathscr{L}^*$ is homotopically Φ-fine, show that $H^*(H^p_\Phi(X; \mathscr{L}^*)) = 0$ for all $p > 0$. [*Hint*: If $\sigma \in C^p_\Phi(X; \mathscr{L}^*)$, cover X by $\{U_\alpha\}$ such that $U_{\alpha_0} = X - |s|$ and $\overline{U}_\alpha \in \Phi$ for $\alpha \neq \alpha_0$, and such that $s|U_\alpha$ is a coboundary, say of $t_\alpha \in C^{p-1}(U_\alpha; \mathscr{L}^*)$, with $t_{\alpha_0} = 0$. Let $t = \sum h^*_\alpha(t_\alpha)$. Show that $d't = h^*(s)$, where d' is the differential of $C^*_\Phi(X; \bullet)$, so that $h^*(\sigma) = 0$ in $H^p_\Phi(X; \mathscr{L}^*)$, and finish the proof.]

(b) Show that the sheaf Δ_* of singular chains defined in I-Exercise 12 is homotopically fine. [*Hint*: Use the generalized operation of subdivision defined in [38, pp. 207–208] on the defining presheaf.]

33. Ⓢ Let L be a principal ideal domain. Let us say that an L-module M has property **F** if for each finite set $\{a_1, \dots, a_n\} \subset M$,
$$N = \{a \in M \mid ka = \sum_{i=1}^{n} k_i a_i \quad \text{for some} \quad k_i \in L \quad \text{and} \quad 0 \neq k \in L\}$$
is *free*. Prove the following statements:

(a) If M is countably generated and has property **F**, then M is free.

(b) If X is locally compact Hausdorff, then $H_c^0(X;L)$ has property **F**. [*Hint*: Reduce this to the compact case and consider reduced cohomology. Cover X by closed sets $K_1, ..., K_k$ with $a_j|K_i \in \tilde{H}^0(K_i;L)$ trivial for all i,j.]

(c) If X is locally compact, Hausdorff, and locally connected, then $H_c^1(X;L)$ has property **F**. [*Hint*: Cover X by the interiors of sets $K_1, ..., K_k$ with each K_i compact for $i > 1$ and with K_1 a neighborhood of ∞. Prove by induction on r that if $\{D_i\}$ is a closed covering of X with $D_i \subset \operatorname{int} K_i$, then the image of N in $H_c^1(D_1 \cup \cdots \cup D_r; L)$ is finitely generated, and hence free by Exercise 28.]

(d) If X is locally compact and separable metric, then $H_c^*(X;L)$ is countably generated. [*Hint*: Use continuity.]

34. Ⓢ If S is a set, let $B(S)$ be the additive group of all bounded functions $f : S \to \mathbb{Z}$. If X is a space, let $C(X)$ be the group of all continuous functions $f : X \to \mathbb{Z}$ where $\mathbb{Z}$ has the discrete topology. Prove the following statements:

(a) $H^0(X;\mathbb{Z}) \approx C(X) \approx {}_{AS}H^0(X;\mathbb{Z})$; generalize this to arbitrary constant coefficient groups.

(b) $B(S) \approx C(\hat{S})$, where $\hat{S}$ denotes the Stone-Čech compactification of the discrete space S.

(c) The following two statements are equivalent:

i. $B(S)$ is free for every S with $\operatorname{card}(S) \leq \eta$.

ii. $C(X)$ is free for every compact Hausdorff space X with a dense set of cardinality $\leq \eta$.

(d) The same as part (c) with "free" replaced by "$\operatorname{Ext}(\bullet, \mathbb{Z}) = 0$."

Remark: Nöbeling [63] has proved that $B(S)$ is free for *all* S (previously, Specker [77] had done this for countable S) and so we conclude that $H_c^0(X;\mathbb{Z})$ is free for all locally compact Hausdorff X. The Universal Coefficient Theorem II-15.3 and Exercise 28 then imply that the same holds for an arbitrary coefficient ring L.

35. Let L be a Dedekind domain (i.e., a domain in which every ideal is projective). Show that 16.12 remains true if condition (c) is replaced by the following two conditions:

(c′) $\mathscr{A} \oplus \mathscr{B} \in \mathfrak{S} \Rightarrow$ both $\mathscr{A}$ and $\mathscr{B}$ are in $\mathfrak{S}$.

(c″) $L_U \in \mathfrak{S}$ for all open sets $U \subset X$.

Thus extend 16.31, 16.32, and 16.33 to Dedekind domains. [*Hint*: Use the known fact that every ideal of a Dedekind domain is finitely generated.]

36. Let L be a principal ideal domain. If $\mathscr{A}$ is an L-module or a sheaf of L-modules and $p \in L$ is a prime, we say that $\mathscr{A}$ has p-torsion if multiplication by $p : \mathscr{A} \to \mathscr{A}$ is not a monomorphism. Let $T(\mathscr{A})$ denote the set of all primes $p \in L$ such that $\mathscr{A}$ has p-torsion. Prove the following statements:

(a) $T(\mathscr{A}) = \bigcup_{x \in X} T(\mathscr{A}_x)$.

(b) $T(\mathscr{A}\mathbin{\widehat{*}}\mathscr{B}) = T(\mathscr{A}) \cap T(\mathscr{B})$. (Prove this first for modules and then generalize to sheaves).

(c) If F is a covariant left exact functor (from sheaves to sheaves or sheaves to modules) that preserves the L-module operations, then

$$T(F(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)) \subset \bigcap_{i=1}^{n} T(\mathscr{A}_i).$$

[Note the cases: $F(\mathscr{A}, \mathscr{B}) = C^p_\Phi(X; \mathscr{A}) * C^q_\Psi(Y; \mathscr{B}))$ and $F(\mathscr{A}, \mathscr{B}) = \Gamma_\Phi(\mathscr{A} \otimes \mathscr{L}) * \Gamma_\Psi(\mathscr{B} \otimes \mathscr{N})$ when $\mathscr{L}$ and $\mathscr{N}$ are torsion-free.]

(d) $T(\mathscr{A} \otimes \mathscr{B}) \subset T(\mathscr{A}) \cup T(\mathscr{B})$.

[*Hint for* (b): If $p \in T(A)$, there is a monomorphism $L_p = L/pL \to A$. Moreover, $L_p * B = \operatorname{Ker}\{p : B \to B\}$.]

37. Ⓢ Give an example of a compact, totally disconnected Hausdorff space X and a locally closed subspace $A \subset X$ such that the canonical map $\varinjlim \mathscr{A}(U) \to \mathscr{A}(A)$, where U ranges over the neighborhoods of A, is not surjective for *any* constant sheaf $\mathscr{A} \neq 0$ on X.

38. Ⓢ If $A \subset X$ and $\mathscr{A}$ is a sheaf on X that is concentrated on $\bar{A}$, then show that $H^p_\Phi(X, A; \mathscr{A}) \approx H^p_{\Phi \cap \bar{A}}(\bar{A}, A; \mathscr{A})$.

39. Ⓢ Let $A \subset X$ and let Φ be a family of supports on X such that each $K \in \Phi \cap \bar{A}$ is in the interior of A *in* $\bar{A}$ (i.e., K has a neighborhood N with $N \cap \bar{A} \subset A$). Show that $C^*_{\Phi \cap \bar{A}}(\bar{A}, A; \mathscr{A}) = 0$ for all sheaves $\mathscr{A}$ on X. If $\mathscr{A}$ is concentrated on $\bar{A}$, show that the restriction $H^*_\Phi(X; \mathscr{A}) \to H^*_{\Phi \cap A}(A; \mathscr{A})$ is an isomorphism. [Note that the conditions on Φ are satisfied when A is locally closed and $\Phi \cap \bar{A} = \Phi \cap A$, and each member of Φ has a neighborhood in Φ.]

40. Ⓢ Let $T = \mathbb{R}/\mathbb{Z}$. Let X be any space and let $\mathscr{F}$ (respectively, $\mathscr{F}_0$) denote the sheaf of germs of continuous functions from X to $\mathbb{R}$ (respectively, T). Show that there is an exact sequence

$$0 \to \mathbb{Z} \to \mathscr{F} \xrightarrow{j} \mathscr{F}_0 \to 0$$

of sheaves, where j is induced by the canonical surjection $\mathbb{R} \to T$. Conclude that if X is paracompact, then $H^1(X; \mathbb{Z})$ is isomorphic to the group $[X; T]$ of homotopy classes of maps from X to T. Generalize this to arbitrary paracompactifying families of supports on X.

41. Ⓢ Let $\mathscr{H}^*(X, A; \mathscr{A})$ be the derived sheaf of $\mathscr{C}^*(X, A; \mathscr{A})$, and let $\mathscr{A}$ be constant.

(a) Show that $\mathscr{H}^0(X, A; \mathscr{A}) \approx \mathscr{A}_{X - \bar{A}}$.

(b) Show, by example, that $\mathscr{H}^q(X, A; \mathscr{A})$ need not be zero for $q > 0$.

(c) If $\mathscr{H}^q(X, A; \mathscr{A}) = 0$ for $q > 0$, show that there is a natural isomorphism $H^*_\Phi(X, A; \mathscr{A}) \approx H^*_\Phi(X; \mathscr{A}_{X - \bar{A}})$ for any Φ.

(d) If there exists an isomorphism $H^*(U, A \cap U; \mathscr{A}) \approx H^*(U; \mathscr{A}_{U - \bar{A}})$ that is natural for open sets $U \subset X$, show that $\mathscr{H}^q(X, A; \mathscr{A}) = 0$ for $q > 0$.

42. Let $\mathscr{A}^*$ and $\mathscr{B}^*$ be differential sheaves on the spaces X and Y respectively. Suppose that $\mathscr{A}^p \widehat{*} \mathscr{B}^q = 0$ for all p, q. Show that there is a natural exact sequence

$$0 \to \bigoplus_{p+q=n} \mathscr{H}^p(\mathscr{A}^*)\widehat{\otimes}\mathscr{H}^q(\mathscr{B}^*) \to \mathscr{H}^n(\mathscr{A}^*\widehat{\otimes}\mathscr{B}^*) \to \bigoplus_{p+q=n+1} \mathscr{H}^p(\mathscr{A}^*)\widehat{*}\mathscr{H}^q(\mathscr{B}^*) \to 0$$

of sheaves on $X \times Y$. In particular, show that $\mathscr{A}^*\widehat{\otimes}\mathscr{B}^*$ is a resolution of $\mathscr{A}\widehat{\otimes}\mathscr{B}$ if $\mathscr{A}^*$ and $\mathscr{B}^*$ are resolutions of $\mathscr{A}$ and $\mathscr{B}$ respectively such that $\mathscr{A}^p \widehat{*} \mathscr{B}^q = 0$ for all p, q. [*Hint*: Consider the algebraic Künneth formula for the double complex $\mathscr{A}^*(U) \otimes \mathscr{B}^*(V)$, where U and V are open in X and Y respectively.]

43. Let Φ be a paracompactifying family of supports on X. If $\{\mathscr{L}_\lambda\}$ is any family of Φ-soft sheaves on X, show that $\prod_\lambda \mathscr{L}_\lambda$ is Φ-soft.

44. Let X be a locally paracompact space and let F be the fixed point set of a homeomorphism of period p on X, where p is a prime. Let $L = \mathbb{Z}_p$ and assume that $\dim_L X < \infty$. If X is clc_L^∞, show that F is also clc_L^∞.

45. Ⓢ Let $\mathscr{L}$ be a sheaf on the Hausdorff space X such that $\mathscr{L}_U$ is flabby for all open sets $U \subset X$. Show that $|s|$ is discrete for all $s \in \mathscr{L}(X)$.

46. Ⓢ If X is a nondiscrete Hausdorff space and if $M \neq 0$ is a constant sheaf on X, show that there does *not* exist a sheaf $\mathscr{L}$ on X containing M as a subsheaf and such that $\mathscr{L} \otimes \mathscr{A}$ is Φ-acyclic for all families Φ of supports and all sheaves $\mathscr{A}$ on X.

47. Let

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{A}^* & \xrightarrow{i} & \mathscr{B}^* & \xrightarrow{j} & \mathscr{C}^* & \to & 0 \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \to & \mathscr{L}^* & \xrightarrow{i'} & \mathscr{M}^* & \xrightarrow{j'} & \mathscr{N}^* & \to & 0 \end{array}$$

be a diagram of differential sheaves (or of ordinary chain complexes) such that the rows are exact and the square is homotopy commutative, i.e., there is a homomorphism $D : \mathscr{B}^* \to \mathscr{N}^*$ of degree -1 such that

$$j'g - hj = dD + Dd.$$

Define a homomorphism $f^* : \mathscr{H}^*(\mathscr{A}^*) \to \mathscr{H}^*(\mathscr{L}^*)$ as follows: If $a \in \mathscr{A}^*$ with $da = 0$ and if $D(i(a)) = j'(m)$, put

$$f^*[a] = [(i')^{-1}(g(i(a)) - dm)],$$

where square brackets denote homology classes. Show that the induced diagram

$$\begin{array}{ccccccccc} \cdots \to \mathscr{H}^p(\mathscr{A}^*) & \xrightarrow{i^*} & \mathscr{H}^p(\mathscr{B}^*) & \xrightarrow{j^*} & \mathscr{H}^p(\mathscr{C}^*) & \xrightarrow{\delta} & \mathscr{H}^{p+1}(\mathscr{A}^*) \to \cdots \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle f^*} \\ \cdots \to \mathscr{H}^p(\mathscr{L}^*) & \xrightarrow{i'^*} & \mathscr{H}^p(\mathscr{M}^*) & \xrightarrow{j'^*} & \mathscr{H}^p(\mathscr{N}^*) & \xrightarrow{\delta} & \mathscr{H}^{p+1}(\mathscr{L}^*) \to \cdots \end{array}$$

is commutative.

Similarly, suppose that

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{A}^* & \xrightarrow{i} & \mathscr{B}^* & \xrightarrow{j} & \mathscr{C}^* & \to & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ 0 & \to & \mathscr{L}^* & \xrightarrow{i'} & \mathscr{M}^* & \xrightarrow{j'} & \mathscr{N}^* & \to & 0 \end{array}$$

has exact rows and homotopy commutative square, that is,

$$gi - i'f = dD' + D'd, \quad \text{where} \quad D' : \mathscr{A}^* \to \mathscr{M}^*.$$

Define $h^* : \mathscr{H}^*(\mathscr{C}^*) \to \mathscr{H}^*(\mathscr{N}^*)$ as follows: If $b \in \mathscr{B}^*$ and $db = i(a)$, put

$$h^*[j(b)] = [j'(g(b) - D'(a))].$$

Again, show that the diagram above of derived sheaves is commutative.

Also show that these maps (f^* in the first case and h^* in the second) depend only on the homotopy class of D or of D'. Suppose that f, g, and h are all given such that both squares are homotopy commutative. Assume that there is a homomorphism $J : \mathscr{A}^* \to \mathscr{N}^*$, of degree -2, such that $j'D' - Di = dJ - Jd$. Then in the first case above, show that f^* is induced by f and in the second case, that h^* is induced by h.

48. Use the fact that $\mathscr{C}^*(X; \bullet)$ is naturally pointwise homotopically trivial (see Section 2) to show that for any differential sheaf $\mathscr{A}^*$, the inclusion $\varepsilon : \mathscr{A}^* \to \mathscr{C}^*(X; \mathscr{A}^*)$ is a pointwise homotopy equivalence. That is, for each $x \in X$ there are natural homomorphisms

$$\begin{cases} \eta_x : \mathscr{C}^*(X; \mathscr{A}^*)_x \to \mathscr{A}^*_x & \text{of degree zero,} \\ D_x : \mathscr{C}^*(X; \mathscr{A}^*)_x \to \mathscr{C}^*(X; \mathscr{A}^*)_x & \text{of degree } -1, \end{cases}$$

with

$$\eta_x \varepsilon = 1 \quad \text{and} \quad 1 - \varepsilon \eta_x = dD_x + D_x d.$$

Thus, for example, $\mathscr{C}^*(X; \mathscr{A}^*)$ is a resolution of $\mathscr{A}$ if $\mathscr{A}^*$ is.

49. Let $f : \mathscr{A}^* \to \mathscr{B}^*$ be a homomorphism of differential sheaves. (We note the special case in which these are ordinary chain complexes with differentials of degree $+1$, for example, the induced map $\Gamma_\Phi(\mathscr{A}^*) \to \Gamma_\Phi(\mathscr{B}^*)$.) Define the *mapping cone* of f to be the differential sheaf $\mathscr{M}^*_f$, where

$$\mathscr{M}^p_f = \mathscr{A}^{p+1} \oplus \mathscr{B}^p$$

with differential given by $d(a, b) = (-da, f(a) + db)$. [Note that $\Gamma_\Phi(\mathscr{M}^*_f)$ is the mapping cone of $f : \Gamma_\Phi(\mathscr{A}^*) \to \Gamma_\Phi(\mathscr{B}^*)$.] Consider the exact sequences

$$0 \to \mathscr{B}^p \xrightarrow{i} \mathscr{M}^p_f \xrightarrow{j} \mathscr{A}^{p+1} \to 0,$$

where $i(b) = (0, b)$ and $j(a, b) = a$. Show that the connecting homomorphism $\delta : \mathscr{H}^p(\mathscr{A}^*) \to \mathscr{H}^p(\mathscr{B}^*)$ of this sequence is just f^*. Note that j *anticommutes* with d.

Suppose now that f is one-one and let $\mathscr{C}^* = \operatorname{Coker} f$, so that the sequence $0 \to \mathscr{A}^* \xrightarrow{f} \mathscr{B}^* \xrightarrow{g} \mathscr{C}^* \to 0$ is exact. Let $\delta' : \mathscr{H}^p(\mathscr{C}^*) \to \mathscr{H}^{p+1}(\mathscr{A}^*)$ be the connecting homomorphism of this sequence. Consider

the homomorphism $h : \mathscr{M}_f^p \to \mathscr{C}^p$ given by $h(a,b) = g(b)$. Show that the diagram

$$\begin{array}{ccccccccc}
\cdots \to \mathscr{H}^p(\mathscr{A}^*) & \xrightarrow{\delta} & \mathscr{H}^p(\mathscr{B}^*) & \xrightarrow{i^*} & \mathscr{H}^p(\mathscr{M}_f^*) & \xrightarrow{j^*} & \mathscr{H}^{p+1}(\mathscr{A}^*) \to \cdots \\
\downarrow 1 & +1 & \downarrow 1 & +1 & \downarrow h^* & -1 & \downarrow 1 \\
\cdots \to \mathscr{H}^p(\mathscr{A}^*) & \xrightarrow[f^*]{} & \mathscr{H}^p(\mathscr{B}^*) & \xrightarrow[g^*]{} & \mathscr{H}^p(\mathscr{C}^*) & \xrightarrow[\delta']{} & \mathscr{H}^{p+1}(\mathscr{A}^*) \to \cdots
\end{array}$$

commutes up to the sign that is indicated. In particular, h^* is an isomorphism.

[Note that if $\mathscr{A}^*$ is Φ-acyclic then passing to sections, we obtain that $h^* : H^p(\Gamma_\Phi(\mathscr{M}_f^*)) \to H^p(\Gamma_\Phi(\mathscr{C}^*))$ is an isomorphism by applying the above results to the case of ordinary chain complexes and using the exactness of $0 \to \Gamma_\Phi(\mathscr{A}^*) \to \Gamma_\Phi(\mathscr{B}^*) \to \Gamma_\Phi(\mathscr{C}^*) \to 0$.]

50. Let $\mathscr{A}$ be a torsion-free sheaf and consider the multiplication $n : \mathscr{A} \to \mathscr{A}$, where $n \neq 0$ is some fixed integer. Let $\mathscr{A}^*$ be a given Φ-acyclic resolution of $\mathscr{A}$ (not necessarily torsion-free) and let $M_n(\mathscr{A}^*)$ denote the mapping cone of $n : \mathscr{A}^* \to \mathscr{A}^*$ (see Exercise 49). Show that there is a natural isomorphism $h^* : H^p(\Gamma_\Phi(M_n(\mathscr{A}^*))) \xrightarrow{\approx} H^p_\Phi(X;\mathscr{A}_n)$, where $\mathscr{A}_n = \mathscr{A}/n\mathscr{A}$, such that the diagram

$$\begin{array}{ccccccccc}
\cdots \to H^p(\Gamma_\Phi(\mathscr{A}^*)) & \xrightarrow{\delta} & H^p(\Gamma_\Phi(\mathscr{A}^*)) & \xrightarrow{i^*} & H^p(\Gamma_\Phi(M_n(\mathscr{A}^*))) & \xrightarrow{j^*} & H^p(\Gamma_\Phi(\mathscr{A}^*)) \to \cdots \\
\downarrow & +1 & \downarrow & +1 & \downarrow h^* & -1 & \downarrow \\
\cdots \to H^p_\Phi(X;\mathscr{A}) & \xrightarrow[n]{} & H^p_\Phi(X;\mathscr{A}) & \xrightarrow[\rho]{} & H^p_\Phi(X;\mathscr{A}_n) & \xrightarrow[\beta]{} & H^{p+1}_\Phi(X;\mathscr{A}) \to \cdots
\end{array}$$

commutes up to the signs indicated. [Here the top row is as in Exercise 49, the vertical maps $H^*(\Gamma_\Phi(\mathscr{A}^*)) \to H^*_\Phi(X;\mathscr{A})$ are the canonical isomorphisms of 5.15, and the bottom sequence is the cohomology sequence of the coefficient sequence $0 \to \mathscr{A} \xrightarrow{n} \mathscr{A} \to \mathscr{A}_n \to 0$.]

[*Hint*: Compare $\mathscr{A}^*$ with an injective resolution and then compare $\mathscr{C}^*(X;\mathscr{A})$ with the same injective resolution and apply Exercise 49.]

51. Ⓢ Let $\mathscr{A}^*$ and $\mathscr{B}^*$ be Φ-acyclic resolutions, of $\mathbb{Z}_2$-modules, of $\mathbb{Z}_2$ on X. Suppose that we are given a homomorphism $\theta : \mathscr{A}^* \otimes \mathscr{A}^* \to \mathscr{B}^*$ of differential sheaves which is symmetric [i.e., $\theta(a \otimes b) = (-1)^{(\deg a)(\deg b)}\theta(b \otimes a)$] and which extends the multiplication map $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \to \mathbb{Z}_2$. Then show that $H^q_\Phi(X;\mathbb{Z}_2) = 0$ for all $q > 0$. Moreover, if Φ is paracompactifying with $E(\Phi) = X$, and if $\mathscr{A}^*$ and $\mathscr{B}^*$ are Φ-soft, then show that $\dim_{\mathbb{Z}_2} X = 0$.

52. Let p be a prime and let $\mathscr{L}^*$ be a sheaf of differential algebras over $\mathbb{Z}_p$ on a space X (i.e., $\mathscr{L}^*$ is a differential sheaf of $\mathbb{Z}_p$-modules and we are given a homomorphism $\theta : \mathscr{L}^* \otimes \mathscr{L}^* \to \mathscr{L}^*$ of differential sheaves preserving degree and *associative*). Suppose that $\mathscr{L}^*$ is a Φ-acyclic resolution of $\mathbb{Z}_p$, that θ is (signed) commutative, and that θ extends multiplication $\mathbb{Z}_p \otimes \mathbb{Z}_p \to \mathbb{Z}_p$. Then show that $H^n_\Phi(X;\mathbb{Z}_p) = 0$ for all $n > 0$. If, moreover, Φ is paracompactifying with $E(\Phi) = X$, and $\mathscr{L}^*$ is Φ-soft, then show that $\dim_{\mathbb{Z}_p} X = 0$.

53. Ⓢ Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on a Hausdorff space X, let $\Phi \subset c$ (but X need not be locally compact), and let $q > 0$. For $\alpha \in H^p_\Phi(X;\mathscr{A})$ and

$\beta \in H^q(X;\mathscr{B})$, show that the cup product $\alpha\beta^n = 0$ for some integer n. [*Hint*: Φ may as well be taken to be $c|K$ for some *compact* set $K \subset X$. Then n may be chosen to depend only on K and β. Use the fact that β is "locally zero."]

54. Ⓢ Let G be a compact Lie group acting on the paracompact space X. Show that over any base ring L, $\dim_L X/G \le \dim_L X$. [*Hint*: Use the fact that every orbit $G(x)$ has a "tube" about it, that is, an invariant neighborhood N with an equivariant retraction $r : N \to G(x)$. Also, the set $S = r^{-1}(x)$ (called a "slice at x") induces a homeomorphism $S/G_x \xrightarrow{\approx} N/G$.]

55. Ⓢ Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be an exact sequence of sheaves on the Hausdorff space X and assume that $\mathscr{A}'$ and $\mathscr{A}''$ are locally constant. Assume one of the following three conditions:

 (a) X is locally compact Hausdorff and $clc^2_{\mathbb{Z}}$.

 (b) X is $clc^2_{\mathbb{Z}}$ and the stalks of $\mathscr{A}'$ are finitely generated.

 (c) X is $clc^1_{\mathbb{Z}}$ and the stalks of $\mathscr{A}'$ are finitely generated and free.

 Then show that $\mathscr{A}$ is locally constant. (Compare I-Exercise 9.) Also, give an example showing that X being locally compact and clc^1 is not sufficient for the conclusion.

56. Ⓢ If X is a locally compact Hausdorff space and $\dim_{\mathbb{Z}} X = n$, then show that $\dim_{\mathbb{Z}} X \times X$ is $2n$ or $2n-1$. (This is due to I. Fary.)

57. Ⓢ Call a space X *rudimentary* if each point $x \in X$ has a minimal neighborhood. (The prototype is the topology on the positive integers given by Exercise 27.) If $\{\mathscr{L}_\lambda \mid \lambda \in \Lambda\}$ is an arbitrary family of sheaves on the rudimentary space X, then show that

$$\boxed{H^p(X;\textstyle\prod_\lambda \mathscr{L}_\lambda) \approx \prod_\lambda H^p(X;\mathscr{L}_\lambda).}$$

58. Ⓢ Let X be a rudimentary space. For $x \in X$ let U_x be the minimal neighborhood of x. If $A \subset X$, then show that A is rudimentary. If $x \in A \subset U_x$, then show that all sheaves on A are acyclic and that A is taut in X.

59. Ⓢ Let $\mathbb{M} = \mathbb{N} \times \mathbb{N}$ where $\mathbb{N}$ is as in Exercise 27. Then a sheaf $\mathscr{A}$ on $\mathbb{M}$ is equivalent to the double inverse system

$$\{A_{i,j} \mid \pi_{i,j} : A_{i,j} \to A_{i-1,j};\ \varpi_{i,j} : A_{i,j} \to A_{i,j-1};\ \pi_{i,j-1}\varpi_{i,j} = \varpi_{i-1,j}\pi_{i,j}\}.$$

 Show that $\mathscr{A}$ admits a flabby resolution $0 \to \mathscr{A} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \mathscr{A}^2 \to 0$, whence

$$\varprojlim\nolimits^n_{i,j} A_{i,j} = H^n(\mathbb{M};\mathscr{A}) = 0 \quad \text{for} \quad n > 2.$$

 Show, however, that for the inverse sequence $\mathscr{A}|\Delta = \{A_{1,1} \leftarrow A_{2,2} \leftarrow \cdots\}$ we have

$$\varprojlim\nolimits^n_{i,j} A_{i,j} \approx \varprojlim\nolimits^n_k A_{k,k},$$

 and hence that $\varprojlim^2_{i,j} A_{i,j} = 0$ also.

60. Let X be locally paracompact and let $\{F_\alpha\}$ be a locally finite closed covering of X with $\dim_L F_\alpha \le n$. Then show that $\dim_L X \le n$.

61. Let K be a totally disconnected compact Hausdorff space. If $\mathscr{A}$ is a sheaf on K with $\Gamma(\mathscr{A}) = 0$, then show that $\mathscr{A} = 0$.

Chapter III

Comparison with Other Cohomology Theories

We return in this chapter to the classical singular, Alexander-Spanier, de Rham, and Čech cohomology theories. It is shown that under suitable restrictions, these theories are equivalent to sheaf-theoretic cohomology. Homomorphisms induced by maps, cup products, and relative cohomology are also discussed at some length. In Section 3 the direct natural transformation between singular theory and de Rham theory, which is important in the applications, is considered.

Throughout this chapter L will denote a given principal ideal domain, which will be taken as the base ring. All sheaves are to be sheaves of L-modules; tensor products are over L; and so on.

Most of this chapter can be read after Section 9 of Chapter II.

1 Singular cohomology

For the moment, let X be a fixed space and let $\mathscr{S}^* = \mathscr{S}^*(X;L)$, in the notation from I-7. For a sheaf $\mathscr{A}$ on X we define the singular cohomology groups of X with coefficients in $\mathscr{A}$ and supports in the paracompactifying family Φ by

$$\boxed{{}_SH^p_\Phi(X;\mathscr{A}) = H^p(\Gamma_\Phi(\mathscr{S}^* \otimes \mathscr{A})).} \tag{1}$$

For $\mathscr{A}$ locally constant with stalks G, we also have the functors ${}_\Delta H^*_\Phi(X;\mathscr{A}) = H^*(\Gamma_\Phi(\mathscr{S}^*(X;\mathscr{A})))$ of Chapter I, which coincide with classical singular theory when Φ is paracompactifying. There is, for $\mathscr{A}$ locally constant, the homomorphism

$$\mu : S^*(U;L) \otimes \mathscr{A}(U) \to S^*(U;\mathscr{A}|U)$$

given by $\mu(f \otimes g)(\sigma) = f(\sigma) \cdot \sigma^*_U(g) \in \Gamma(\sigma^*\mathscr{A})$, where $\sigma^*_U : \mathscr{A}(U) \to (\sigma^*\mathscr{A})(\sigma^{-1}U) = \Gamma(\sigma^*\mathscr{A})$ is the map (4) on page 12. Over an open set U on which $\mathscr{A}$ is constant, this is just the canonical map

$$S^*(U;L) \otimes G = \mathrm{Hom}(\Delta_*(U), L) \otimes G \to \mathrm{Hom}(\Delta_*(U), G) = S^*(U;G).$$

Since $\Delta_*(U)$ is free, this is obviously an isomorphism when G is finitely generated. Therefore

$${}_SH^*_\Phi(X;\mathscr{A}) \approx {}_\Delta H^*_\Phi(X;\mathscr{A})$$

when $\mathscr{A}$ is locally constant with finitely generated stalks. This is not true if the stalks are not finitely generated; see Exercise 7. However, for a general locally constant sheaf $\mathscr{A}$ the map μ is a map of differential presheaves, and so it induces a map of differential sheaves

$$\bar{\mu} : \mathscr{S}^*(X;L) \otimes \mathscr{A} \to \mathscr{S}^*(X;\mathscr{A})$$

and hence a natural map

$$\boxed{\mu^* : {}_SH^*_\Phi(X;\mathscr{A}) \to {}_\Delta H^*_\Phi(X;\mathscr{A})} \tag{2}$$

to which we will return later.

The cup product of singular cochains $S^p(U;L) \otimes S^q(U;L) \to S^{p+q}(U;L)$ induces a product $\mathscr{S}^p \otimes \mathscr{S}^q \to \mathscr{S}^{p+q}$, which is a homomorphism of differential sheaves when $\mathscr{S}^* \otimes \mathscr{S}^*$ is given the total degree and differential. Thus $\mathscr{S}^0$ is a sheaf of rings with unit and each $\mathscr{S}^p$ (and hence $\mathscr{S}^p \otimes \mathscr{A}$) is an $\mathscr{S}^0$-module.

The sheaf $\mathscr{S}^0$ is clearly the same sheaf as $\mathscr{C}^0(X;L)$ and hence is flabby, and consequently is Φ-fine for any paracompactifying family Φ. Thus each $\mathscr{S}^p \otimes \mathscr{A}$ is also Φ-fine. Similarly, $\mathscr{S}^*(X;\mathscr{A})$ is Φ-fine for Φ paracompactifying and $\mathscr{A}$ locally constant.

Now, the stalks of $\mathscr{S}^p$ are torsion-free so that $\mathscr{S}^p \otimes \mathscr{A}$ and $\Gamma_\Phi(\mathscr{S}^p \otimes \mathscr{A})$ are *exact* functors of $\mathscr{A}$ when Φ is paracompactifying. Thus $\mathscr{A} \mapsto {}_SH^*_\Phi(X;\mathscr{A})$ is a connected sequence of functors. The sequence

$$0 \to \Gamma_\Phi(\mathscr{A}) \xrightarrow{\varepsilon} \Gamma_\Phi(\mathscr{S}^0 \otimes \mathscr{A}) \xrightarrow{d} \Gamma_\Phi(\mathscr{S}^1 \otimes \mathscr{A}) \xrightarrow{d} \cdots$$

is of order two (see II-1). Hence ε takes $\Gamma_\Phi(\mathscr{A}) = H^0_\Phi(X;\mathscr{A})$ into $\mathrm{Ker}(d) = {}_SH^0_\Phi(X;\mathscr{A})$, yielding a natural transformation of functors

$$H^0_\Phi(X;\mathscr{A}) \to {}_SH^0_\Phi(X;\mathscr{A}).$$

Thus, by II-6.2, we obtain a unique extension (compatible with connecting homomorphisms)

$$\boxed{\theta : H^*_\Phi(X;\mathscr{A}) \to {}_SH^*_\Phi(X;\mathscr{A}).}$$

If X is *HLC*, then by II-1, $\mathscr{S}^*$ is a *resolution* of L, and since $\mathscr{S}^*$ is torsion free, $\mathscr{S}^* \otimes \mathscr{A}$ is a resolution of $\mathscr{A}$ for any $\mathscr{A}$. By II-4.3 it follows that ${}_SH^p_\Phi(X;\mathscr{A}) = 0$ for $p > 0$ and $\mathscr{A}$ flabby and also that θ is an isomorphism in degree zero for Φ paracompactifying. Thus, if X is *HLC* and Φ is paracompactifying, then ${}_SH^*_\Phi(X;\mathscr{A})$ is a *fundamental* connected sequence of functors, whence θ is an isomorphism by II-6.2.[1]

We shall be concerned with the properties of θ for *general* X, with Φ paracompactifying.

[1]By II-6.2 the inverse of θ is the homomorphism ρ of II-5.15.

Again, for X general and Φ paracompactifying, the singular cup product induces a chain map

$$(\mathscr{S}^p \otimes \mathscr{A}) \otimes (\mathscr{S}^q \otimes \mathscr{B}) \to \mathscr{S}^{p+q} \otimes (\mathscr{A} \otimes \mathscr{B})$$

and hence a homomorphism

$$\cup : {}_SH^p_\Phi(X;\mathscr{A}) \otimes {}_SH^q_\Psi(X;\mathscr{B}) \to {}_SH^{p+q}_{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B})$$

satisfying properties analogous to those in II-7.1.

We claim that θ preserves these products. That is, if Φ and Ψ are paracompactifying, then the two maps

$$H^*_\Phi(X;\mathscr{A}) \otimes H^*_\Psi(X;\mathscr{B}) \to {}_SH^*_{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B})$$

defined by $\alpha\otimes\beta \mapsto \theta(\alpha\cup\beta)$ and $\alpha\otimes\beta\mapsto\theta(\alpha)\cup\theta(\beta)$, coincide. To check this in degree zero, note that the obvious commutative diagram

$$\begin{array}{ccc} L\otimes L & \xrightarrow[\approx]{\times} & L \\ \Big\downarrow{\scriptstyle \varepsilon\otimes\varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathscr{S}^0\otimes\mathscr{S}^0 & \xrightarrow{\cup} & \mathscr{S}^0 \end{array}$$

induces, upon tensoring with $\mathscr{A}\otimes\mathscr{B}$, the commutative diagram

$$\begin{array}{ccc} L\otimes\mathscr{A}\otimes L\otimes\mathscr{B} & \xrightarrow[\approx]{} & L\otimes\mathscr{A}\otimes\mathscr{B} \\ \Big\downarrow{\scriptstyle \varepsilon\otimes1\otimes\varepsilon\otimes1} & & \Big\downarrow{\scriptstyle \varepsilon\otimes1\otimes1} \\ \mathscr{S}^0\otimes\mathscr{A}\otimes\mathscr{S}^0\otimes\mathscr{B} & \longrightarrow & \mathscr{S}^0\otimes\mathscr{A}\otimes\mathscr{B} \end{array}$$

and this induces the commutative diagram

$$\begin{array}{ccccc} \Gamma_\Phi(\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) & \xrightarrow{\cup} & \Gamma_{\Phi\cap\Psi}(\mathscr{A}\otimes\mathscr{B}) & \xrightarrow{=} & \Gamma_{\Phi\cap\Psi}(\mathscr{A}\otimes\mathscr{B}) \\ \Big\downarrow{\scriptstyle \varepsilon\otimes\varepsilon} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\ \Gamma_\Phi(\mathscr{S}^0\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{S}^0\otimes\mathscr{B}) & \to & \Gamma_{\Phi\cap\Psi}(\mathscr{S}^0\otimes\mathscr{A}\otimes\mathscr{S}^0\otimes\mathscr{B}) & \to & \Gamma_{\Phi\cap\Psi}(\mathscr{S}^0\otimes\mathscr{A}\otimes\mathscr{B}) \end{array}$$

in which the composition along the bottom is the degree zero cup product. By the definition of θ this implies that $\theta(\alpha\cup\beta)=\theta(\alpha)\cup\theta(\beta)$ in degree zero. The contention then follows from II-6.2 applied to the two natural transformations $\alpha\otimes\beta\mapsto\theta(\alpha\cup\beta)$ and $\alpha\otimes\beta\mapsto\theta(\alpha)\cup\theta(\beta)$.

Now let $f : X \to Y$ be a map and let Φ and Ψ be paracompactifying families on X and Y respectively, with $\Phi \supset f^{-1}\Psi$. Then f induces a homomorphism $S^*(U;L)\to S^*(f^{-1}(U);L)$, $U\subset Y$ open and therefore an f-cohomomorphism of the induced sheaves

$$f^* : \mathscr{S}^*(Y;L) \rightsquigarrow \mathscr{S}^*(X;L).$$

Tensoring this with the f-cohomomorphism $\mathscr{A}\rightsquigarrow f^*\mathscr{A}$ for a sheaf $\mathscr{A}$ on Y, we obtain a functorial f-cohomomorphism

$$\mathscr{S}^*(Y;L)\otimes\mathscr{A} \rightsquigarrow \mathscr{S}^*(X;L)\otimes f^*\mathscr{A}.$$

This induces, in turn, $f^* : \Gamma_\Psi(\mathscr{S}^*(Y;L)\otimes\mathscr{A}) \to \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes f^*\mathscr{A})$, which extends the canonical map $\Gamma_\Psi(\mathscr{A}) \to \Gamma_\Phi(f^*\mathscr{A})$. (That is, we have compatibility with the augmentations $\mathscr{A} \rightarrowtail \mathscr{S}^*(X;L)\otimes\mathscr{A}$ and $f^*\mathscr{A} \rightarrowtail \mathscr{S}^*(X;L)\otimes f^*\mathscr{A}$.) This gives the commutative diagram

$$\begin{array}{ccccc} 0 \to & \Gamma_\Psi(\mathscr{A}) & \xrightarrow{\varepsilon} & \Gamma_\Psi(\mathscr{S}^0(Y;L)\otimes\mathscr{A}) \\ & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ 0 \to & \Gamma_\Phi(f^*\mathscr{A}) & \xrightarrow{\varepsilon} & \Gamma_\Phi(\mathscr{S}^0(X;L)\otimes f^*\mathscr{A}), \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} H^0_\Psi(Y;\mathscr{A}) & \xrightarrow{\theta} & {}_SH^0_\Psi(Y;\mathscr{A}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^0_\Phi(X;f^*\mathscr{A}) & \xrightarrow{\theta} & {}_SH^0_\Phi(X;f^*\mathscr{A}). \end{array}$$

Consider the diagram

$$\begin{array}{ccc} H^*_\Psi(Y;\mathscr{A}) & \xrightarrow{\theta} & {}_SH^*_\Psi(Y;\mathscr{A}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^*_\Phi(X;f^*\mathscr{A}) & \xrightarrow{\theta} & {}_SH^*_\Phi(X;f^*\mathscr{A}). \end{array}$$

We claim that this diagram commutes. This is so in degree zero as just shown. Considering the upper left and lower right parts of the diagram as connected sequences of functors of $\mathscr{A}$, it is easily seen that both compositions $f^*\circ\theta$ and $\theta\circ f^*$ commute with connecting homomorphisms. Thus the contention follows from II-6.2.

Now we wish to study relationships with subspaces. Let $F \subset X$ be closed and put $U = X - F$. Let Φ be a paracompactifying family of supports on X.

The exact sequence $0 \to \mathscr{A}_U \to \mathscr{A} \to \mathscr{A}_F \to 0$ induces an exact sequence $0 \to \mathscr{S}^*(X;L)\otimes\mathscr{A}_U \to \mathscr{S}^*(X;L)\otimes\mathscr{A} \to \mathscr{S}^*(X;L)\otimes\mathscr{A}_F \to 0$. But $\mathscr{S}^*(X;L)\otimes\mathscr{A}_U \approx (\mathscr{S}^*(X;L)\otimes\mathscr{A})_U$. Therefore, also, $\mathscr{S}^*(X;L)\otimes\mathscr{A}_F \approx (\mathscr{S}^*(X;L)\otimes\mathscr{A})_F$.

The restriction homomorphism $S^*(V;L) \to S^*(V\cap F;L)$, for $V \subset X$ open, induces an epimorphism $(\mathscr{S}^*(X;L)\otimes\mathscr{A})|F \twoheadrightarrow \mathscr{S}^*(F;L)\otimes\mathscr{A}$ of sheaves on F. The kernel of this is an $\mathscr{S}^0(X;L)|F$-module and hence is $\Phi|F$-soft. Thus we have an epimorphism

$$\Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}_F) = \Gamma_{\Phi|F}((\mathscr{S}^*(X;L)\otimes\mathscr{A})|F) \twoheadrightarrow \Gamma_{\Phi|F}(\mathscr{S}^*(F;L)\otimes\mathscr{A}|F)$$

of chain complexes. The map $\Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}) \to \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}_F)$ is also onto. Define the chain complex $K^*_\Phi(X,F;\mathscr{A})$ to be the kernel of the epimorphism $\Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}) \twoheadrightarrow \Gamma_{\Phi|F}(\mathscr{S}^*(F;L)\otimes\mathscr{A}|F)$.

We define the *relative* singular cohomology with coefficients in the sheaf $\mathscr{A}$ by

$$\boxed{{}_SH^*_\Phi(X,F;\mathscr{A}) = H^*(K^*_\Phi(X,F;\mathscr{A})).}$$

For $\mathscr{A}$ locally constant we have the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & K^*_\Phi(X,F;\mathscr{A}) & \to & \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}) & \to & \Gamma_{\Phi|F}(\mathscr{S}^*(F;L)\otimes\mathscr{A}|F) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & S^*_\Phi(X,F;\mathscr{A}) & \to & S^*_\Phi(X;\mathscr{A}) & \to & S^*_{\Phi|F}(F;\mathscr{A}) & \to 0 \end{array}$$

in which the two vertical maps on the right are induced by $\bar{\mu}$ and the one on the left is forced by commutativity and exactness of the rows. The 5-lemma shows that the vertical map on the left induces a homology isomorphism if both the $\bar{\mu}$ maps do.

To relate these relative groups to the absolute groups and to the sheaf-theoretic cohomology groups, we consider the homomorphisms θ for the sheaves $\mathscr{A}_U$, $\mathscr{A}$, $\mathscr{A}_F$, and the following diagram of chain groups:

$$\begin{array}{ccccccccc} 0 \to & \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}_U) & \to & \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}) & \to & \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}_F) & \to 0 \\ & \downarrow & & \downarrow{=} & & \downarrow{f} & \\ 0 \to & K^*_\Phi(X,F;\mathscr{A}) & \to & \Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}) & \to & \Gamma_{\Phi|F}(\mathscr{S}^*(F;L)\otimes\mathscr{A}|F) & \to 0. \end{array}$$

This induces the following commutative diagram:

$$\begin{array}{ccccccccc} & H^p_{\Phi|U}(U;\mathscr{A}|U) & & & & H^p_{\Phi|F}(F;\mathscr{A}|F) & \\ & \| & & & & \| & \\ \cdots \to & H^p_\Phi(X;\mathscr{A}_U) & \to & H^p_\Phi(X;\mathscr{A}) & \to & H^p_\Phi(X;\mathscr{A}_F) & \to \cdots \\ & \downarrow{\theta} & & \downarrow{\theta} & & \downarrow{\theta} & \\ \cdots \to & {}_SH^p_\Phi(X;\mathscr{A}_U) & \to & {}_SH^p_\Phi(X;\mathscr{A}) & \to & {}_SH^p_\Phi(X;\mathscr{A}_F) & \to \cdots \\ & \downarrow & & \downarrow{=} & & \downarrow{f^*} & \\ \cdots \to & {}_SH^p_\Phi(X,F;\mathscr{A}) & \to & {}_SH^p_\Phi(X;\mathscr{A}) & \to & {}_SH^p_{\Phi|F}(F;\mathscr{A}|F) & \to \cdots \end{array}$$

The bottom row, for $\mathscr{A} = L$, is just the singular cohomology sequence of the pair (X,F). The row of vertical maps marked "θ" are isomorphisms if X is *HLC*. The composition of vertical maps on the right is the "θ" map for F by the uniqueness part of II-6.2. Also, f^*, and hence each of the vertical maps on the bottom row, is an isomorphism if X and F are both *HLC*, as also follows from II-6.2. Note that $(\mathscr{S}^*(X;L)\otimes\mathscr{A})|U \approx \mathscr{S}^*(U;L)\otimes\mathscr{A}$ and $\Gamma_\Phi(\mathscr{S}^*(X;L)\otimes\mathscr{A}_U) = \Gamma_\Phi((\mathscr{S}^*(X;L)\otimes\mathscr{A})_U) = \Gamma_{\Phi|U}((\mathscr{S}^*(X;L)\otimes\mathscr{A})|U) = \Gamma_{\Phi|U}(\mathscr{S}^*(U;L)\otimes\mathscr{A})$, so that ${}_SH^p_\Phi(X;\mathscr{A}_U) = {}_SH^p_{\Phi|U}(U;\mathscr{A}|U)$.

It follows that if X and F are both *HLC*, then all vertical maps in the last diagram are isomorphisms, and in particular, the relative singular cohomology of (X,F) "depends" only on $X-F=U$;[2] i.e., invariance under relative homeomorphisms is satisfied. Of course, this is false without the *HLC* condition on both X and F.

Since CW-complexes are locally contractible, and hence *HLC*, we now know that sheaf-theoretic cohomology agrees with singular cohomology (including cup products) on pairs of CW-complexes. (This also follows from

[2]Strictly speaking, it also depends on $\Phi|U$, but in the most important case of compact supports on locally compact spaces, $\Phi|U = c$ is also intrinsic to U.

uniqueness theorems for cohomology theories on CW-pairs, but the present result is stronger because it covers the case of twisted coefficients.)

For *HLC* pairs and constant coefficients, one can do much more using the methods employed at the end of the next section; see 2.1.

If X is *HLC*, $\mathscr{A}$ is locally constant, and Φ is paracompactifying, then $\mathscr{S}^*(X;L)\otimes\mathscr{A}$ and $\mathscr{S}^*(X;\mathscr{A})$ are both Φ-acyclic resolutions of $\mathscr{A}$, and it follows that μ^* of (2) is an isomorphism. Since μ preserves cup products,[3] so does μ^*. Therefore, for X *HLC*, $\mathscr{A}$ locally constant, and Φ paracompactifying, we have the natural multiplicative isomorphisms

$$H^*_\Phi(X;\mathscr{A}) \underset{\approx}{\xrightarrow{\theta}} {}_SH^*_\Phi(X;\mathscr{A}) \underset{\approx}{\xleftarrow{\mu^*}} {}_\Delta H^*_\Phi(X;\mathscr{A}).$$

We summarize our discussion in the following theorem.

1.1. Theorem. *There exist the natural multiplicative transformations of functors (of X as well as of $\mathscr{A}$)*

$$H^*_\Phi(X;\mathscr{A}) \xrightarrow{\theta} {}_SH^*_\Phi(X;\mathscr{A}) \xleftarrow{\mu^*} {}_\Delta H^*_\Phi(X;\mathscr{A})$$

in which the groups ${}_\Delta H^*_\Phi(X;\mathscr{A})$, and hence μ^*, are defined only for locally constant $\mathscr{A}$ and are the classical singular cohomology groups when Φ is paracompactifying. The map μ^* is an isomorphism when $\mathscr{A}$ has finitely generated stalks. Both θ and μ^* are isomorphisms when X is HLC and Φ is paracompactifying. Both natural transformations extend to closed pairs of spaces with the same conclusions. □

1.2. Example. Let X be the union of the 2-spheres of radius $1/n$ all tangent to the xy-plane at the origin x_0. (See II-10.10.) Let W be the same point set with the CW-topology. Then the identity map $(W,x_0)\to(X,x_0)$ is a relative homeomorphism, since the complement is a countable topological sum of 2-planes in both cases. However, by the foregoing results, ${}_\Delta H^n(W,x_0;\mathbb{Z})\approx H^n(W,x_0;\mathbb{Z})\approx\bigoplus_{i=1}^\infty H^n_c(\mathbb{R}^2;\mathbb{Z})$ vanishes for $n>2$. On the other hand, ${}_\Delta H^n(X,x_0;\mathbb{Z})\approx{}_\Delta\tilde{H}^n(X;\mathbb{Z})$ is nonzero for arbitrarily large n, as shown in [3]. ◇

1.3. Example. Let X be the Cantor set. Then by continuity II-14.6 we have $H^0(X;\mathbb{Z})\approx\varinjlim\mathbb{Z}^{2^n}$, which is countable. On the other hand, ${}_\Delta H^0(X;\mathbb{Z})\approx\operatorname{Hom}(H_0(X;\mathbb{Z}),\mathbb{Z})\approx\operatorname{Hom}(\bigoplus_X\mathbb{Z},\mathbb{Z})\approx\prod_X\mathbb{Z}$, which is uncountable. ◇

Another such example is the "topologist's sine curve" of II-10.9.

[3]This follows directly from the definition of the cup product in singular theory.

2 Alexander-Spanier cohomology

Put $\mathscr{A}^* = \mathscr{A}^*(X;L)$ as defined in I-7. For any sheaf $\mathscr{B}$ on X we define the Alexander-Spanier cohomology with coefficients in $\mathscr{B}$ and supports in Φ by

$$\boxed{{}_AH^*_\Phi(X;\mathscr{B}) = H^*(\Gamma_\Phi(\mathscr{A}^*\otimes\mathscr{B})).} \tag{3}$$

When $\mathscr{B} = L$ and Φ is paracompactifying, this coincides, by I-7, with the classical Alexander-Spanier cohomology. This is also true for $\mathscr{B} = G$ constant and Φ paracompactifying, as we shall see presently.

By II-1, $\mathscr{A}^*$ is a resolution of L, and $\mathscr{A}^*$ is torsion free. Thus, $\mathscr{A}^*\otimes\mathscr{B}$ is a resolution of $\mathscr{B}$ and is an *exact* functor of $\mathscr{B}$. The cup product $A^p(U;L)\otimes A^q(U;L)\to A^{p+q}(U;L)$, defined by $(f\cup g)(x_0,\dots,x_{p+q}) = f(x_0,\dots,x_p)g(x_p,\dots,x_{p+q})$, induces a product $\mathscr{A}^p\otimes\mathscr{A}^q\to\mathscr{A}^{p+q}$ (a homomorphism of differential sheaves) and also $(\mathscr{A}^p\otimes\mathscr{B})\otimes(\mathscr{A}^q\otimes\mathscr{C})\to\mathscr{A}^{p+q}\otimes(\mathscr{B}\otimes\mathscr{C})$. Thus $\mathscr{A}^0$ is a sheaf of rings with unit, and each $\mathscr{A}^p\otimes\mathscr{B}$ is an $\mathscr{A}^0$-module. Now $\mathscr{A}^0\approx\mathscr{C}^0(X;L)$, and hence it is flabby. Thus, when Φ is paracompactifying, $\mathscr{A}^p\otimes\mathscr{B}$ is Φ-fine.

Since $\mathscr{A}^*\otimes\mathscr{B}$ is a resolution of $\mathscr{B}$, we have the homomorphism

$$\boxed{\rho: {}_AH^*_\Phi(X;\mathscr{B})\to H^*_\Phi(X;\mathscr{B})} \tag{4}$$

of II-5.15.

If Φ is paracompactifying, then $\mathscr{A}^*\otimes\mathscr{B}$ is Φ-fine, so that ρ is an isomorphism. (Its inverse is the analogue of the map θ of Section 1.) For Φ paracompactifying, a development similar to that of Section 1 can be made. It is somewhat simpler since θ is now always an isomorphism.

In particular, for $F\subset X$ closed, $U = X - F$, and Φ paracompactifying, there is the natural commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots\to & H^p_{\Phi|U}(U;\mathscr{B}) & \to & H^p_\Phi(X;\mathscr{B}) & \to & H^p_{\Phi|F}(F;\mathscr{B}) & \to\cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots\to & {}_AH^p_\Phi(X,F;\mathscr{B}) & \to & {}_AH^p_\Phi(X;\mathscr{B}) & \to & {}_AH^p_{\Phi|F}(F;\mathscr{B}) & \to\cdots
\end{array}$$

where the relative Alexander-Spanier cohomology is defined to be the homology of the kernel of the canonical surjection

$$\Gamma_\Phi(\mathscr{A}^*(X;L)\otimes\mathscr{B})\to\Gamma_{\Phi|F}(\mathscr{A}^*(F;L)\otimes\mathscr{B}).$$

If $f: U^{p+1}\to L$ is an Alexander-Spanier p-cochain on the open set $U\subset X$ and if $\sigma:\Delta_p\to U$ is a singular p-simplex in U, put $(\lambda f)(\sigma) = f(\sigma(e_0),\dots,\sigma(e_p))$, where the e_i are the vertices of the standard p-simplex Δ_p. This defines a natural map $\lambda: A^*(U;L)\to S^*(U;L)$, which induces a homomorphism $\mathscr{A}^*\to\mathscr{S}^*$ of differential sheaves. This, in turn, induces a chain map $\Gamma_\Phi(\mathscr{A}^*\otimes\mathscr{B})\to\Gamma_\Phi(\mathscr{S}^*\otimes\mathscr{B})$ and hence a natural homology homomorphism

$$\boxed{\lambda^*: {}_AH^*_\Phi(X;\mathscr{B})\to {}_SH^*_\Phi(X;\mathscr{B}).}$$

When Φ is paracompactifying, the compatibility properties of λ^*, together with II-6.2, imply that $\lambda^* = \theta\rho$, where θ is as in Section 1 and ρ is as in (4).

We now turn to the case of *constant coefficients* in the L-module G. There is a natural map

$$A^*(U; L) \otimes G \to A^*(U; G).$$

Note that when G is finitely generated, this is already an isomorphism. This induces a homomorphism $\mathscr{A}^* \otimes G \to \mathscr{A}^*(X; G)$. Also, $\mathscr{A}^*(X; G)$ is an $\mathscr{A}^0(X; L)$-module and is a resolution of G. We have the maps

$$\Gamma_\Phi(\mathscr{A}^* \otimes G) \to \Gamma_\Phi(\mathscr{A}^*(X; G)) \leftarrow \bar{A}^*_\Phi(X; G),$$

where $\bar{A}^*_\Phi = A^*_\Phi / A^*_0$. When Φ is paracompactifying, the first map induces an isomorphism in homology by II-4.2, while the second map is already an isomorphism by I-7. Thus, for Φ paracompactifying and G constant,

$$\boxed{H^*_\Phi(X; G) \approx {}_AH^*_\Phi(X; G) \approx {}_{AS}H^*_\Phi(X; G).}$$

Moreover, these isomorphisms preserve the cup product, the first by the functorial method used in Section 1 (or by II-7.5) and the second by its definition.

Now suppose that $B \subset X$ is arbitrary. Let $\bar{A}^*_\Phi(X, B; G)$ be the kernel of the canonical surjection $\bar{A}^*_\Phi(X; G) \twoheadrightarrow \bar{A}^*_{\Phi \cap B}(B; G)$. The relative Alexander-Spanier cohomology group is defined by

$$\boxed{{}_{AS}H^p_\Phi(X, B; G) = H^p(\bar{A}^*_\Phi(X, B; G)).}$$

For $U \subset X$ open, the restriction $A^*(U; G) \to A^*(B \cap U; G)$ is onto and hence, passing to the limit over neighborhoods U of $x \in B$, the induced map $\mathscr{A}^*(X; G)_x \to \mathscr{A}^*(B; G)_x$ is also onto. Thus the canonical cohomomorphism $\mathscr{A}^*(X; G) \rightsquigarrow A^*(B; G)$ is surjective in the sense of II-12. We have the commutative diagram

$$\begin{array}{ccc} \bar{A}^*_\Phi(X; G) & \longrightarrow & \bar{A}^*_{\Phi \cap B}(B; G) \\ \downarrow & & \downarrow \\ \Gamma_\Phi(\mathscr{A}^*(X; G)) & \xrightarrow{r^*} & \Gamma_{\Phi \cap B}(\mathscr{A}^*(B; G)) \end{array}$$

in which the top map is always onto and the vertical maps are isomorphisms provided that Φ is paracompactifying. Thus $\operatorname{Ker} r^* \approx \bar{A}^*_\Phi(X, B; G)$, and by II-12.17,

$$\boxed{H^p_\Phi(X, B; G) \approx {}_{AS}H^p_\Phi(X, B; G)}$$

when Φ and $\Phi \cap B$ are both paracompactifying.

In a similar manner, when X and B are both *HLC*, $\mathscr{A}$ is locally constant, and Φ and $\Phi \cap B$ are both paracompactifying, one can show that

$$\boxed{H^p_\Phi(X, B; \mathscr{A}) \approx {}_\Delta H^p_\Phi(X, B; \mathscr{A}) \approx {}_SH^p_\Phi(X, B; \mathscr{A}).}$$

We summarize the main results:

2.1. Theorem. *There exist the natural multiplicative transformations of functors (of X as well as of $\mathscr{A}$)*

$$\begin{array}{ccccc} {}_{AS}H^*_\Phi(X;\mathscr{A}) & \xrightarrow{\mu^*} & {}_{A}H^*_\Phi(X;\mathscr{A}) & \xrightarrow{\rho} & H^*_\Phi(X;\mathscr{A}) \\ & & \downarrow{\scriptstyle\lambda^*} & & \\ & & {}_{S}H^*_\Phi(X;\mathscr{A}) & & \end{array}$$

in which the groups ${}_{AS}H^*_\Phi(X;\mathscr{A})$, and hence μ^*, are defined only for constant $\mathscr{A}$ and are the classical Alexander-Spanier cohomology groups. The map μ^* is an isomorphism when $\mathscr{A} = L$ or when Φ is paracompactifying. The map ρ is an isomorphism when Φ is paracompactifying. The map λ^* is an isomorphism when X is HLC and Φ is paracompactifying. All three natural transformations extend to closed pairs of spaces with the same conclusions. The isomorphism $\rho\mu^*$ extends to arbitrary pairs (X, B) if Φ and $\Phi \cap B$ are paracompactifying and $\mathscr{A}$ is constant. The isomorphism $\lambda^*\rho^{-1}$ extends to arbitrary HLC pairs (X, B) when Φ and $\Phi \cap B$ are paracompactifying and $\mathscr{A}$ is locally constant. □

3 de Rham cohomology

Let X be a C^∞-manifold and let $L = \mathbb{R}$. Let $\mathscr{A}$ be any sheaf of $\mathbb{R}$-modules and define the de Rham cohomology with coefficients in $\mathscr{A}$ by

$$\boxed{{}_{\Omega}H^*_\Phi(X;\mathscr{A}) = H^*(\Gamma_\Phi(\Omega^* \otimes \mathscr{A})).}$$

Now, Ω^* is a resolution of $\mathbb{R}$, so that $\Omega^* \otimes \mathscr{A}$ is a resolution of $\mathscr{A}$. The exterior product $\wedge : \Omega^p(U) \otimes \Omega^q(U) \to \Omega^{p+q}(U)$ induces a product $(\Omega^p \otimes \mathscr{A}) \otimes (\Omega^q \otimes \mathscr{B}) \to \Omega^{p+q} \otimes (\mathscr{A} \otimes \mathscr{B})$. Thus Ω^0 is a sheaf of rings with unit and each $\Omega^p \otimes \mathscr{A}$ is an Ω^0-module. [Recall that $\Omega^0(U)$ is the ring of C^∞ real-valued functions on U.]

For any *paracompactifying* family Φ on X and $K, K' \in \Phi$ with $K \subset \operatorname{int} K'$ there is an $f \in \Omega^0(X)$ with $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \notin K'$. Let $g \in \Gamma(\Omega^0|K)$. We can extend g to a neighborhood, say K', of K by II-9.5. Then $f \cdot g$ is zero on the boundary of K' and thus can be extended to all of X. It follows that Ω^0 is Φ-soft, and hence each $\Omega^p \otimes \mathscr{A}$ is Φ-fine for any paracompactifying family Φ.

By II-5.15, there is a canonical map

$$\boxed{\rho : {}_{\Omega}H^*_\Phi(X;\mathscr{A}) \to H^*_\Phi(X;\mathscr{A})}$$

for arbitrary Φ, and this is an *isomorphism* when Φ is paracompactifying. Moreover, ρ preserves products when Φ is paracompactifying by II-7.1. (This is also true for arbitrary Φ, but we have not proved it.)

For many applications it is important to have a more direct "de Rham isomorphism": ${}_{S}H^*_\Phi(X;\mathscr{A}) \approx {}_{\Omega}H^*_\Phi(X;\mathscr{A})$, Φ paracompactifying. We shall briefly describe this.

We use singular theory based on C^∞ singular simplices. Let $\omega \in \Omega^p(X)$. We let $k(\omega)$ denote the singular p-cochain defined as follows. Let $\sigma : \Delta_p \to X$ be a C^∞ singular simplex. Then define

$$k(\omega)(\sigma) = \int_{\Delta_p} \sigma^*(\omega).$$

Stokes' theorem may be interpreted as saying that

$$k(d\omega)(\sigma) = \int_{\Delta_p} \sigma^*(d\omega) = \int_{\Delta_p} d\sigma^*(\omega) = \int_{\partial\Delta_p} \sigma^*(\omega) = k(\omega)(\partial\sigma) = (d(k(\omega))(\sigma),$$

so that $k(d\omega) = dk(\omega)$. That is,

$$k : \Omega^*(X) \to S^*(X;\mathbb{R})$$

is a chain map.[4]

Clearly $|k(\omega)| = |\omega|$, so that $k : \Omega^*_\Phi(X) \to S^*_\Phi(X;\mathbb{R})$ for any family Φ. Thus k induces a homomorphism

$$k^* : {}_\Omega H^*_\Phi(X) \to {}_\Delta H^*_\Phi(X;\mathbb{R}).$$

We claim that k^* is an isomorphism when Φ is paracompactifying and, in fact, that $k^* = \theta\rho$, where θ is as in Section 1.

To do this, we first generalize k^* as follows. The map $k : \Omega^*(U) \to S^*(U;\mathbb{R})$ induces a sheaf homomorphism $\Omega^* \to \mathscr{S}^*(X;\mathbb{R})$, which extends to a map $\Omega^* \otimes \mathscr{A} \to \mathscr{S}^*(X;\mathbb{R}) \otimes \mathscr{A}$ inducing an extension of k to $k : \Gamma_\Phi(\Omega^* \otimes \mathscr{A}) \to \Gamma_\Phi(\mathscr{S}^*(X;L) \otimes \mathscr{A})$. Thus we have, in general, a natural homomorphism

$$\boxed{k^* : {}_\Omega H^*_\Phi(X;\mathscr{A}) \to {}_S H^*_\Phi(X;\mathscr{A})}$$

coinciding with the former map for $\mathscr{A} = \mathbb{R}$ and Φ paracompactifying.

Now, when Φ is paracompactifying, $k : \Omega^* \otimes \mathscr{A} \to \mathscr{S}^*(X;\mathbb{R}) \otimes \mathscr{A}$ is a homomorphism of *resolutions* of $\mathscr{A}$ by Φ-soft sheaves. Consequently k^* is an *isomorphism* by II-4.2. This isomorphism k^* also commutes with connecting homomorphisms, and it follows from II-6.2 that $k^* = \theta\rho$.

We summarize:

3.1. Theorem. *For smooth manifolds X and sheaves $\mathscr{A}$ of $\mathbb{R}$-modules, there are natural multiplicative transformations of functors (of X as well as of $\mathscr{A}$)*

$$\begin{array}{ccc} H^*_\Phi(X;\mathscr{A}) & \xrightarrow{\ \theta\ } & {}_S H^*_\Phi(X;\mathscr{A}) \\ & {\scriptstyle\rho}\nwarrow \quad \nearrow{\scriptstyle k^*} & \\ & {}_\Omega H^*_\Phi(X;\mathscr{A}) & \end{array}$$

in which all three maps are isomorphisms when Φ is paracompactifying. The map k^* is induced by the classical integration of differential forms over singular simplices. □

[4]For a reasonably complete exposition of these matters see [19, V].

4 Čech cohomology

In I-7 we defined Čech cohomology with coefficients in a presheaf and proved one substantial result in the classical case of coefficients in a constant presheaf. In this section, we further develop the Čech approach to cohomology with coefficients in a sheaf as well as in a presheaf and study the connections between the two. The reader is advised to review the material in I-7 as to our notational conventions.

Let A be a presheaf on X and $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ an open covering of X. We shall use σ^n to denote an ordered n-simplex $\langle \alpha_0, \ldots, \alpha_n \rangle$ of the nerve $N(\mathfrak{U})$ of $\mathfrak{U}$, and U_{σ^n} to denote $U_{\alpha_0,\ldots,\alpha_n} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$. Let $c \in \check{C}^n_\Phi(X; A)$. Then the support $|c|$ of c is characterized by

$$\boxed{x \notin |c| \;\Leftrightarrow\; \exists V_x \;\ni\; 0 = c(\sigma^n)|U_{\sigma^n} \cap V_x \in A(U_{\sigma^n} \cap V_x)\; \forall \sigma^n \in N(\mathfrak{U}),}$$

where V_x is an open neighborhood of x.

4.1. Proposition. *The functor $\check{C}^n(\mathfrak{U}; \bullet)$ of presheaves is exact.*

Proof. It is essential to understand that this applies to exact sequences of *presheaves* and not to sheaves, since an exact sequence of sheaves is not generally right exact as a sequence of presheaves.

Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of presheaves; that is, for all $U \subset X$ open, $0 \to A'(U) \to A(U) \to A''(U) \to 0$ is exact. Then the induced sequence

$$0 \to \check{C}^n(\mathfrak{U}; A') \to \check{C}^n(\mathfrak{U}; A) \to \check{C}^n(\mathfrak{U}; A'') \to 0$$

has the form

$$0 \to \prod_{\sigma^n} A'(U_{\sigma^n}) \to \prod_{\sigma^n} A(U_{\sigma^n}) \to \prod_{\sigma^n} A''(U_{\sigma^n}) \to 0$$

which is exact. □

4.2. Theorem. *If every member of Φ has a neighborhood in Φ then the functor $\check{C}^n_\Phi(X; \bullet)$ of presheaves is exact.*

Proof. By passing to the direct limit over the coverings $\mathfrak{U} = \{U_x; x \in X\}$, we see that, for an exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$0 \to \check{C}^n(X; A') \to \check{C}^n(X; A) \to \check{C}^n(X; A'') \to 0$$

is exact by 4.1. It is also clear that

$$0 \to \check{C}^n_\Phi(X; A') \to \check{C}^n_\Phi(X; A) \xrightarrow{j} \check{C}^n_\Phi(X; A'')$$

is exact, and it remains to show that j is surjective. Thus, suppose that $c''_\mathfrak{U} \in \check{C}^n_\Phi(\mathfrak{U}; A'')$ represents a given element of $\check{C}^n_\Phi(X; A'')$, where $\mathfrak{U} = \{U_x; x \in$

$X\}$, and that $K = |c''_{\mathfrak{U}}| \in \Phi$. Let $K' \in \Phi$ be a neighborhood of K. Then for each $x \in X - K$ there is a neighborhood $V_x \subset U_x$ such that $c''_{\mathfrak{U}}(\sigma^n)|U_{\sigma^n} \cap V_x = 0$ in $A''(U_{\sigma^n} \cap V_x)$ for all σ^n. For $x \in K$ let $V_x = U_x \cap \operatorname{int} K'$. Then for the refinement $\mathfrak{V} = \{V_x\}$, we see that the image $c''_{\mathfrak{V}}$ of $c''_{\mathfrak{U}}$ has $c''_{\mathfrak{V}}(\sigma^n) = 0$ whenever $\sigma^n = \{x_0, \ldots, x_n\}$ has some vertex $x_i \notin K$. If $c''_{\mathfrak{V}} = j_{\mathfrak{V}}(c_{\mathfrak{V}})$ for some $c_{\mathfrak{V}} \in \check{C}^n(\mathfrak{V}; A)$, then define $\bar{c}_{\mathfrak{V}} \in \check{C}^n(\mathfrak{V}; A)$ by

$$\bar{c}_{\mathfrak{V}}(\sigma^n) = \begin{cases} c_{\mathfrak{V}}(\sigma^n), & \text{if all vertices } x_i \text{ of } \sigma^n \text{ are in } K, \\ 0, & \text{if some vertex } x_i \text{ of } \sigma^n \text{ is in } X - K. \end{cases}$$

Then, also, $j_{\mathfrak{V}}(\bar{c}_{\mathfrak{V}}) = c''_{\mathfrak{V}}$ and $|\bar{c}_{\mathfrak{V}}| \subset K' \in \Phi$. □

It follows, of course, that if $0 \to A' \to A \to A'' \to 0$ is an exact sequence of presheaves, then there is an induced long exact sequence

$$\cdots \to \check{H}^n_\Phi(X; A') \to \check{H}^n_\Phi(X; A) \to \check{H}^n_\Phi(X; A'') \to \check{H}^{n+1}_\Phi(X; A') \to \cdots$$

of Čech cohomology groups when Φ is paracompactifying. Presently we shall show that this also applies to short exact sequences of *sheaves.*

4.3. Proposition. *If Φ is paracompactifying, then any given cohomology class $\gamma \in \check{H}^n_\Phi(X; A)$ is represented by a cocycle $c_{\mathfrak{U}} \in \check{C}^n_\Phi(\mathfrak{U}; A)$ for some locally finite covering $\mathfrak{U}$ of X.*

Proof. Let γ be represented by the cocycle $c_{\mathfrak{U}} \in \check{C}^n_\Phi(\mathfrak{U}; A)$ for some open covering $\mathfrak{U} = \{U_x\}$ and let $K = |c| \in \Phi$. Let $K' \in \Phi$ be a neighborhood of K and $K'' \in \Phi$ a neighborhood of K'. For $x \in X - K$ there is a neighborhood $V_x \subset U_x - K$ of x such that $0 = c(\sigma^n)|V_x \cap U_{\sigma^n} \in A(V_x \cap U_{\sigma^n})$ for all σ^n. Let $\{W_\alpha; \alpha \in I\}$ be a locally finite open (in X) covering of K' in K'' that refines the covering $\{U_x \cap \operatorname{int} K''\}$ and is such that if $W_\alpha \cap (X - K') \neq \varnothing$ then $W_\alpha \subset V_x$ for some x. Then $\mathfrak{V} = \{V_x; x \in X - K'\} \cup \{W_\alpha; \alpha \in I\}$ is a refinement of $\mathfrak{U}$. The projection $c_{\mathfrak{V}}$ of $c_{\mathfrak{U}}$ has the property that $c_{\mathfrak{V}}(\sigma^n) = 0$ unless *all* vertices α_i of σ^n have $V_{\alpha_i} \subset K'$, since otherwise $V_{\alpha_i} \subset V_x$ for some x. Let $\mathfrak{W} = \{W_\alpha; \alpha \in I\} \cup \{X - K' = W_\infty\}$ (where $\infty \notin I$) and let $c_{\mathfrak{W}} \in \check{C}^n_\Phi(\mathfrak{W}; A)$ be given by

$$c_{\mathfrak{W}}(\sigma^n) = \begin{cases} c_{\mathfrak{V}}(\sigma^n), & \text{if no vertex of } \sigma^n \text{ is } \infty, \\ 0, & \text{if some vertex of } \sigma^n \text{ is } \infty. \end{cases}$$

Then $\mathfrak{V}$ refines both $\mathfrak{U}$ and $\mathfrak{W}$. Moreover, $c_{\mathfrak{W}}$ projects to $c_{\mathfrak{V}} \in \check{C}^n_\Phi(\mathfrak{V}; A)$. We have $dc_{\mathfrak{W}}(\sigma^{n+1}) = dc_{\mathfrak{V}}(\sigma^{n+1}) = 0$ if no vertex of $\sigma^{n+1} = \{\alpha_0, \ldots, \alpha_{n+1}\}$ is ∞. Also, $dc_{\mathfrak{W}}(\sigma^{n+1}) = 0$ if σ^{n+1} has at least two such vertices. If only the vertex $\alpha_k = \infty$, then

$$dc_{\mathfrak{W}}(\sigma^{n+1}) = \pm c_{\mathfrak{V}}(\alpha_0, \ldots, \widehat{\alpha_k}, \ldots, \alpha_{n+1})|W_{\alpha_0} \cap \cdots \cap W_\infty \cap \cdots \cap W_{n+1} = 0,$$

since $W_\infty \cap W_{\alpha_i} = \varnothing$ unless $W_{\alpha_i} \cap X - K' \neq \varnothing$, in which case $W_{\alpha_i} \subset V_x$ for some x, whence $dc_{\mathfrak{W}}(\sigma^{n+1}) = \pm c_{\mathfrak{V}}(\alpha_0, \ldots, \widehat{\alpha_k}, \ldots, \alpha_{n+1})|W_{\sigma^{n+1}} = 0$. Since $\mathfrak{W}$ is locally finite, $c_{\mathfrak{W}}$ is the required cocycle representative of γ. □

4.4. Theorem. *If Φ is paracompactifying and A is a presheaf on X such that $\mathscr{S}\!\mathit{heaf}(A) = 0$, then $\check{H}^n_\Phi(X; A) = 0$ for all n.*

Proof. Let $\gamma \in \check{H}^n_\Phi(X; A)$ be represented by the cocycle $c_{\mathfrak{U}} \in \check{C}^n_\Phi(\mathfrak{U}; A)$ where $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ is a locally finite open covering of X. For each $x \in X$ there is an open neighborhood V_x of x such that $V_x \cap U_\alpha \neq \varnothing$ for only a finite number of α. Then V_x can be further restricted so that each $c_{\mathfrak{U}}(\sigma^n)|U_{\sigma^n} \cap V_x = 0$ and such that $\mathfrak{V} = \{V_x\}$ refines $\mathfrak{U}$. But then the projection $c_{\mathfrak{V}} \in \check{C}^n_\Phi(\mathfrak{V}; A)$ of $c_{\mathfrak{U}}$ is the zero cocycle, and so $\gamma = 0$. □

4.5. Corollary. *If Φ is paracompactifying on X, A is a presheaf on X, and $\mathscr{A} = \mathscr{S}\!\mathit{heaf}(A)$, then the canonical map $\theta : A \to \mathscr{A}$ induces an isomorphism*

$$\boxed{\check{H}^*_\Phi(X; A) \xrightarrow{\approx} \check{H}^*_\Phi(X; \mathscr{A}).}$$

Proof. Let $K = \operatorname{Ker}\theta$ and $I = \operatorname{Im}\theta$. Then the exact sequences

$$0 \to K \to A \to I \to 0$$

and

$$0 \to I \to \mathscr{A} \to \mathscr{A}/I \to 0$$

of *presheaves* induce long exact cohomology sequences. Since $\mathscr{S}\!\mathit{heaf}(K) = 0 = \mathscr{S}\!\mathit{heaf}(\mathscr{A}/I)$, the claimed isomorphism follows. □

4.6. Corollary. *If Φ is paracompactifying and $0 \to \mathscr{A}' \xrightarrow{i} \mathscr{A} \xrightarrow{j} \mathscr{A}'' \to 0$ is an exact sequence of sheaves on X, then there is an induced long exact sequence*

$$\cdots \to \check{H}^n_\Phi(X; \mathscr{A}') \xrightarrow{i^*} \check{H}^n_\Phi(X; \mathscr{A}) \xrightarrow{j^*} \check{H}^n_\Phi(X; \mathscr{A}'') \xrightarrow{\delta^*} \check{H}^{n+1}_\Phi(X; \mathscr{A}') \to \cdots$$

of Čech cohomology groups.

Proof. If $A''(U) = \mathscr{A}(U)/i\mathscr{A}'(U)$ then the exact sequence $0 \to \mathscr{A}' \to \mathscr{A} \to A'' \to 0$ of presheaves induces a long exact cohomology sequence. But $\mathscr{S}\!\mathit{heaf}(A'') = \mathscr{A}''$ by definition, and so $\check{H}^n_\Phi(X; A'') \approx \check{H}^n_\Phi(X; \mathscr{A}'')$ by 4.5. □

4.7. Proposition. *If $\mathscr{A}$ is a sheaf, then for any open covering $\mathfrak{U}$ of X, the presheaf $V \mapsto \check{C}^n(\mathfrak{U} \cap V; \mathscr{A})$ is a sheaf. If $\mathscr{A}$ is flabby, then so are the $\check{C}^n(\mathfrak{U} \cap \bullet; \mathscr{A})$.*

Proof. We have

$$\check{C}^n(\mathfrak{U}\cap V;\mathscr{A}) = \prod_{\sigma^n}\mathscr{A}(U_{\sigma^n}\cap V) = \prod_{\sigma^n}(i_{\sigma^n}(\mathscr{A}|U_{\sigma^n}))(V),$$

where $i_{\sigma^n} : U_{\sigma^n} \hookrightarrow X$. Since the direct product of sheaves is a sheaf and the direct image of a flabby sheaf is flabby, the result follows. □

4.8. Lemma. *If $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ is a covering of X that includes the set X, then $\check{H}^n(\mathfrak{U}; A) = 0$ for $n > 0$ and any presheaf A.*

Proof. Let $\mathfrak{V} = \{X\}$. Then $\mathfrak{U}$ and $\mathfrak{V}$ refine one another, whence $\check{C}^*(\mathfrak{U}; A)$ is chain equivalent to $\check{C}^*(\mathfrak{V}; A)$, which vanishes in positive degrees. □

4.9. Theorem. *If $\mathscr{A}$ is a sheaf on X and $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ is any open covering of X, then $\check{\mathscr{C}}^*(\mathfrak{U};\mathscr{A})$ is a resolution of $\mathscr{A}$.*

Proof. Let V be a neighborhood of a given point $x \in X$, so small that $V \subset U_\alpha$ for some $\alpha \in I$. We have the sequence

$$0 \to \mathscr{A}(V) \xrightarrow{\varepsilon} \check{C}^0(\mathfrak{U}\cap V;\mathscr{A}) \xrightarrow{d^0} \check{C}^1(\mathfrak{U}\cap V;\mathscr{A}) \xrightarrow{d^1} \check{C}^2(\mathfrak{U}\cap V;\mathscr{A}) \to \cdots$$

and $\operatorname{Ker} d^p = \operatorname{Im} d^{p-1}$ for $p > 0$ by the lemma. The remainder of the exactness comprises the conditions (S1) and (S2) of I-1. Passage to the direct limit over neighborhoods V of x shows that

$$0 \to \mathscr{A}_x \to \check{\mathscr{C}}^0(\mathfrak{U};\mathscr{A})_x \to \check{\mathscr{C}}^1(\mathfrak{U};\mathscr{A})_x \to \cdots$$

is exact. □

4.10. Corollary. *Let $\mathscr{A}$ be a sheaf on X and let $\mathfrak{U}$ be an open covering of X. Then $\check{H}^0_\Phi(\mathfrak{U};\mathscr{A}) \approx \Gamma_\Phi(\mathscr{A})$. If $\mathscr{A}$ is flabby, then $\check{H}^n_\Phi(\mathfrak{U};\mathscr{A}) = 0$ for $n > 0$.*

Proof. Since $V \mapsto \check{C}^n(\mathfrak{U}\cap V;\mathscr{A})$ is a sheaf by 4.7, it is $\check{\mathscr{C}}^*(\mathfrak{U};\mathscr{A})$. By 4.9, $0 \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\check{\mathscr{C}}^0(\mathfrak{U};\mathscr{A})) \to \Gamma_\Phi(\check{\mathscr{C}}^1(\mathfrak{U};\mathscr{A}))$ is exact, whence $\Gamma_\Phi(\mathscr{A}) \approx H^0(\Gamma_\Phi(\check{\mathscr{C}}^*(\mathfrak{U};\mathscr{A}))) = H^0(\check{C}^*_\Phi(\mathfrak{U};\mathscr{A})) = \check{H}^0_\Phi(\mathfrak{U};\mathscr{A})$. If $\mathscr{A}$ is flabby then 4.7, 4.9, and II-4.3 show that

$$0 \to \Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\check{\mathscr{C}}^0(\mathfrak{U};\mathscr{A})) \to \Gamma_\Phi(\check{\mathscr{C}}^1(\mathfrak{U};\mathscr{A})) \to \cdots$$

is exact, so that $\check{H}^n_\Phi(\mathfrak{U};\mathscr{A}) = H^n(\Gamma_\Phi(\check{\mathscr{C}}^*(\mathfrak{U};\mathscr{A}))) = 0$ for $n > 0$. □

By passing to the limit over $\mathfrak{U}$ we have:

4.11. Corollary. *Let $\mathscr{A}$ be a sheaf on X. Then $\check{H}^0_\Phi(X;\mathscr{A}) \approx \Gamma_\Phi(\mathscr{A})$. If $\mathscr{A}$ is flabby, then $\check{H}^n_\Phi(X;\mathscr{A}) = 0$ for $n > 0$.* □

4.12. Corollary. *For Φ paracompactifying and sheaves $\mathscr{A}$ on X there is a natural isomorphism*

$$\boxed{\check{H}^*_\Phi(X;\mathscr{A}) \approx H^*_\Phi(X;\mathscr{A}).}$$

Proof. By the previous corollary and 4.6, $\check{H}^n_\Phi(X;\mathscr{A})$ is a fundamental connected sequence of functors of sheaves $\mathscr{A}$. Thus the result follows from II-6.2. □

4.13. Theorem. *Let $\mathscr{A}$ be a sheaf on X and let $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ be an open covering of X having the property that $H^p(U_{\sigma^n};\mathscr{A}) = 0$ for $p > 0$ and all $\sigma^n \in N(\mathfrak{U})$, all n. Then there is a canonical isomorphism*

$$\boxed{H^*(X;\mathscr{A}) \approx \check{H}^*(\mathfrak{U};\mathscr{A}).}$$

Proof. Let $\mathscr{A}^q = \mathscr{C}^q(X;\mathscr{A})$ and consider the complex $\check{C}^p(\mathfrak{U};\mathscr{A}^*)$ for p fixed. Since $\check{C}^p(\mathfrak{U};\bullet)$ is an exact functor of presheaves, we have

$$\begin{aligned} H^q(\check{C}^p(\mathfrak{U};\mathscr{A}^*)) &= \check{C}^p(\mathfrak{U};H^q(\Gamma(\mathscr{A}^*|\bullet))) \\ &= \check{C}^p(\mathfrak{U};H^q(\bullet;\mathscr{A})) \\ &= \begin{cases} \check{C}^p(\mathfrak{U};\mathscr{A}), & \text{for } q = 0, \\ 0, & \text{for } q \neq 0 \end{cases} \end{aligned}$$

by the hypothesis that $H^q(U_{\sigma^n};\mathscr{A}) = 0$ for $q > 0$.

Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \Gamma(\mathscr{A}) & \to & \Gamma(\mathscr{A}^0) & \to & \Gamma(\mathscr{A}^1) & \to & \Gamma(\mathscr{A}^2) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \check{C}^0(\mathfrak{U};\mathscr{A}) & \to & \check{C}^0(\mathfrak{U};\mathscr{A}^0) & \to & \check{C}^0(\mathfrak{U};\mathscr{A}^1) & \to & \check{C}^0(\mathfrak{U};\mathscr{A}^2) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \check{C}^1(\mathfrak{U};\mathscr{A}) & \to & \check{C}^1(\mathfrak{U};\mathscr{A}^0) & \to & \check{C}^1(\mathfrak{U};\mathscr{A}^1) & \to & \check{C}^1(\mathfrak{U};\mathscr{A}^2) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \check{C}^2(\mathfrak{U};\mathscr{A}) & \to & \check{C}^2(\mathfrak{U};\mathscr{A}^0) & \to & \check{C}^2(\mathfrak{U};\mathscr{A}^1) & \to & \check{C}^2(\mathfrak{U};\mathscr{A}^2) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

in which the rows are exact, excepting the first, as just shown and the columns are exact, excepting the first, by 4.10 and 4.9. A diagram chase shows that the homology of the first (nontrivial) row is isomorphic to that of the first column. □

4.14. Let $f : X \to Y$ be a map and $\mathscr{A}$ a sheaf on Y. Then $f^* : \mathscr{A} \rightsquigarrow f^*\mathscr{A}$ induces $f_U : \mathscr{A}(U) \to (f^*\mathscr{A})(f^{-1}U)$. If $\mathfrak{U} = \{U_\alpha; \alpha \in I\}$ is an open covering of Y and $f^{-1}(\mathfrak{U}) = \{f^{-1}(U) \mid U \in \mathfrak{U}\}$, indexed by the same set I, then we can define

$$f_{\mathfrak{U}} : \check{C}^n_\Phi(\mathfrak{U};\mathscr{A}) \to \check{C}^n_{f^{-1}\Phi}(f^{-1}\mathfrak{U}; f^*\mathscr{A})$$

by $f_{\mathfrak{U}}(c)(\sigma^n) = f_{U_{\sigma^n}}(c(\sigma^n))$. This is clearly a chain map, and so it induces a map

$$f^*_{\mathfrak{U}} : \check{H}^n_\Phi(\mathfrak{U};\mathscr{A}) \to \check{H}^n_{f^{-1}\Phi}(f^{-1}\mathfrak{U}; f^*\mathscr{A}).$$

Following this with the canonical map

$$\check{H}^n_{f^{-1}\Phi}(f^{-1}\mathfrak{U}; f^*\mathscr{A}) \to \check{H}^n_{f^{-1}\Phi}(X; f^*\mathscr{A}),$$

we obtain maps

$$\check{H}^n_\Phi(\mathfrak{U}; \mathscr{A}) \to \check{H}^n_{f^{-1}\Phi}(X; f^*\mathscr{A})$$

compatible with refinements. Therefore, these induce homomorphisms

$$\boxed{\check{f}^* : \check{H}^n_\Phi(Y; \mathscr{A}) \to \check{H}^n_{f^{-1}\Phi}(X; f^*\mathscr{A}).}$$

It is clear that for $n = 0$ this is just the canonical map

$$\Gamma_\Phi(\mathscr{A}) \to \Gamma_{f^{-1}\Phi}(f^*\mathscr{A}).$$

Therefore, by II-6.2, $\check{f}^*$ becomes $f^* : H^n_\Phi(Y; \mathscr{A}) \to H^n_{f^{-1}\Phi}(X; f^*\mathscr{A})$ under the isomorphisms of 4.12.

4.15. For presheaves A and B on X, we can define a cup product

$$\cup : \check{C}^n_\Phi(X; A) \otimes \check{C}^m_\Psi(X; B) \to \check{C}^{n+m}_{\Phi\cap\Psi}(X; A \otimes B),$$

where $(A \otimes B)(U) = A(U) \otimes B(U)$, by

$$c_1 \cup c_2(\alpha_0, \ldots, \alpha_{n+m}) = (c_1(\alpha_0, \ldots, \alpha_n) \otimes c_2(\alpha_n, \ldots, \alpha_{n+m}))|U_{\alpha_0,\ldots,\alpha_{n+m}}.$$

This induces a cup product

$$\cup : \check{H}^n_\Phi(X; A) \otimes \check{H}^m_\Psi(X; B) \to \check{H}^{n+m}_{\Phi\cap\Psi}(X; A \otimes B).$$

If $\mathscr{A} = \mathscr{S}\!\mathit{heaf}(A)$ and $\mathscr{B} = \mathscr{S}\!\mathit{heaf}(B)$, then $\mathscr{A} \otimes \mathscr{B} = \mathscr{S}\!\mathit{heaf}(A \otimes B)$ by definition, whence this becomes a cup product

$$\cup : \check{H}^n_\Phi(X; \mathscr{A}) \otimes \check{H}^m_\Psi(X; \mathscr{B}) \to \check{H}^{n+m}_{\Phi\cap\Psi}(X; \mathscr{A} \otimes \mathscr{B})$$

when Φ and Ψ are paracompactifying. It is not hard to see, by using II-7.1, that this coincides with the cup product for sheaf cohomology via 4.12.

Exercises

1. Complete the discussion of Section 2 with regard to homomorphisms induced by maps.
2. Investigate the behavior of de Rham cohomology with respect to differentiable mappings.
3. Let G be a compact Lie group acting differentiably on a C^∞-manifold M. There is an induced action of G on $\Omega^*(M)$ and on ${}_\Omega H^*(M) \approx H^*(M; \mathbb{R})$. Denoting invariant elements by a superscript G, show that the inclusion $\Omega^*(M)^G \hookrightarrow \Omega^*(M)$ induces an isomorphism

$$H^*(\Omega^*(M)^G) \xrightarrow{\approx} H^*(M)^G.$$

[This is nontrivial. Note that $H^*(M)^G = H^*(M)$ when G is connected.] [*Hint*: Let $J : \Omega^*(M) \hookrightarrow \Omega^*(M)$ be the inclusion. By integration over G, define $I : \Omega^*(M) \to \Omega^*(M)^G$ so that $I \circ J = 1$ and such that for $\alpha \in H^p(M)^G$ and $\omega \in \Omega^*(M)$ representing α, $I(\omega)$ also represents α.]

4. Ⓢ If X is hereditarily paracompact and G is a constant sheaf, show that $\mathscr{S}^*(X;G)$ and $\mathscr{A}^*(X;G)$ are flabby. Thus $H^*(\Gamma_\Phi(\mathscr{A}^*(X;G))) \approx H^*_\Phi(X;G)$ for *any* family Φ of supports.

5. A continuous surjection $f : X \to Y$ is called "ductile" if for each $y \in Y$, there exists a neighborhood $V \subset Y$ of y which contracts to y through U in such a way that this contraction can be covered by a homotopy of $f^{-1}(V)$; see [16]. Let ${}_fS^*(U;\mathbb{Z}) = \operatorname{Hom}(f_*\Delta^c_*(f^{-1}(U)),\mathbb{Z}))$ where $f_*\Delta^c_*(f^{-1}(U))$ is the image in $\Delta^c_*(U)$ of the classical singular chain complex of $f^{-1}(U)$ under f. If $S^*(W;\mathbb{Z}) = \operatorname{Hom}(\Delta^c_*(W),\mathbb{Z})$, as usual, then the inclusion $f_*\Delta^c_*(f^{-1}(U)) \hookrightarrow \Delta^c_*(U)$ induces a homomorphism $S^*(U;\mathbb{Z}) \to {}_fS^*(U;\mathbb{Z})$ of presheaves and hence a map $\mathscr{S}^* \to {}_f\mathscr{S}^*$ of the generated sheaves. If Φ is a paracompactifying family of supports on Y and if f is ductile, show that the induced map

$$_\Delta H^*_\Phi(Y;\mathbb{Z}) = H^*(S^*_\Phi(Y;\mathbb{Z})) \to H^*({}_fS^*_\Phi(Y;\mathbb{Z}))$$

is an isomorphism, and that for any sheaf $\mathscr{A}$ on Y, the map

$$H^*(\Gamma_\Phi(\mathscr{S}^* \otimes \mathscr{A})) \to H^*(\Gamma_\Phi({}_f\mathscr{S}^* \otimes \mathscr{A}))$$

is an isomorphism. (That is, for ductile maps, the singular cohomology of Y can be computed using only those singular simplices of Y that are images of singular simplices of X.)

6. Let X be a complex manifold. Let $\mathscr{A}^{p,q}$ denote the sheaf of germs of C^∞ differentiable forms of type (p,q) on X and let Ω^p denote the sheaf of germs of holomorphic p-forms of X. (This is not to be confused with the notation used in the real case.) The operator d of exterior differentiation has the decomposition $d = d' + d''$, where $d' : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q}$ and $d'' : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}$, $(d')^2 = 0 = (d'')^2$, and $d'd'' + d''d' = 0$. It can be shown that via d'', $\mathscr{A}^{p,*}$ is a resolution of Ω^p for each p. Using this fact, prove that (with the obvious notation) there is a natural isomorphism

$$H^q_\Phi(X;\Omega^p \otimes \mathscr{B}) \approx {}''H^q(\Gamma_\Phi(\mathscr{A}^{p,*} \otimes \mathscr{B})),$$

where Φ is paracompactifying, $\mathscr{B}$ is any sheaf of complex vector spaces, and the tensor products are over the complex numbers. (This result is known as the theorem of Dolbeault [32].)

7. Ⓢ Construct a compact space X with singular homology groups

$$H_1(X;\mathbb{Z}) \approx \bigoplus_{n=2}^{\infty} \mathbb{Z}_n, \qquad H_2(X;\mathbb{Z}) = 0.$$

Let $\mathbb{Z}$ be the base ring, let $\mathbb{Q}$ denote the rationals as a $\mathbb{Z}$-module, and show that ${}_\Delta H^2(X;\mathbb{Q}) = 0$ while

$${}_SH^2(X;\mathbb{Q}) \approx {}_SH^2(X;\mathbb{Z}) \otimes \mathbb{Q} \approx {}_\Delta H^2(X;\mathbb{Z}) \otimes \mathbb{Q} \neq 0.$$

In particular, unlike ${}_\Delta H^*$, the ${}_SH^*$ *depend on the base ring.*

8. Show that $HLC \Rightarrow clc^\infty_\mathbb{Z}$.

9. If $F \subset X$ is closed and $\mathscr{A}$ is a sheaf on X, show that $\check{H}^n(X;\mathscr{A}_F) \approx \check{H}^n(F;\mathscr{A}|F)$.

10. Ⓢ Let X be a rudimentary space; see II-Exercise 57. Then show that $\check{H}^n(X;A) \approx H^n(X;\mathscr{A})$ for all n and any presheaf A on X with $\mathscr{A} = \mathscr{S}\!\mathit{heaf}(A)$.

11. Ⓢ Let the Čech cohomology groups ${}_{LF}\check{H}^n(X;A)$ be defined using only *locally finite* coverings of X. If $\mathbb{N}$ is the rudimentary space of II-Exercise 27, show that ${}_{LF}\check{H}^n(\mathbb{N};A) = 0$ for all $n > 0$ and all presheaves A on $\mathbb{N}$. (Compare Exercise 10.)

12. Define and study relative Čech cohomology groups $\check{H}^n_\Phi(X,B;\mathscr{A})$.

13. Ⓢ For the sheaf $\mathscr{A}_n$ on $\mathbb{S}^1$ described in I-1.14, compute $\check{H}^1(\mathbb{S}^1;\mathscr{A}_n)$.

14. Ⓢ Use constant coefficients in some group throughout. Let X be a compact metric space and $0 \neq \gamma \in H^n(X)$. Show that there is a number $\varepsilon > 0$, depending on γ, such that if $f : X \twoheadrightarrow Y$ is a surjective map to a compact Hausdorff space Y with diam $f^{-1}(y) < \varepsilon$ for all $y \in Y$, then $\gamma = f^*(\gamma')$ for some $\gamma' \in H^n(Y)$.

15. For sheaves $\mathscr{A}$ on X define a natural map $h^k : \check{H}^k(\mathfrak{U};\mathscr{A}) \to H^k(X;\mathscr{A})$. If for every intersection U of at most $m+1$ members of $\mathfrak{U}$ we have $H^q(U;\mathscr{A}) = 0$ for $0 < q < n$, then show that h^k is an isomorphism for $k < \min\{n,m\}$ and a monomorphism for $k = \min\{n,m\}$.

16. Ⓢ Construct an example showing that 4.2 is false without the assumption that each member of Φ has a neighborhood in Φ.

Chapter IV

Applications of Spectral Sequences

In this chapter we shall assume that the reader is familiar with the theory of spectral sequences, especially with the spectral sequences of double complexes. This basic knowledge is applied specifically to the theory of sheaves in Sections 1 and 2. See Appendix A for an outline of the parts of the theory of spectral sequences we shall need.

In Section 3 we define an important generalization of the direct image functor, the direct image relative to a support family. Its derived functors, which yield the so-called Leray sheaf, are studied in Section 4. It should be noted that the direct image functor f_Ψ generalizes the functor Γ_Ψ since they are equivalent when f is the map to a point.

Section 5 is concerned with a construction dealing with support families and is primarily notational.

In Section 6 the Leray spectral sequence of an arbitrary map is derived and a few remarks about it are given. The cup product in this spectral sequence is also discussed. This spectral sequence is the central result of this chapter, perhaps of the whole book, and the remainder of the chapter is devoted to its consequences.

The important special case of a locally trivial bundle projection is discussed in Section 7, and it is shown that under suitable conditions, the Leray sheaf (the coefficient sheaf in the Leray spectral sequence) is locally constant and has the cohomology of the fiber as stalks. We also obtain a generalization of the Künneth formula to the case in which one of the factors X of $X \times Y$ is an arbitrary space, provided, however, that the other factor Y is locally compact Hausdorff and clc_L^∞ and is provided with compact supports and constant coefficients in a principal ideal domain. These results are then applied to the case of vector bundles over an arbitrary space, and such things as the Thom isomorphism and the Gysin and Wang exact sequences are derived.

Some important consequences that the Leray spectral sequence and the fundamental theorems have for dimension theory are discussed in Section 8. That section also contains a strengthening of the Vietoris mapping theorem due to Skljarenko.

In Section 9 the Leray spectral sequence is applied to spaces with compact groups of transformations acting on them, and the Borel and Cartan spectral sequences are derived.

In Section 10 we apply the discussion of Section 7 to give a general definition of the Stiefel-Whitney, Chern, and symplectic Pontryagin characteristic classes, and to derive some of their properties. Applications to the study of characteristic classes of the normal bundle to the fixed point set of a differentiable transformation group are discussed at the end of this section.

In Sections 11 and 12 we derive the Fary spectral sequence of a map with a filtered base space. This is applied in Section 13 to derive the Smith-Gysin sequence of a sphere fibration with singularities. In Section 14 we derive Oliver's transfer map and use it, as he did, to prove the Conner conjecture.

1 The spectral sequence of a differential sheaf

Let $\mathscr{L}^*$ be a differential sheaf on X. We are concerned with two cases:

(a) $\mathscr{L}^* = 0$ *for* $q < q_0$.

(b) $\dim_{\Phi,L} X < \infty$ *for a given ground ring* L *and family* Φ *of supports.*

If we are in case (b) with $\dim_{\Phi,L} X \leq n$, then

$$0 \to \mathscr{A} \to \mathscr{C}^0(X;\mathscr{A}) \to \cdots \to \mathscr{C}^{n-1}(X;\mathscr{A}) \to \mathscr{Z}^n(X;\mathscr{A}) \to 0$$

is a resolution of $\mathscr{A}$ by Φ-acyclic sheaves and is an exact functor of $\mathscr{A}$. In the discussion below, $C^*_\Phi(X;\mathscr{A})$ should be replaced by Γ_Φ of this resolution when we are in case (b).

Let

$$\boxed{L^{p,q} = C^p_\Phi(X;\mathscr{L}^q).}$$

We take $d' : L^{p,q} \to L^{p+1,q}$ to be the differential of the complex $C^*_\Phi(X;\mathscr{L}^q)$ and $(-1)^p d'' : L^{p,q} \to L^{p,q+1}$ to be the differential induced by the coefficient homomorphism $\mathscr{L}^q \to \mathscr{L}^{q+1}$. Let $d = d' + d''$ be the total differential and L^* the total complex, where

$$\boxed{L^n = \bigoplus_{p+q=n} L^{p,q}.}$$

There are two spectral sequences, ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$, of the double complex $L^{*,*}$ converging[1] to graded groups associated with filtrations on $H^{p+q}(L^*)$. We abbreviate this statement by the notation $E_2^{p,q} \Longrightarrow H^{p+q}(L^*)$; see Appendix A.

[1]The first spectral sequence converges because of condition (a) or (b).

In the first spectral sequence we have ${}'E_2^{p,q} = {}'H^p({}''H^q(L^{*,*}))$ and in the second ${}''E_2^{p,q} = {}''H^p({}'H^q(L^{*,*}))$, where ${}'H$ and ${}''H$ are computed using d' and d'' respectively.

Now

$$''H^q(L^{*,*}) = C^*_\Phi(X; \mathscr{H}^q(\mathscr{L}^*))$$

since $C^*_\Phi(X; \bullet)$ is an exact functor. Thus

$$\boxed{{}'E_2^{p,q} = H^p_\Phi(X; \mathscr{H}^q(\mathscr{L}^*)).} \tag{1}$$

1.1. In the second spectral sequence ${}'H^q(L^{*,*}) = H^q_\Phi(X; \mathscr{L}^*)$, so that

$$\boxed{{}''E_2^{p,q} = H^p(H^q_\Phi(X; \mathscr{L}^*)).}$$

Note that in particular, ${}''E_2^{p,0} = H^p(\Gamma_\Phi(\mathscr{L}^*))$.

In the second spectral sequence we have the edge homomorphism

$$H^p(\Gamma_\Phi(\mathscr{L}^*)) = {}''E_2^{p,0} \twoheadrightarrow {}''E_\infty^{p,0} \rightarrowtail H^p(L^*). \tag{2}$$

Recall that this is induced by the chain map

$$\Gamma_\Phi(\mathscr{L}^p) \rightarrowtail L^p, \tag{3}$$

given by $\Gamma_\Phi(\mathscr{L}^p) \rightarrowtail C^0_\Phi(X; \mathscr{L}^p) = L^{0,p} \rightarrowtail L^p$; see Appendix A.

Suppose that we are given a homomorphism $\mathscr{L}^* \to \mathscr{M}^*$ of differential sheaves. Putting $M^{p,q} = C^p_\Phi(X; \mathscr{M}^q)$, we have a map $L^{p,q} \to M^{p,q}$ of double complexes and hence a corresponding map of spectral sequences. Since (2) is induced by the chain map (3), it follows that the diagram

$$\begin{array}{ccc} H^p(\Gamma_\Phi(\mathscr{L}^*)) & \to & H^p(L^*) \\ \downarrow & & \downarrow \\ H^p(\Gamma_\Phi(\mathscr{M}^*)) & \to & H^p(M^*) \end{array} \tag{4}$$

commutes.

Let $0 \to \mathscr{L}^* \to \mathscr{M}^* \to \mathscr{N}^* \to 0$ be an exact sequence of differential sheaves. Since $C^*_\Phi(X; \bullet)$ is an exact functor, the sequence

$$0 \to L^{p,q} \to M^{p,q} \to N^{p,q} \to 0$$

(with the obvious notation) is exact. We shall *assume* that the sequence

$$0 \to \Gamma_\Phi(\mathscr{L}^*) \to \Gamma_\Phi(\mathscr{M}^*) \to \Gamma_\Phi(\mathscr{N}^*) \to 0$$

is also exact. This is the case, for example, when $\mathscr{L}^*$ is Φ-acyclic.[2] Then it follows that the diagram

$$\begin{array}{ccc} H^p(\Gamma_\Phi(\mathscr{N}^*)) & \to & H^p(N^*) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H^{p+1}(\Gamma_\Phi(\mathscr{L}^*)) & \to & H^{p+1}(L^*) \end{array} \tag{5}$$

is commutative, where the horizontal maps are the edge homomorphisms (2) and the vertical maps are the obvious connecting homomorphisms.

[2] Also see exercise 14.

1.2. We wish to obtain naturality relations such as (4) and (5) for the edge homomorphisms of the first spectral sequence. In order to obtain results of the generality needed, we must go into this situation somewhat more deeply than we did for the second spectral sequence.

In the remainder of this section we shall *assume* either that $\dim_{\Phi,L} X < \infty$ or that all differential sheaves considered are bounded below; that is, that condition (a) or (b) is satisfied. This assumption is needed to ensure the convergence of the "first" spectral sequences considered.

Let $\mathscr{L}^*$ be a differential sheaf such that

$$\mathscr{H}^q(\mathscr{L}^*) = 0 \quad \text{for} \quad q < 0.$$

Let $\mathscr{L}$ be some given sheaf and let $\mathscr{L} \to \mathscr{H}^0(\mathscr{L}^*)$ be a given homomorphism. Since $'E_2^{p,q} = H^p_\Phi(X; \mathscr{H}^q(\mathscr{L}^*)) = 0$ for $q < 0$ we have the edge homomorphism

$$H^p_\Phi(X;\mathscr{L}) \to H^p_\Phi(X;\mathscr{H}^0(\mathscr{L}^*)) = {'E_2^{p,0}} \twoheadrightarrow {'E_\infty^{p,0}} \rightarrowtail H^p(L^*). \tag{6}$$

Since we have not assumed that $\mathscr{L}^*$ vanishes in negative degrees, we cannot immediately conclude that (6) is induced by a chain map. In order to provide such a chain map, we make the following construction. Let $\mathscr{L}_0^*$ be the differential sheaf defined by

$$\mathscr{L}_0^q = \begin{cases} 0, & \text{for} \quad q < 0, \\ \operatorname{Coker}\{\mathscr{L}^{-1} \to \mathscr{L}^0\}, & \text{for} \quad q = 0, \\ \mathscr{L}^q, & \text{for} \quad q > 0, \end{cases}$$

with differentials induced from those of $\mathscr{L}^*$. We have the canonical map

$$\mathscr{L}^* \to \mathscr{L}_0^*, \tag{7}$$

and it is clear that the induced map $\mathscr{H}^q(\mathscr{L}^*) \to \mathscr{H}^q(\mathscr{L}_0^*)$ is an isomorphism for all q. Thus we have the isomorphism

$$H^p_\Phi(X;\mathscr{H}^q(\mathscr{L}^*)) \xrightarrow{\approx} H^p_\Phi(X;\mathscr{H}^q(\mathscr{L}_0^*)) \tag{8}$$

for all p, q.

The homomorphism (7) induces a map of the "first" spectral sequences, which, by (8), is an isomorphism from E_2 on. Thus, with the obvious notation, we have the commutative diagram

$$\begin{array}{ccccc} H^p_\Phi(X;\mathscr{L}) & \to & H^p_\Phi(X;\mathscr{H}^0(\mathscr{L}^*)) & \to & H^p(L^*) \\ \| & & \downarrow\approx & & \downarrow\approx \\ H^p_\Phi(X;\mathscr{L}) & \to & H^p_\Phi(X;\mathscr{H}^0(\mathscr{L}_0^*)) & \to & H^p(L_0^*) \end{array} \tag{9}$$

in which the horizontal maps are edge homomorphisms. Note that the given map $\mathscr{L} \to \mathscr{H}^0(\mathscr{L}^*) \approx \mathscr{H}^0(\mathscr{L}_0^*)$ now arises from a homomorphism

$\mathscr{L} \to \mathscr{L}_0^0$. We also know that the edge homomorphism on the bottom of (9) arises from the chain map

$$C_\Phi^p(X;\mathscr{L}) \to L_0^p$$

given by $C_\Phi^p(X;\mathscr{L}) \to C_\Phi^p(X;\mathscr{L}_0^0) = L_0^{p,0} \rightarrowtail L_0^p$; see Appendix A. It follows that the edge homomorphisms (6) satisfy, via (9), naturality relations similar to (4) and (5). Explicitly, let $\mathscr{M}^*$ be a differential sheaf with $\mathscr{H}^q(\mathscr{M}^*) = 0$ for $q < 0$ and let $\mathscr{M} \to \mathscr{H}^0(\mathscr{M}^*)$ be given. Then for any homomorphism $\mathscr{L}^* \to \mathscr{M}^*$ of differential sheaves and any compatible map $\mathscr{L} \to \mathscr{M}$, the diagram

$$\begin{array}{ccc} H_\Phi^p(X;\mathscr{L}) & \to & H^p(L^*) \\ \downarrow & & \downarrow \\ H_\Phi^p(X;\mathscr{M}) & \to & H^p(M^*) \end{array} \tag{10}$$

commutes.

Similarly, suppose we are also given $\mathscr{N}^*$ with $\mathscr{H}^q(\mathscr{N}^*) = 0$ for $q < 0$ and a compatible map $\mathscr{N} \to \mathscr{H}^0(\mathscr{N}^*)$. Suppose also that we have an exact sequence $0 \to \mathscr{L}^* \to \mathscr{M}^* \to \mathscr{N}^* \to 0$ of differential sheaves compatible with an exact sequence $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$, that is, such that the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{N} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \mathscr{H}^0(\mathscr{L}^*) & \to & \mathscr{H}^0(\mathscr{M}^*) & \to & \mathscr{H}^0(\mathscr{N}^*) & \end{array}$$

commutes. Then the diagram

$$\begin{array}{ccc} H_\Phi^p(X;\mathscr{N}) & \to & H^p(N^*) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H_\Phi^{p+1}(X;\mathscr{L}) & \to & H^{p+1}(L^*) \end{array} \tag{11}$$

commutes.

Remark: The spectral sequences considered in this section are not changed, from E_2 on, if we replace $C_\Phi^*(X;\bullet)$ by $\Gamma_\Phi(\mathscr{R}^*(X;\bullet))$, where $\mathscr{R}^*(X;\bullet)$ is any exact functorial resolution by Φ-acyclic sheaves. To see this, consider the natural homomorphisms

$$\mathscr{R}^*(X;\bullet) \to \mathscr{C}^*(\mathscr{R}^*(X;\bullet)) \leftarrow \mathscr{C}^*(X;\bullet),$$

where the middle term is considered as a functorial resolution with the total degree. Then apply these resolutions to $\mathscr{L}^*$, take Γ_Φ, and consider the spectral sequences of the resulting double complexes. The contention follows immediately. Of course, in case (b), we must assume that $\mathscr{R}^*(X;\bullet)$ has finite length. For example, we could take $\mathscr{R}^*(X;\bullet) = \mathscr{F}^*(X;\bullet)$ of II-2. If Φ is paracompactifying, we could take $\mathscr{R}^*(X;\bullet) = \mathscr{R}^* \otimes (\bullet)$, where $\mathscr{R}^*$ is a Φ-fine torsion-free resolution of the ground ring, such as the Alexander-Spanier sheaf.

2 The fundamental theorems of sheaves

The previous section has two cases of importance, corresponding to the degeneracy of one or the other of the spectral sequences.

Degeneracy of the second spectral sequence

Assume that

$$''E_2^{p,q} = H^p(H^q_\Phi(X;\mathscr{L}^*)) = 0 \quad \text{for} \quad q > 0.$$

This holds, for example, when each $\mathscr{L}^p$ is Φ-acyclic. We also have that

$$''E_2^{p,0} = H^p(\Gamma_\Phi(\mathscr{L}^*)).$$

It follows that the canonical homomorphisms $H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^p(L^*)$ of (2) are isomorphisms. The first spectral sequence then provides the following result:

2.1. Theorem. (First fundamental theorem.) *Let $\mathscr{L}^*$ be a differential sheaf on X. Assume either that $\dim_{\Phi,L} X < \infty$ or that $\mathscr{L}^*$ is bounded below (i.e., $\mathscr{L}^q = 0$ for $q < q_0$, for some q_0). Also assume that*

$$H^p(H^q_\Phi(X;\mathscr{L}^*)) = 0 \quad \textit{for} \quad q > 0.$$

Then there is a spectral sequence with

$$\boxed{E_2^{p,q} = H^p_\Phi(X;\mathscr{H}^q(\mathscr{L}^*)) \Longrightarrow H^{p+q}(\Gamma_\Phi(\mathscr{L}^*)).}$$

□

Note also that if $\mathscr{H}^q(\mathscr{L}^*) = 0$ for $q < 0$ and if $\mathscr{L} \to \mathscr{H}^0(\mathscr{L}^*)$ is a given homomorphism, then the edge homomorphism provides a canonical map

$$H^p_\Phi(X;\mathscr{L}) \to H^p(\Gamma_\Phi(\mathscr{L}^*)),$$

which satisfies the naturality relations resulting from formulas (4), (5), (10), and (11) of Section 1.

2.2. Theorem. (Second fundamental theorem.) *Let $h : \mathscr{L}^* \to \mathscr{M}^*$ be a homomorphism of differential sheaves and assume that $\dim_{\Phi,L} X < \infty$ or that both $\mathscr{L}^*$ and $\mathscr{M}^*$ are bounded below. Also assume that for some $0 < N \le \infty$ the induced map $h^* : \mathscr{H}^q(\mathscr{L}^*) \to \mathscr{H}^q(\mathscr{M}^*)$ of derived sheaves is an isomorphism for $q < N$ and that*

$$H^*(H^q_\Phi(X;\mathscr{L}^*)) = 0 = H^*(H^q_\Phi(X;\mathscr{M}^*)) \quad \textit{for} \quad q > 0.$$

Then the induced map $H^n(\Gamma_\Phi(\mathscr{L}^*)) \to H^n(\Gamma_\Phi(\mathscr{M}^*))$ is an isomorphism for each $n < N$.

Proof. If $N = \infty$ (as suffices for most applications) then the proof is immediate from the fact[3] that a homomorphism of regularly filtered complexes that induces an isomorphism on the E_2 terms of the spectral sequences also induces an isomorphism on the homology of the complexes. In general, we must digress to study a purely combinatorial construction.

For integers $r \geq 2$, define the collection

$$S_r = \{1, 1+(r-1), 1+(r-1)+(r-2), \dots, 1+(r-1)+\cdots+2+1, \Rrightarrow\}$$

of integers, where the symbol $\Rrightarrow$ indicates that all integers beyond that point are included in S_r. Then

$$S_{r+1} = \{1, 1+r, 1+r+(r-1), \dots, 1+r+(r-1)+\cdots+2+1, \Rrightarrow\}.$$

Now, for $p \geq 0$, define $k_r(p) = N - p + \#(S_r \cap [0,p])$. Note that $m \in S_r \Leftrightarrow m + r \in S_{r+1}$ for $m \geq 0$. This implies that $k_{r+1}(p+r) - k_r(p)$ is constant in $p \geq 0$. Therefore

$$\begin{aligned} k_{r+1}(p+r) - k_r(p) &= k_{r+1}(r) - k_r(0) \\ &= -r + \#(S_{r+1} \cap [0,r]) - \#(S_r \cap [0,0]) \\ &= -r + 1 - 0 \\ &= -r + 1. \end{aligned}$$

Thus we have the partial recursion formula

$$k_{r+1}(p+r) = k_r(p) - r + 1 \quad \text{for } p \geq 0. \tag{12}$$

Also note that

$$\begin{aligned} k_{r+1}(p) &\leq k_r(p), \\ k_2(p) &= N, \\ k_\infty(p) &= N - p + 1 \quad \text{for } p > 0, \\ k_\infty(0) &= N. \end{aligned}$$

For $r \geq 2$, we claim that $E_r^{p,q}(\mathscr{L}^*) \to E_r^{p,q}(\mathscr{M}^*)$ is an isomorphism for $q < k_r(p)$. The proof is by induction on r. It is true for $r = 2$ since $k_2(p) = N$ for all p. Suppose it is true for r. Then we must show that

$$q \geq k_r(p) \;\Rightarrow\; q - r + 1 \geq k_{r+1}(p+r)$$

when $p \geq 0$ (regarding differentials leading out of $E_r^{p,q}$) and

$$q + r - 1 \geq k_r(p-r) \;\Rightarrow\; q \geq k_{r+1}(p)$$

when $p - r \geq 0$ (regarding differentials leading into $E_r^{p,q}$). But these follow immediately from (12). (We also need that $k_{r+1}(p) \leq k_r(p)$, which has also been noted.)

In particular, $E_\infty^{p,q}(\mathscr{L}^*) \to E_\infty^{p,q}(\mathscr{M}^*)$ is an isomorphism for $p + q \leq N$ if $p > 0$ and for $q \leq N - 1$ if $p = 0$. Therefore the total terms map isomorphically for total degree less than N. □

[3] See A-4.

Remark: The case $N < \infty$ can also be attacked using a mapping cone argument; see II-Exercise 49. However the present method applies more generally since it is a pure spectral sequence argument. It also gives slightly better conclusions. Also note that Exercise 33 sharpens 2.2, perhaps only by an amount $< \varepsilon$.

The following result is needed in Section 12:

2.3. Proposition. *Let $\mathscr{L}^*$ be a Φ-soft differential sheaf on X with Φ paracompactifying, and let $A \subset X$ be locally closed. Then the spectral sequence* 2.1 *of the differential sheaf $\mathscr{L}^*_A$ on X with supports in Φ is canonically isomorphic to the spectral sequence* 2.1 *of the differential sheaf $\mathscr{L}^*|A$ on A with supports in $\Phi|A$ (from E_2 on).*

Proof. It is sufficient to prove this for the cases in which A is either open or closed. Note that since $\mathscr{B} \mapsto \mathscr{B}|A$ and $\mathscr{B} \mapsto \mathscr{B}_A$ are exact functors, we have

$$\mathscr{H}^*(\mathscr{L}^*|A) = \mathscr{H}^*(\mathscr{L}^*)|A \quad \text{and} \quad \mathscr{H}^*(\mathscr{L}^*_A) = \mathscr{H}^*(\mathscr{L}^*)_A.$$

If $U \subset X$ is open, then the map

$$C^*_{\Phi|U}(U; \mathscr{L}^*|U) \to C^*_{\Phi|U}(X; \mathscr{L}^*_U) \hookrightarrow C^*_{\Phi}(X; \mathscr{L}^*_U)$$

(using that $\mathscr{L}^*|U = \mathscr{L}^*_U|U$) induces a map of spectral sequences that on the E_2 terms reduces to the natural isomorphism

$$H^p_{\Phi|U}(U; \mathscr{H}^q(\mathscr{L}^*)|U) \xrightarrow{\approx} H^p_{\Phi}(X; \mathscr{H}^q(\mathscr{L}^*)_U)$$

of II-10.2. It follows that this map of spectral sequences is an isomorphism from E_2 on, as was to be shown.

If $F \subset X$ is closed, then the map

$$C^*_{\Phi}(X; \mathscr{L}^*_F) \to C^*_{\Phi|F}(F; \mathscr{L}^*|F)$$

of II-8 induces a map of spectral sequences that on the E_2 terms is the isomorphism

$$H^p_{\Phi}(X; \mathscr{H}^q(\mathscr{L}^*)_F) \xrightarrow{\approx} H^p_{\Phi|F}(F; \mathscr{H}^q(\mathscr{L}^*)|F)$$

of II-10.2. Again, it follows that the spectral sequences in question are isomorphic from E_2 on. □

Degeneracy of the first spectral sequence

Here we assume that $\mathscr{H}^q(\mathscr{L}^*) = 0$ for $q \neq 0$. Also assume that either (a) or (b) of Section 1 holds. Let $\mathscr{L} = \mathscr{H}^0(\mathscr{L}^*)$. Note that these conditions hold, for example, when $\mathscr{L}^*$ is a resolution of $\mathscr{L}$.

We have that

$$'E_2^{p,q} = \begin{cases} 0, & \text{for } q \neq 0, \\ H_\Phi^p(X; \mathscr{L}), & \text{for } q = 0, \end{cases}$$

and it follows that the canonical homomorphism

$$H_\Phi^p(X; \mathscr{L}) \to H^p(L^*)$$

of (6) is an isomorphism. From the second spectral sequence we conclude the following:

2.4. Theorem. (Third fundamental theorem.) *Let $\mathscr{L}^*$ be a differential sheaf such that $\mathscr{H}^q(\mathscr{L}^*) = 0$ for $q \neq 0$. Assume either that* $\dim_{\Phi,L} X < \infty$ *or that $\mathscr{L}^*$ is bounded below. Then with $\mathscr{L} = \mathscr{H}^0(\mathscr{L}^*)$, there is a spectral sequence with*

$$\boxed{E_2^{p,q} = H^p(H_\Phi^q(X; \mathscr{L}^*)) \Longrightarrow H_\Phi^{p+q}(X; \mathscr{L}).}$$ □

In particular, the edge homomorphism gives a canonical homomorphism

$$\rho : H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H_\Phi^p(X; \mathscr{L})$$

satisfying the naturality relations resulting from formulas (4), (5), (10), and (11) of Section 1. If $\mathscr{L}^*$ is a resolution of $\mathscr{L}$, then it follows from the naturality that this ρ is identical to the homomorphism ρ of II-5.15.

Also, there is the following generalization of II-4.1:

2.5. Theorem. (Fourth fundamental theorem.) *Let $\mathscr{L}^*$ be a differential sheaf such that $\mathscr{H}^q(\mathscr{L}^*) = 0$ for $0 \neq q < N$ and $H^*(H_\Phi^q(X; \mathscr{L}^*)) = 0$ for $q > 0$. Assume either that* $\dim_{\Phi,L} X < \infty$ *or that $\mathscr{L}^*$ is bounded below. Then with $\mathscr{L} = \mathscr{H}^0(\mathscr{L}^*)$, the edge homomorphism*

$$\xi^n : H_\Phi^n(X; \mathscr{L}) = E_2^{n,0} \twoheadrightarrow E_\infty^{n,0} \rightarrowtail H^n(\Gamma_\Phi(\mathscr{L}^*))$$

in the spectral sequence 2.1 *is an isomorphism for $n < N$ and a monomorphism for $n = N$.*[4] *In particular, for $N = \infty$ there is an isomorphism*

$$\boxed{H^p(\Gamma_\Phi(\mathscr{L}^*)) \approx H_\Phi^p(X; \mathscr{L}).}$$

Proof. Since $E_2^{p,q} = 0$ for $0 \neq q < N$, ξ^n is an isomorphism for $n < N$. Also, $E_2^{N,0} \twoheadrightarrow E_\infty^{N,0}$ is an isomorphism, whence ξ^N is a monomorphism. □

We shall conclude this section with six examples of applications.

[4]Also see Exercise 32.

2.6. Example. Let ${}_S\mathscr{H}^*(X;\mathscr{A}) = \mathscr{H}^*(\mathscr{S}^*(X;L)\otimes\mathscr{A})$ be the singular cohomology sheaf. If Φ is paracompactifying, then $\mathscr{S}^*(X;L)\otimes\mathscr{A}$ is Φ-fine, whence 2.1 yields a spectral sequence with

$$\boxed{E_2^{p,q} = H^p_\Phi(X;{}_S\mathscr{H}^q(X;\mathscr{A})) \Longrightarrow {}_SH^{p+q}_\Phi(X;\mathscr{A}).}$$

There is also a similar spectral sequence with the subscript S replaced by Δ when $\mathscr{A}$ is locally constant. The edge homomorphism $H^p_\Phi(X;\mathscr{A}) \xrightarrow{\varphi} H^p_\Phi(X;{}_S\mathscr{H}^0(X;\mathscr{A})) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \rightarrowtail {}_SH^p_\Phi(X;\mathscr{A})$ is the map θ of III-1. If X is HLC^n_L, then ${}_\Delta\mathscr{H}^q(X;L) = 0$ for $0 \neq q \leq n$ and ${}_\Delta\mathscr{H}^0(X;L) \approx L$. Therefore ${}_\Delta H^k_\Phi(X;L) \approx H^k_\Phi(X;L)$ for $k \leq n$. In particular,

$$\boxed{HLC^n_L \Rightarrow clc^n_L.}$$

For some explicit cases see Exercises 22 to 25 and their solutions. ◇

2.7. Example. For the Alexander-Spanier sheaf $\mathscr{A}^*(X;L)\otimes\mathscr{B}$ we obtain from 2.4 a spectral sequence with

$$\boxed{E_2^{p,q} = H^p(H^q_\Phi(X;\mathscr{A}^*(X;L)\otimes\mathscr{B})) \Longrightarrow H^{p+q}_\Phi(X;\mathscr{B}).}$$

This reduces to the isomorphism ${}_AH^p_\Phi(X;\mathscr{B}) = H^p(\Gamma_\Phi(\mathscr{A}^*(X;L)\otimes\mathscr{B})) \approx H^p_\Phi(X;\mathscr{B})$ when Φ is paracompactifying or when $\mathscr{B} = L$ and X is hereditarily paracompact; see III-Exercise 4. ◇

2.8. Example. There is a spectral sequence with

$$\boxed{E_2^{p,q} = H^p_\Phi(X;\mathscr{H}^q(X,A;\mathscr{A})) \Longrightarrow H^{p+q}_\Phi(X,A;\mathscr{A}).}$$

(See II-Exercise 41.) ◇

2.9. Example. Let $\varDelta_*$ be the sheaf of germs of singular chains with coefficients in the given base ring L, as defined in I-Exercise 12, and let ${}_\Delta\mathscr{H}_*(X;L)$ denote its derived sheaf. Note that ${}_\Delta\mathscr{H}_*(X;L) = \mathscr{S}\!heaf(U \mapsto {}_\Delta H^c_*(X, X-U;L))$, where ${}_\Delta H^c_*(X,X-U;L)$ is the classical relative singular homology group based on finite chains. Suppose that Φ is a paracompactifying family of supports on X and that $\dim_{\Phi,L} X < \infty$. Let $\mathscr{L}^*$ be the differential sheaf defined by $\mathscr{L}^q = \varDelta_{-q}$. Since $\mathscr{L}^*$ is homotopically fine by II-Exercise 32b, it follows from II-Exercise 32a and 2.1 that there is a spectral sequence with

$$\boxed{E_2^{p,q} = H^p_\Phi(X;{}_\Delta\mathscr{H}_{-q}(X;L)) \Longrightarrow {}_\Delta H^\Phi_{-p-q}(X;L),}$$

where the right-hand side is $H_{-p-q}(\Gamma_\Phi(\varDelta_*))$, which is the classical singular homology group based on locally finite chains with supports in Φ.[5]

[5]That is, the union of the images of the simplices in a chain is in Φ.

If X is a topological n-manifold, then it is clear that ${}_\Delta\mathscr{H}_q(X;L) = 0$ for $q \neq n$. Moreover, the "orientation sheaf" $\mathscr{O} = {}_\Delta\mathscr{H}_n(X;L)$ is locally constant (constant if X is orientable, by definition), with stalks isomorphic to L. It follows that for an n-manifold X there is the isomorphism

$$H^p_\Phi(X;\mathscr{O}) \approx {}_\Delta H^\Phi_{n-p}(X;L).$$

This is the Poincaré duality theorem. It is easy to generalize this theorem so that it will apply to arbitrary coefficient sheaves. One may also obtain extensions to relative groups. Since this is done in considerably more generality in Chapter V, we shall not pursue the subject further here. ◇

2.10. Example. Let $\mathbb{N}$ be the positive integers with the topology as in II-Exercise 27. Let $A_1 \supset A_2 \supset \cdots$ be a decreasing sequence of subsets of a space X and let Φ be a family of supports on X. Assume either that Φ is paracompactifying and the A_i are arbitrary, or that Φ is arbitrary and each A_i is closed. Put $K = \bigcap A_n$ and *assume* that $K = \bigcap \bar{A}_n$ and that $\bar{A}_1 \in \Phi$. Let $\mathscr{A}^*$ be a flabby differential sheaf on X. For a given index p consider the inverse system $\{\Gamma_{\Phi|A_n}(\mathscr{A}^p)\}$. As in the exercise, this is a sheaf $\mathscr{L}^p$ on $\mathbb{N}$. We claim that it is acyclic. According to the exercise, to prove this we must show that for every sequence $\{a_1', a_2', \ldots\}$ of sections of $\mathscr{A}^p$ with $|a_n'| \subset A_n$ there exists such a sequence $\{a_1, a_2, \ldots\}$ with $a_{n+1} = a_n - a_n'$ for all n. Since $\mathscr{A}^p$ is Φ-soft, we can find a section $b_1 \in \Gamma_\Phi(\mathscr{A}^p)$ that coincides with a_1' on $X - A_2$. (When Φ is paracompactifying make $b_1 = a_1'$ on $X - \operatorname{int} A_2$.) Then $|b_1| \subset A_1$ and $|b_1 - a_1'| \subset A_2$. Next, we can find a section b_2 that coincides with $a_1' + a_2' - b_1$ on $X - A_3$. Then $|b_2| \subset A_2$ and $|b_2 + b_1 - a_1' - a_2'| \subset A_3$. Continue inductively in the obvious manner. Define $b = b_1 + b_2 + \cdots$ on $X - K$. (This makes sense because each point $x \in X - K = X - \bigcap \bar{A}_n = \bigcup (X - \bar{A}_n)$ has a neighborhood disjoint from all but a finite number of the A_n.) Since $\mathscr{A}^p$ is flabby, b extends to all of X with support necessarily in Φ. Now put $a_n = b - a_1' - \cdots - a_{n-1}'$. Then $a_{n+1} = a_n - a_n'$, and at least outside of K,

$$|a_n| = |b - a_1' - \cdots - a_{n-1}'| \subset |b_n + b_{n+1} + \cdots| \cup A_n \subset A_n,$$

as required.

Now the spectral sequence of 2.1 has

$$E_2^{p,q} = H^p(\mathbb{N};\mathscr{H}^q(\mathscr{L}^*)) \Longrightarrow H^{p+q}(\Gamma(\mathscr{L}^*)).$$

For the limit term we have

$$H^n(\Gamma(\mathscr{L}^*)) = H^n(\varprojlim \Gamma_{\Phi|A_i}(\mathscr{A}^*)) = H^n(\Gamma_{\Phi|K}(\mathscr{A}^*)),$$

by II-Exercise 27. Also, $E_2^{p,q} = 0$ for $p \neq 0, 1$ since $\dim \mathbb{N} = 1$. We also have

$$E_2^{0,q} = H^0(\mathbb{N};\{H^q(\Gamma_{\Phi|A_i}(\mathscr{A}^*))\}) = \varprojlim H^q(\Gamma_{\Phi|A_i}(\mathscr{A}^*))$$

and

$$E_2^{1,q} = H^1(\mathbb{N}; \{H^q(\Gamma_{\Phi|A_i}(\mathscr{A}^*))\}) = \varprojlim{}^1 H^q(\Gamma_{\Phi|A_i}(\mathscr{A}^*))$$

by the exercise. Since all differentials are zero and only two columns are nonzero, the spectral sequence degenerates into the exact sequences

$$0 \to E_2^{1,n-1} \to H^n(\Gamma_{\Phi|K}(\mathscr{A}^*)) \to E_2^{0,n} \to 0,$$

that is,

$$\boxed{\varprojlim{}^1 H^{n-1}(\Gamma_{\Phi|A_i}(\mathscr{A}^*)) \rightarrowtail H^n(\Gamma_{\Phi|K}(\mathscr{A}^*)) \twoheadrightarrow \varprojlim H^n(\Gamma_{\Phi|A_i}(\mathscr{A}^*)).}$$

Now let us specialize to the case in which each A_n is open, Φ is paracompactifying, and $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$. Then we have $H^n(\Gamma_{\Phi|K}(\mathscr{A}^*)) \approx H^n_{\Phi|K}(X;\mathscr{A}) \approx H^n(X, X-K;\mathscr{A})$ by II-12.1 and II-12.9 since K has a neighborhood in Φ. Also, $H^n(\Gamma_{\Phi|A_i}(\mathscr{A}^*)) \approx H^n_{\Phi|A_i}(A_i;\mathscr{A})$ since A_i is open. Therefore, for Φ paracompactifying, if $U_1 \supset U_2 \supset \dots$ is a decreasing sequence of open sets with $K = \bigcap U_i = \bigcap \overline{U}_i$ and $\overline{U}_1 \in \Phi$, then we get the exact sequence

$$\boxed{0 \to \varprojlim{}^1 H^{n-1}_{\Phi|U_i}(U_i;\mathscr{A}) \to H^n(X, X-K;\mathscr{A}) \to \varprojlim H^n_{\Phi|U_i}(U_i;\mathscr{A}) \to 0.} \qquad (13)$$

Particularly note the case in which X is locally compact Hausdorff, $\Phi = c$, and K is a point. In this case the exact sequence becomes

$$\boxed{0 \to \varprojlim{}^1 H^{n-1}_c(U;\mathscr{A}) \to H^n(X, X-\{x\};\mathscr{A}) \to \varprojlim H^n_c(U;\mathscr{A}) \to 0,}$$

where U ranges over the open neighborhoods of x and we assume that x has a countable neighborhood basis. Note that if $K = \bigcap \overline{U}_i = \varnothing$, then the middle term, whence each term, of (13) is zero. Thus this discussion is essentially an elaboration of the proof of the sum theorem II-16.40 in cohomological dimension theory. In V-5.15 we will have an application of these remarks to homology theory. ◇

2.11. Example. Let Φ be a family of supports on X and let $A_1 \subset A_2 \subset \cdots$ be an increasing sequence of subspaces with $X = \bigcup \operatorname{int} A_i$. Assume either that each A_i is open or that Φ is paracompactifying and each A_i is closed. Let $\mathscr{A}^*$ be a flabby resolution of a given sheaf $\mathscr{A}$ on X. Then one can treat the inverse system $\mathscr{L}^p = \{\Gamma_{\Phi \cap A_i}(\mathscr{A}^p|A_i)\}$ as in 2.10. This situation is somewhat easier since $\mathscr{L}^p$ is flabby on $\mathbb{N}$ in both cases; i.e., each restriction map $\Gamma_{\Phi\cap A_{i+1}}(\mathscr{A}^p|A_{i+1}) \to \Gamma_{\Phi\cap A_i}(\mathscr{A}^p|A_i)$ is surjective. As in 2.10 we get a spectral sequence

$$E_2^{p,q} = H^p(\mathbb{N}; \{H^q_{\Phi\cap A_i}(A_i;\mathscr{A})\}) \Longrightarrow H^{p+q}_\Psi(X;\mathscr{A})$$

in either of our two cases, where

$$\Psi = \{K \subset X \text{ closed} \mid K \cap A_i \in \Phi \cap A_i \text{ all } i\}.$$

This degenerates to the exact sequence

$$\boxed{0 \to \varprojlim{}^1 H^{n-1}_{\Phi\cap A_i}(A_i;\mathscr{A}) \to H^n_\Psi(X;\mathscr{A}) \to \varprojlim H^n_{\Phi\cap A_i}(A_i;\mathscr{A}) \to 0.}$$

In particular, if $\Phi = cld$, then there is an exact sequence

$$\boxed{0 \to \varprojlim{}^1 H^{n-1}(A_i;\mathscr{A}) \to H^n(X;\mathscr{A}) \to \varprojlim H^n(A_i;\mathscr{A}) \to 0} \tag{14}$$

whenever each A_i is open, or when X is paracompact and each A_i is closed with $X = \bigcup \operatorname{int} A_i$.

For example, consider the union X of the mapping cylinders of $\mathbb{S}^1 \xrightarrow{2} \mathbb{S}^1 \xrightarrow{3} \mathbb{S}^1 \xrightarrow{4} \cdots$ Let A_i be the union of the first i of these. Then $H^2(A_i;\mathbb{Z}) = 0$ and the inverse system

$$\{H^1(A_1;\mathbb{Z}) \leftarrow H^1(A_2;\mathbb{Z}) \leftarrow \cdots\}$$

has the form $\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3} \cdots$ In V-5.15 we shall show that $\varprojlim^1$ of this is $\operatorname{Ext}(\mathbb{Q},\mathbb{Z})$ and that this is an uncountable rational vector space. Therefore

$$H^2(X;\mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Q},\mathbb{Z}),$$

which is uncountable. On the other hand, the inverse sequence

$$\{H^1(A_1;\mathbb{Q}) \leftarrow H^1(A_2;\mathbb{Q}) \leftarrow \cdots\}$$

is flabby, and so

$$H^2(X;\mathbb{Q}) = 0.$$

If K is the one-point compactification of X and x is the point at infinity, then by (41) on page 136 we have that

$$H^3_x(K;\mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Q},\mathbb{Z}) \neq 0$$

even though $\dim_{\mathbb{Z}} K = 2$ and K is separable metric, contractible, and locally contractible.

As another example consider the space X of II-13.2. Let N_n be an open square of side $1/n$ about the origin, and put $K_n = X - N_n$. Then $H^1(K_n;\mathbb{Z}) = 0$. We deduce from (14) that

$$\varprojlim{}^1 H^0(K_n;\mathbb{Z}) \approx H^1(X;\mathbb{Z}),$$

which is uncountable. Note that $H^0(K_n;\mathbb{Z})$ is free abelian. Also note that the reasoning that showed that $H^1(X;\mathbb{Z})$ is uncountable is also valid over a countable field L as base ring. Thus we conclude that unlike Ext and Tor, $\varprojlim^1$ does not generally vanish over a field as base ring. However, an inverse sequence of *finite-dimensional* vector spaces is Mittag-Leffler, and so $\varprojlim^1$ does vanish in that case.

As a nonexample consider $X = \mathbb{S}^1$ and the arcs $A_n = \{e^{2\pi i\theta} \mid 0 \leq \theta \leq 1 - 1/n\}$. Then $\varprojlim^1 H^0(A_n;\mathbb{Z}) = 0 = \varprojlim H^1(A_n;\mathbb{Z})$, but $H^1(X;\mathbb{Z}) \neq 0$. This shows that the condition $X = \bigcup \operatorname{int} A_i$ is essential. ◇

3 Direct image relative to a support family

Let $f : X \to Y$ and let Ψ be a family of supports on X. To simplify notation we let $U^\bullet = f^{-1}(U)$ for $U \subset Y$.

3.1. Definition. *For an open set $U \subset Y$ let $\Psi(U)$ be the family of relatively closed subsets A of $U^\bullet$ such that each point $y \in U$ has a neighborhood N with $A \cap N^\bullet \in \Psi \cap N^\bullet$.*

Note that if this condition holds for N then it holds for any smaller neighborhood of y, and so N can be taken to be open in this definition.

Taking $N \subset U$, which is no loss of generality, we remark that the condition $A \cap N^\bullet \in \Psi \cap N^\bullet$ is equivalent to $\overline{A \cap N^\bullet} \in \Psi$. To see this, note that if $\overline{A \cap N^\bullet} \in \Psi$ then $A \cap N^\bullet = \overline{A \cap N^\bullet} \cap N^\bullet \in \Psi \cap N^\bullet$ since $\overline{A \cap N^\bullet} \cap N^\bullet \subset \bar{A} \cap U^\bullet = A \cap U^\bullet \subset A$. Conversely, if $A \cap N^\bullet \in \Psi \cap N^\bullet$, then $A \cap N^\bullet = K \cap N^\bullet$ for some $K \in \Psi$, and so $\overline{A \cap N^\bullet} \subset \overline{K} = K \in \Psi$.

Also note that if Y is regular, then N, and hence $N^\bullet$, can be taken to be closed, in which case $\Psi \cap N^\bullet = \Psi | N^\bullet$, which may make it easier to understand this family.

Also note that if Y is compact, then $\Psi(Y) = \Psi$ because a family of supports is closed under finite unions.

If $\mathscr{A}$ is a sheaf on X, then the presheaf $U \mapsto \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet)$ is clearly a conjunctive monopresheaf, and hence a sheaf.

3.2. Definition. *Let $\mathscr{A}$ be a sheaf on X and let Ψ be a family of supports on X. Then the "direct image of $\mathscr{A}$ with respect to Ψ" is defined to be $f_\Psi \mathscr{A}$, where*

$$\boxed{(f_\Psi \mathscr{A})(U) = \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet).}$$

Note that $\Psi(U) \supset \Psi \cap U^\bullet$, so that there is the inclusion

$$\Gamma_{\Psi \cap U^\bullet}(\mathscr{A}|U^\bullet) \hookrightarrow \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet) = (f_\Psi \mathscr{A})(U). \tag{15}$$

For a section $s \in \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet)$ and for any point $y \in U$ there is, by definition, an open neighborhood N of y such that $|s| \cap N^\bullet \in \Psi \cap N^\bullet$. Then $s|N^\bullet \in \Gamma_{\Psi \cap N^\bullet}(\mathscr{A}|N^\bullet)$. It follows that the map (15) of presheaves induces an isomorphism of the generated sheaves. Thus $f_\Psi \mathscr{A}$ can be described as the sheaf *generated* by the monopresheaf

$$U \mapsto \Gamma_{\Psi \cap U^\bullet}(\mathscr{A}|U^\bullet),$$

a useful fact. However, this presheaf is not conjunctive.

Obviously, the functor f_Ψ is left exact.

3.3. Proposition. *If $\mathscr{L}$ is a flabby sheaf on X and if Φ is a paracompactifying family of supports on Y, then $f_\Psi \mathscr{L}$ is Φ-soft for any family Ψ of supports on X.*[6]

[6] Also see II-5.7 and Exercise 1.

Proof. Let $s \in (f_\Psi \mathscr{L})(K)$, where $K \in \Phi$. By II-9.5, there is an open neighborhood U of K and an extension $s' \in (f_\Psi \mathscr{L})(U) = \Gamma_{\Psi(U)}(\mathscr{L}|U^\bullet)$ of s. Since K has a paracompact neighborhood and since paracompact spaces are normal, there is an open neighborhood V of K with $\overline{V} \subset U$ and $\overline{V} \in \Phi$. Let $s'' = s'|V^\bullet \in \Gamma_{\Psi(V)}(\mathscr{L}|V^\bullet)$ and let t be the zero section of $\mathscr{L}$ over $X - (\overline{V}^\bullet \cap |s'|)$.

Since s'' and t agree where both are defined and since $\mathscr{L}$ is flabby, there is a section $s^\bullet \in \mathscr{L}(X)$ extending both s'' and t. Then $|s^\bullet| \subset \overline{V}^\bullet \cap |s'| \in \Psi(Y)$. Thus we may regard $s^\bullet$ as an element of $(f_\Psi \mathscr{L})(Y)$, and as a section of $f_\Psi \mathscr{L}$ it has support in $\overline{V} \in \Phi$. Thus $s^\bullet$ is the required extension of s to $\Gamma_\Phi(f_\Psi \mathscr{L})$. □

3.4. Proposition. *Let $\mathscr{S}$ and $\mathscr{R}$ be sheaves of rings on X and Y respectively, and suppose that there exists an f-cohomomorphism $\mathscr{R} \rightsquigarrow \mathscr{S}$ preserving the ring structures. Then for any $\mathscr{S}$-module $\mathscr{A}$ on X, $f_\Psi \mathscr{A}$ is an $\mathscr{R}$-module. In particular, if $\mathscr{A}$ is a $\mathscr{C}^0(X;L)$-module (where L is the base ring), then $f_\Psi \mathscr{A}$ is a $\mathscr{C}^0(Y;L)$-module.*

Proof. Note that $(f_\Psi \mathscr{A})(U) = \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet)$ is a module over $\mathscr{S}(U^\bullet)$ via the cup product

$$\mathscr{S}(U^\bullet) \otimes \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet) \to \Gamma_{\Psi(U)}((\mathscr{S} \otimes \mathscr{A})|U^\bullet) \to \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet),$$

where the last map is induced by the module product $\mathscr{S} \otimes \mathscr{A} \to \mathscr{A}$. Thus $(f_\Psi \mathscr{A})(U)$ is also an $\mathscr{R}(U)$-module, via the f-cohomomorphism $\mathscr{R}(U) \to \mathscr{S}(U^\bullet)$, whence $f_\Psi \mathscr{A}$ is an $\mathscr{R}$-module as claimed. □

3.5. We shall need the following elementary observations concerning the naturality of the direct image under cohomomorphisms. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{h} & Y_2 \end{array}$$

be a commutative diagram of maps. Let $\mathscr{A}_i$ be sheaves on X_i $(i = 1, 2)$ and let $k : \mathscr{A}_2 \rightsquigarrow \mathscr{A}_1$ be a g-cohomomorphism. Let Ψ_i be a family of supports on X_i with $g^{-1}(\Psi_2) \subset \Psi_1$.

For $U_2 \subset Y_2$, let $U_2^\bullet = f_2^{-1}(U_2)$, $U_1 = h^{-1}(U_2)$ and $U_1^\bullet = f_1^{-1}(U_1) = g^{-1}(U_2^\bullet)$. The homomorphisms

$$\Gamma_{\Psi_2 \cap U_2^\bullet}(\mathscr{A}_2|U_2^\bullet) \to \Gamma_{\Psi_1 \cap U_1^\bullet}(\mathscr{A}_1|U_1^\bullet)$$

induced by k form an h-cohomomorphism of presheaves and hence give rise to an h-cohomomorphism

$$k^* : f_{2,\Psi_2}\mathscr{A}_2 \rightsquigarrow f_{1,\Psi_1}\mathscr{A}_1$$

of induced sheaves.

If $\mathscr{A}_i'$ are also sheaves on X_i and

$$\begin{array}{ccc}\mathscr{A}_2 & \overset{k}{\rightsquigarrow} & \mathscr{A}_1 \\ \downarrow & & \downarrow \\ \mathscr{A}_2' & \overset{k'}{\rightsquigarrow} & \mathscr{A}_1'\end{array}$$

is a commutative diagram of homomorphisms and g-cohomomorphisms, then the diagram

$$\begin{array}{ccc}f_{2,\Psi_2}(\mathscr{A}_2) & \overset{k^*}{\rightsquigarrow} & f_{1,\Psi_1}(\mathscr{A}_1) \\ \downarrow & & \downarrow \\ f_{2,\Psi_2}(\mathscr{A}_2') & \overset{k'^*}{\rightsquigarrow} & f_{1,\Psi_1}(\mathscr{A}_1')\end{array}$$

also commutes.

The remaining material of this section is not used in this chapter[7] and is presented to aid the reader's understanding of the these matters. The following is actually a special case of 4.2, but this direct treatment may be useful.

3.6. Proposition. *Let $f : X \to Y$ be Ψ-closed for a family Ψ of supports on X. Then for any sheaf $\mathscr{A}$ on X, the sheaf $f_\Psi \mathscr{A}$ is concentrated on $f(X)$.*

Proof. We have that $f_\Psi\mathscr{A} = \mathscr{S}\!heaf(U \mapsto \Gamma_{\Psi\cap U^\bullet}(\mathscr{A}|U^\bullet))$. Let $y \in U$ with $y \notin f(X)$, and let $s \in \Gamma_{\Psi\cap U^\bullet}(\mathscr{A}|U^\bullet)$. Then $|s| = K \cap U^\bullet$ for some $K \in \Psi$. Now, $f(K)$ is closed and $y \notin f(K)$. Thus there is an open neighborhood $V \subset U$ of y with $V^\bullet \cap K = \varnothing$. Therefore $s|V^\bullet = 0$, showing that s induces $0 \in (f_\Psi\mathscr{A})_y = \varinjlim \Gamma_{\Psi\cap U^\bullet}(\mathscr{A}|U^\bullet)$. Consequently, $(f_\Psi\mathscr{A})_y = 0$. □

3.7. Corollary. *Let $i : A \hookrightarrow X$ be the inclusion of a subspace and let Ψ be a family of supports on A such that each member of Ψ is closed in X. Also assume that every point of A has a neighborhood, in A, that is in Ψ.*[8] *Then $i_\Psi\mathscr{A} \approx \mathscr{A}^X$.*

Proof. By 3.6, $i_\Psi\mathscr{A}$ vanishes outside of A. Now $U^\bullet = U \cap A$ and

$$i_\Psi\mathscr{A} = \mathscr{S}\!heaf\,(U \mapsto \Gamma_{\Psi\cap U^\bullet}(\mathscr{A}|U \cap A))\,.$$

Thus $(i_\Psi\mathscr{A})|A = \mathscr{S}\!heaf\,(U \cap A \mapsto \Gamma_{\Psi\cap U^\bullet}(\mathscr{A}|U \cap A))$. Since every point of A has a neighborhood, in A, in Ψ, this is the same as the sheaf generated by the presheaf $U \cap A \mapsto \Gamma(\mathscr{A}|U \cap A)$, which is just $\mathscr{A}$. But $\mathscr{A}^X$ is the unique sheaf inducing $\mathscr{A}$ on A and zero on $X - A$. □

3.8. Corollary. *If $i : A \hookrightarrow X$ is the inclusion of a locally closed subspace A of a locally compact Hausdorff space X, then we have $i_c\mathscr{A} \approx \mathscr{A}^X$.* □

[7]The last corollary is used in Chapter V.

[8]This implies that A is locally closed in X.

4 The Leray sheaf

Let $A \subset X$, let Ψ be a family of supports on X, let $\mathscr{A}$ be a sheaf on X, and let $f : X \to Y$ be a map.

4.1. Definition. *The "Leray sheaf of the map f mod $f|A$" is the sheaf*

$$\boxed{\mathscr{H}^*_\Psi(f, f|A; \mathscr{A}) = \mathscr{S}\!\mathit{heaf}\,(U \mapsto H^*_{\Psi \cap U^\bullet}(U^\bullet, U^\bullet \cap A; \mathscr{A}|U^\bullet))}$$

on Y.

For $A = \varnothing$, $\mathscr{H}^0_\Psi(f; \mathscr{A}) = \mathscr{S}\!\mathit{heaf}\,\big(U \mapsto \Gamma_{\Psi(U)}(\mathscr{A}|U^\bullet) = (f_\Psi \mathscr{A})(U)\big)$, and so

$$\boxed{\mathscr{H}^0_\Psi(f; \mathscr{A}) = f_\Psi \mathscr{A}.}$$

Now, $f_\Psi \mathscr{C}^*(X, A; \mathscr{A})$ is generated by the presheaf

$$\begin{aligned} U \mapsto\ & \Gamma_{\Psi \cap U^\bullet}(\mathscr{C}^*(X, A; \mathscr{A})|U^\bullet) \\ =\ & \Gamma_{\Psi \cap U^\bullet}(\mathscr{C}^*(U^\bullet, U^\bullet \cap A; \mathscr{A}|U^\bullet)) \\ =\ & C^*_{\Psi \cap U^\bullet}(U^\bullet, U^\bullet \cap A; \mathscr{A}|U^\bullet). \end{aligned}$$

Thus its derived sheaf is the sheaf generated by the presheaf

$$U \mapsto H^*_{\Psi \cap U^\bullet}(U^\bullet, U^\bullet \cap A; \mathscr{A}|U^\bullet),$$

and so it is the Leray sheaf $\mathscr{H}^*_\Psi(f, f|A; \mathscr{A})$. That is, the Leray sheaf is the derived sheaf

$$\boxed{\mathscr{H}^*_\Psi(f, f|A; \mathscr{A}) = \mathscr{H}^*(f_\Psi \mathscr{C}^*(X, A; \mathscr{A})).}$$

More generally,

$$\boxed{\mathscr{H}^*_\Psi(f, f|A; \mathscr{A}) \approx \mathscr{H}^*(f_\Psi \mathscr{A}^*),}$$

where $\mathscr{A}^*$ is any flabby differential sheaf of the form

$$\mathscr{A}^* = \operatorname{Ker}\{k : \mathscr{L}^* \to i\mathscr{N}^*\}, \tag{16}$$

where $\mathscr{L}^*$ is a flabby resolution of $\mathscr{A}$ on X, $\mathscr{N}^*$ is a flabby resolution of $\mathscr{A}|A$ on A, and $k : \mathscr{L}^* \rightsquigarrow \mathscr{N}^*$ is a surjective i-cohomomorphism, where $i : A \hookrightarrow X$. For example, $\mathscr{A}^*$ could be taken to be $\mathscr{F}^*(X, A; \mathscr{A})$; see II-12.17.

For $A = \varnothing$, the Leray sheaf is denoted by $\mathscr{H}^*_\Psi(f; \mathscr{A})$, and by the preceding remarks, it is the derived sheaf

$$\boxed{\mathscr{H}^*_\Psi(f; \mathscr{A}) \approx \mathscr{H}^*(f_\Psi \mathscr{A}^*)}$$

for any flabby (e.g., injective) resolution $\mathscr{A}^*$ of $\mathscr{A}$. Therefore $\mathscr{H}^p_\Psi(f; \mathscr{A})$ is just the pth right derived functor of the left exact functor f_Ψ.

The exact cohomology sequences of the pairs $(U^\bullet, U^\bullet \cap A)$ induce an exact sequence

$$\boxed{\cdots \to \mathscr{H}^p_\Psi(f, f|A; \mathscr{A}) \to \mathscr{H}^p_\Psi(f; \mathscr{A}) \to \mathscr{H}^p_\Psi(f|A; \mathscr{A}) \to \mathscr{H}^{p+1}_\Psi(f, f|A; \mathscr{A}) \to \cdots} \tag{17}$$

of sheaves on Y.

For $y \in Y$, let $y^\bullet = f^{-1}(y)$. The restriction map

$$H^*_{\Psi \cap U^\bullet}(U^\bullet, U^\bullet \cap A; \mathscr{A}|U^\bullet) \to H^*_{\Psi \cap y^\bullet}(y^\bullet, y^\bullet \cap A; \mathscr{A}|y^\bullet)$$

induces a homomorphism

$$\boxed{r^*_y : \mathscr{H}^*_\Psi(f, f|A; \mathscr{A})_y \to H^*_{\Psi \cap y^\bullet}(y^\bullet, y^\bullet \cap A; \mathscr{A}).} \tag{18}$$

It will turn out to be of great importance to obtain sufficient conditions for r^*_y to be an isomorphism. Since r^*_y maps the sequence (17) into the cohomology sequence of the pair $(y^\bullet, y^\bullet \cap A)$, sufficient conditions in the absolute case, for both f and $f|A$, will also be sufficient for the relative case.

4.2. Proposition. *If f is Ψ-closed and $y^\bullet = f^{-1}(y)$ is Ψ-taut in X (e.g., if Ψ is paracompactifying or if Ψ contains a neighborhood of $y^\bullet$ and $y^\bullet$ is compact and relatively Hausdorff in X), then*

$$r^*_y : \mathscr{H}^*_\Psi(f; \mathscr{A})_y \to H^*_{\Psi \cap y^\bullet}(y^\bullet; \mathscr{A})$$

is an isomorphism for all $\mathscr{A}$. (Also see Section 7.)

Proof. This follows immediately from II-10.6. □

4.3. Consider the situation of 3.5. Let $A_i \subset X_i$, $i = 1, 2$, be such that $g(A_1) \subset A_2$ and let $\mathscr{A}^*_i = \mathscr{C}^*(X_i, A_i; \mathscr{A}_i)$. The g-cohomomorphism $k : \mathscr{A}_2 \rightsquigarrow \mathscr{A}_1$ extends to one $\mathscr{A}^*_2 \rightsquigarrow \mathscr{A}^*_1$ of differential sheaves. Thus we have the induced h-cohomomorphism

$$f_{2,\Psi_2}(\mathscr{A}^*_2) \rightsquigarrow f_{1,\Psi_1}(\mathscr{A}^*_1)$$

of differential sheaves and hence an h-cohomomorphism

$$\mathscr{H}^*_{\Psi_2}(f_2, f_2|A_2; \mathscr{A}_2) \rightsquigarrow \mathscr{H}^*_{\Psi_1}(f_1, f_1|A_1; \mathscr{A}_1) \tag{19}$$

of graded sheaves.

Again, in the absolute case, consider the special case of an inclusion $B \subset Y$ and $B^\bullet = f^{-1}(B) \subset X$. The cohomomorphism (19) gives rise to a homomorphism

$$r^* : \mathscr{H}^*_\Psi(f; \mathscr{A})|B \to \mathscr{H}^*_{\Psi \cap B^\bullet}(f|B^\bullet; \mathscr{A}|B^\bullet). \tag{20}$$

If B is open, then it is clear that r^* is an isomorphism, but this is false for general B. However, we have the following result:

4.4. Proposition. *Let $f : X \to Y$ be Ψ-closed and let $B \subset Y$ be a subspace such that for each $y \in B$, $f^{-1}(y)$ is Ψ-taut in X. Then the natural map*

$$r^* : \mathscr{H}^*_\Psi(f;\mathscr{A})|B \to \mathscr{H}^*_{\Psi\cap B^\bullet}(f|B^\bullet;\mathscr{A}|B^\bullet).$$

is an isomorphism.

Proof. This follows immediately from 4.2 and II-12.14, which imply that r^* in (20) is an isomorphism on each stalk. □

4.5. Let $\mathscr{A}$ be a sheaf on X and $\mathscr{B}$ a sheaf on Y, where $f : X \to Y$. For open sets $U \subset Y$, the cup product on $U^\bullet = f^{-1}(U)$ induces the map

$$\begin{aligned} H^p_\Psi(U^\bullet, U^\bullet \cap A;\mathscr{A}) \otimes \mathscr{B}(U) &\to H^p_\Psi(U^\bullet, U^\bullet \cap A;\mathscr{A}) \otimes (f^*\mathscr{B})(U^\bullet) \\ &= H^p_\Psi(U^\bullet, U^\bullet \cap A;\mathscr{A}) \otimes H^0(U^\bullet; f^*\mathscr{B}) \\ &\to H^p_\Psi(U^\bullet, U^\bullet \cap A;\mathscr{A} \otimes f^*\mathscr{B}) \end{aligned}$$

of presheaves. This, in turn, induces a homomorphism

$$\boxed{\mathscr{H}^p_\Psi(f, f|A;\mathscr{A}) \otimes \mathscr{B} \to \mathscr{H}^p_\Psi(f, f|A;\mathscr{A} \otimes f^*\mathscr{B})} \qquad (21)$$

of the generated sheaves.

If f is Ψ-closed and if $y^\bullet = f^{-1}(y)$ is Ψ-taut in X, then on the stalks at y, (21) for A empty is just the cup product

$$H^p_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}) \otimes M \to H^p_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A} \otimes M),$$

where $M = \mathscr{B}_y$.

From these remarks and the Universal Coefficient Theorem II-15.3 we deduce:

4.6. Proposition. *Let $f : X \to Y$ be a closed map such that each $y^\bullet = f^{-1}(y)$ is compact and relatively Hausdorff in X. Let $\mathscr{A}$ be a sheaf on X and $\mathscr{B}$ a sheaf on Y such that $\mathscr{A} \widehat{*} \mathscr{B} = 0$ and such that $H^{p+1}(y^\bullet;\mathscr{A}) * \mathscr{B}_y = 0$ for all $y \in Y$. Then the homomorphism*

$$\mathscr{H}^p(f;\mathscr{A}) \otimes \mathscr{B} \to \mathscr{H}^p(f;\mathscr{A} \otimes f^*\mathscr{B})$$

of (21) *is an isomorphism.*[9] □

4.7. Again, let $f : X \to Y$ and $A \subset X$. Let Ψ and Θ be families of supports on X and let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X. For open sets $U \subset Y$, the cup product

$$H^p_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}) \otimes H^q_{\Theta\cap U^\bullet}(U^\bullet, U^\bullet \cap A;\mathscr{B}) \xrightarrow{\cup} H^{p+q}_{\Psi\cap\Theta\cap U^\bullet}(U^\bullet, U^\bullet \cap A;\mathscr{A} \otimes \mathscr{B})$$

[9] Also see Exercise 4.

on $U^\bullet = f^{-1}(U)$ is compatible with restrictions and therefore defines a cup product of the generated sheaves:

$$\boxed{\mathscr{H}^p_\Psi(f;\mathscr{A}) \otimes \mathscr{H}^q_\Theta(f,f|A;\mathscr{B}) \xrightarrow{\cup} \mathscr{H}^{p+q}_{\Psi\cap\Theta}(f,f|A;\mathscr{A}\otimes\mathscr{B}).}$$

Note that in particular, for A empty and $p = 0 = q$ this provides a natural map

$$\boxed{f_\Psi(\mathscr{A}) \otimes f_\Theta(\mathscr{B}) \to f_{\Psi\cap\Theta}(\mathscr{A}\otimes\mathscr{B}),}$$

which is, of course, easy to describe directly from Definition 3.2. The reader should be aware of the fact that in the absolute case, this cup product satisfies an analogue of the uniqueness theorem II-7.1 since II-6.2 generalizes word for word to the case of functors from the category of sheaves on X to the category of sheaves on Y. (Specializing Y to be a point retrieves the original situation.)

4.8. Example. Here is a simple, but not completely trivial, example of a Leray sheaf. Let $X = \mathbb{RP}^2$ considered as the unit disk in $\mathbb{R}^2$ with antipodal points on the boundary identified. The circle group $G = \mathbb{S}^1$ acts on X by rotations in the plane. Let $Y = X/G$ be the orbit space, which can be considered as the cross section consisting of the interval $[0,1]$ on the real axis. Let $f : X \to Y$ be the orbit map. The point 0 corresponds to a fixed point; the orbit corresponding to 1 is a circle wrapping around twice, and all other orbits are free. From this it is clear that the Leray sheaf $\mathscr{H}^1(f;\mathbb{Z})$ is the subsheaf of the constant sheaf $\mathbb{Z}$ on $[0,1]$ that is 0 at 0, $2\mathbb{Z}$ at 1, and $\mathbb{Z}$ at interior points. (For example, a generator of $H^1(f^{-1}(\frac{1}{2},1];\mathbb{Z})$ gives a section of $\mathscr{H}^1(f;\mathbb{Z})$ over $(\frac{1}{2},1]$ that induces a generator of the stalk at 1 since $f^{-1}(1)$ is a deformation retract of $f^{-1}(\frac{1}{2},1]$, but it gives twice a generator in the stalk at $x \neq 1$ since $H^1(f^{-1}(\frac{1}{2},1];\mathbb{Z}) \to H^1(f^{-1}(x);\mathbb{Z})$ does that.) Since all orbits are connected, $\mathscr{H}^0(f;\mathbb{Z}) \approx \mathbb{Z}$, and of course, $\mathscr{H}^p(f;\mathbb{Z}) = 0$ for $p \neq 0,1$. ◇

4.9. Example. Let $X = (\mathbb{I}\times\mathbb{I} - \{\langle 0,\frac{1}{2}\rangle\})/\{\langle 0,y\rangle \sim \langle 0,1-y\rangle \mid 0 \le y < \frac{1}{2}\}$, a square with the midpoint of one side deleted and the remaining two rays of that side identified. Let $Y = \mathbb{I}$ and let $f : X \to Y$ be the projection $f(x,y) = x$. Then the Leray sheaf $\mathscr{H}^1(f;\mathbb{Z})$ vanishes away from 0, but the stalk at $0 \in Y$ is $\mathscr{H}^1(f;\mathbb{Z})_0 \approx \mathbb{Z}$ since $H^1(f^{-1}[0,\varepsilon);\mathbb{Z}) \approx \mathbb{Z}$. Therefore $\mathscr{H}^1(f;\mathbb{Z}) \approx \mathbb{Z}_{\{0\}}$ is nonzero even though $H^1(f^{-1}(y);\mathbb{Z}) = 0$ for all $y \in Y$. Note that each $f^{-1}(y)$ is taut, but f is not closed. Now, X and Y are locally compact and f is c-closed, of course. The Leray sheaf $\mathscr{H}^1_c(f;\mathbb{Z}) = 0$ since, for example, $f^{-1}[0,\varepsilon] \approx \mathbb{D}^2 - \{0\} \approx \mathbb{S}^1 \times (0,1]$ and $H^1_c(\mathbb{S}^1 \times (0,1];\mathbb{Z}) \approx H^1(\mathbb{S}^1 \times [0,1], \mathbb{S}^1 \times \{0\};\mathbb{Z}) = 0$. By similar calculations, $\mathscr{H}^0_c(f;\mathbb{Z}) = \mathbb{Z}_{(0,1]}$. By 4.2 these stalks of $\mathscr{H}^*_c(f;\mathbb{Z})$ are the cohomology groups with compact supports of the corresponding fibers, as the reader may verify. ◇

In some situations it is possible to prove some general niceness results about the Leray sheaf. The material in the remainder of this section is based largely on the work of Dydak and Walsh [35].

4.10. Definition. *Let L be a principal ideal domain. A presheaf S of L-modules on X is said to be locally finitely generated if for each point $x \in X$ and open neighborhood U of x, there is an open neighborhood $V \subset U$ of x such that* $\operatorname{Im}(r_{V,U} : S(U) \to S(V))$ *is finitely generated over L. A sheaf $\mathscr{S}$ is said to be locally finitely generated if $\mathscr{S} = \mathscr{S}\!\mathit{heaf}(S)$ for some locally finitely generated presheaf S.*

4.11. Theorem. *Let L be a principal ideal domain. Suppose that S is a locally finitely generated presheaf of L-modules on the separable metric space Y. Let $\mathscr{S} = \mathscr{S}\!\mathit{heaf}(S)$ and suppose that $\mathscr{S}$ has finitely generated stalks over L. Let G be a given L-module and put $Y_G = \{y \in Y \mid \mathscr{S}_y \approx G\}$. Then Y_G is covered by a countable collection of sets Y_i, closed in Y, such that each $\mathscr{S}|Y_i$ contains a constant subsheaf $\mathscr{G}$ with stalks G such that $\mathscr{G}_y$ is a direct summand of $\mathscr{S}_y$ for each $y \in Y_i$.*

Proof. Let $\{U_i\}$ be a countable basis for the topology of Y. Call a triple i, j, k of indices *admissible* if $\overline{U}_k \subset U_j$, $\overline{U}_j \subset U_i$, and $G_{i,j} = \operatorname{Im} r_{U_j,U_i}$ is finitely generated. For $y \in U_j$ let $r_{y,i,j} : G_{i,j} \to \mathscr{S}_y$ denote the canonical map. Then for each $y \in Y$ and neighborhood U of y, there exists an admissible triple i, j, k such that $y \in U_k$, $U_i \subset U$, and $r_{y,i,j} : G_{i,j} \to \mathscr{S}_y$ is an isomorphism onto $\mathscr{S}_y$ (since $\mathscr{S}_y$ is finitely generated). For an admissible triple i, j, k, let

$$B_{i,j} = \{y \in Y_G \cap U_j \mid r_{y,i,j} \text{ is an isomorphism}\}$$

and

$$K_{i,j,k} = \overline{B_{i,j} \cap U_k}.$$

Suppose that $x \in K_{i,j,k}$. Then there is an admissible triple p, q, r such that $x \in U_r$, $U_p \subset U_j$, and $r_{x,p,q} : G_{p,q} \to \mathscr{S}_x$ is an isomorphism. Since $x \in \overline{B_{i,j} \cap U_k}$, there is a point $y \in B_{i,j} \cap U_r$. Since $U_p \subset U_j$, there is the commutative diagram

$$\begin{array}{ccc} G_{i,j} & \xrightarrow{\approx} & \mathscr{S}_y \approx G \\ {\scriptstyle r_{x,i,j}}\big\downarrow & \searrow & \big\uparrow \\ \mathscr{S}_x & \xleftarrow[\approx]{} & G_{p,q} \end{array}$$

showing that G is a direct summand of $\mathscr{S}_x$. By construction, the closed sets $K_{i,j,k}$ cover Y_G. The maps $r_{x,i,j} : G_{i,j} \to \mathscr{S}_x$ form a homomorphism of the constant sheaf $\mathscr{G}$ with stalks G to $\mathscr{S}|K_{i,j,k}$, and we have shown this to be a monomorphism onto a direct summand on each stalk. □

4.12. Corollary. *With the hypotheses of* 4.11, *suppose that Y is locally compact or complete metrizable and assume that the stalks of $\mathscr{S}$ are mutually isomorphic. Then there is an open dense subspace U of Y over which $\mathscr{S}$ is locally constant.*

Proof. By the Baire category theorem some Y_i must have nonempty interior. Thus $\bigcup \operatorname{int} Y_i$ is dense in Y and $\mathscr{S}$ is locally constant on it. □

4.13. Theorem. *If $f : X \to Y$ is a proper closed map between separable metric spaces and L is a principal ideal domain then the Leray sheaf $\mathscr{H}^n(f;L)$ is locally finitely generated over L.*[10]

Proof. We may assume that X is a subspace of the Hilbert cube $\mathbb{I}^\infty$; see [19, I-9]. Consider the presheaf S on Y given by $S(U) = H^n(N_{d(U)}(U^\bullet);L)$, where $d(U)$ is the diameter of U and $N_{d(U)}(U^\bullet)$ is the set of points in $\mathbb{I}^\infty$ of distance less than $d(U)$ from $U^\bullet$. Let $\mathscr{S} = \mathscr{S}\!\mathit{heaf}(S)$.

We claim that the sets $N_{d(U)}(U^\bullet)$ form a neighborhood basis of $y^\bullet$ in $\mathbb{I}^\infty$. If not, then there is an open neighborhood V of $y^\bullet$ in $\mathbb{I}^\infty$ containing no set of the form $N_{1/n}(f^{-1}(N_{1/2n}(y)))$. Let $z_n \in N_{1/n}(f^{-1}(N_{1/2n}(y))) - V$ and let $x_n \in f^{-1}(N_{1/2n}(y))$ with $\operatorname{dist}(x_n, z_n) < 1/n$. Then $f(x_n) \in N_{1/2n}(y)$, so that $f(x_n) \to y$. It follows that $\{y, f(x_1), f(x_2), \ldots\}$ is compact. Since f is proper, some subsequence $\{x_{n_i}\}$ must converge to a point $x \in X$. Then $f(x) = \lim f(x_{n_i}) = y$, so that $x \in y^\bullet$. Since $\operatorname{dist}(x_n, z_n) < 1/n$, $\lim z_{n_i} = x \in y^\bullet$, contrary to the fact that no z_{n_i} is in V.

Thus $\mathscr{S}_y = H^n(y^\bullet;L)$. Now, the canonical map $H^n(N_{d(U)}(U^\bullet);L) \to H^n(y^\bullet;L)$ factors through $H^n(U^\bullet;L)$, and the latter generate the Leray sheaf $\mathscr{H}^n(f;L)$. This gives a map $\mathscr{S} \to \mathscr{H}^n(f;L)$, which is an isomorphism on each stalk. Therefore, $\mathscr{S} \approx \mathscr{H}^n(f;L)$. Since the Hilbert cube $\mathbb{I}^\infty$ is clc_L, it follows from II-17.5 that the presheaf S is locally finitely generated. □

4.14. Corollary. *Let $f : X \to Y$ be a proper closed map between separable metric spaces such that the stalks $H^*(y^\bullet;L)$ of $\mathscr{H}^*(f;L)$ are finitely generated for all $y \in Y$. Let Y_G be as in* 4.11. *Then Y_G is covered by a countable family of sets Y_i, closed in Y, such that each $\mathscr{H}^*(f;L)|Y_i$ contains a constant subsheaf $\mathscr{G}$, with stalks G, that is a stalkwise direct summand.*

□

4.15. Example. Let $Y = [-1, 1]$,

$$X = \{\langle x,0\rangle \mid -1 \le x \le 1\} \cup \{\langle x,1\rangle \mid -1 \le x \le 0\} \cup \{\langle x,x\rangle \mid 0 \le x \le 1\},$$

and $f\langle x,y\rangle = x$. Then $Y_{L\oplus L} = Y$. However, $\mathscr{H} = \mathscr{H}^0(f;L)$ is not constant. Instead, there is an exact sequence $0 \to L \to \mathscr{H} \to L_{[-1,0]} \oplus L_{(0,1]} \to 0$. No Y_i, as in 4.14, can have $\{0\}$ in its interior. The collection $Y_0 = [-1,0]$ and $Y_n = [2^{-n}, 2^{-n+1}]$ for $n > 0$ works, and each $\mathscr{H}|Y_n$ is constant. ◇

4.16. Example. Let X be the Cantor set and let $f : X \to Y = [0,1]$ be the identification of the end points in each complementary interval. Then f is 2-to-1 over a countable set $Q = Y_{L\oplus L}$ and 1-to-1 over $P = Y - Q = Y_L$.

[10] This is also true without separability. One uses the space $C(X)$ of continuous bounded real-valued functions on X instead of the Hilbert cube. Since $C(X)$ is not locally compact, II-17.5 does not apply directly, but its proof does.

Now, for $Q = \bigcup Q_i$ as in 4.14, the Q_i can simply be the points of Q. In any case there must be infinitely many Q_i since they are closed and contained in Q. For P we could take $P_1 = Y$. In any covering $P \subset P_1 \cup P_2 \cup \cdots$ with P_i closed in Y, some P_i must have nonempty interior by the Baire category theorem. Let U be an open interval inside some such P_i and let $\Phi = c|U$. Then the exact coefficient sequence $0 \to L \to \mathscr{H} \to \mathscr{Q} \to 0$ induces the exact sequence

$$0 \to \Gamma_\Phi(L) \to \Gamma_\Phi(\mathscr{H}) \to \Gamma_\Phi(\mathscr{Q}) \to H^1_\Phi(Y; L) \to H^1_\Phi(Y; \mathscr{H}).$$

The right-hand term is $H^1_\Phi(Y; fL) \approx H^1_{f^{-1}\Phi}(X; L) = 0$ by II-11.1, and the previous term is $H^1_{c|U}(Y; L) \approx H^1_c(U; L) \approx L$. Thus $\Gamma_\Phi(\mathscr{H}) \to \Gamma_\Phi(\mathscr{Q})$ is not surjective, and so $\mathscr{Q}|P_i$, and hence $L|P_i$, cannot be a direct summand of $\mathscr{H}|P_i$ even though it is a stalkwise direct summand. ◇

5 Extension of a support family by a family on the base space

Let $f : X \to Y$ with Ψ and Φ being families of supports on X and Y respectively.

5.1. Definition. *Let the "extension $\Phi(\Psi)$ of Ψ by Φ" be the family of supports on X defined by*

$$\boxed{\Phi(\Psi) = \{K \in \Psi(Y) \mid \overline{f(K)} \in \Phi\} = \Psi(Y) \cap f^{-1}(\Phi).}$$

Intuitively, $\Phi(\Psi)$ is the result of "spreading Ψ out over Φ." The reason for considering this construction is the following:

5.2. Proposition. *For any sheaf $\mathscr{A}$ on X we have*

$$\boxed{\Gamma_\Phi(f_\Psi\mathscr{A}) = \Gamma_{\Phi(\Psi)}(\mathscr{A})}$$

under the defining equality $(f_\Psi\mathscr{A})(Y) = \Gamma_{\Psi(Y)}(\mathscr{A}) \subset \mathscr{A}(X)$.

Proof. Let $s \in (f_\Psi\mathscr{A})(Y) = \Gamma_{\Psi(Y)}(\mathscr{A})$ and $K = |s| \subset X$ as a section of $\mathscr{A}$. Then the support of s as a section of $f_\Psi\mathscr{A}$ is clearly $\overline{f(K)}$. By Definition 5.1, $\overline{f(K)} \in \Phi \Leftrightarrow K \in \Phi(\Psi)$, since $K \in \Psi(Y)$, and the result follows. □

Most cases of interest are those in which Φ, Ψ are one of the four combinations of *cld*, c on locally compact Hausdorff spaces X and Y and map $f : X \to Y$. The following table indicates the intuitive meaning of $\Phi(\Psi)$ in these four cases:

Φ (on Y)	Ψ (on X)	$\Phi(\Psi)$ (on X)
closed	closed	closed
compact	compact	compact
closed	compact	fiberwise compact
compact	closed	basewise compact

The following is immediate from the definitions.[11]

5.3. Proposition. *If $f : X \to Y$ is a map between locally compact Hausdorff spaces, then*

(a) $c(Y) = \{K \subset X \textit{ closed} \mid f|K : K \to Y \textit{ is proper}\}$,

(b) $\Phi(c) = \{K \in f^{-1}(\Phi) \mid f|K : K \to Y \textit{ is proper}\}$,

(c) $\Psi \subset \Phi(c) \Leftrightarrow (f(\Psi) \subset \Phi \textit{ and } f|K : K \to Y \textit{ is proper for all } K \in \Psi)$. □

We collect some further miscellaneous facts about this construction:

5.4. Proposition. *Let $f : X \to Y$, let Φ, Ξ be families of supports on Y and Ψ, Θ families of supports on X. Then, with $A^\bullet = f^{-1}(A)$,*

1. $\Psi \subset \Psi(Y)$,
2. $\Psi \subset \Theta \Rightarrow \Psi(Y) \subset \Theta(Y)$ *and* $\Phi(\Psi) \subset \Phi(\Theta)$,
3. $\Phi(cld) = f^{-1}\Phi$,
4. $cld(\Psi) = \Psi(Y)$,
5. $\Psi(Y)(Y) = \Psi(Y)$,
6. $\Phi(\Phi(\Psi)) = \Phi(\Psi)$,
7. $\Phi(\Psi) \cap f^{-1}(\Xi) = (\Phi \cap \Xi)(\Psi)$,
8. f *Ψ-closed*, $A \subset Y \Rightarrow \Phi(\Psi)|A^\bullet = (\Phi|A)(\Psi) = (\Phi|A)(\Psi \cap A^\bullet)$,
9. $\Phi(\Psi) \cap \Xi(\Theta) = (\Phi \cap \Xi)(\Psi \cap \Theta)$.

Proof. Items 1–5 are clear. For 6, note that $\Phi(\Psi) \subset \Psi(Y)$ by definition. Thus $\Phi(\Psi)(Y) \subset \Psi(Y)(Y) = \Psi(Y)$ by 2 and 5. Therefore

$$\Phi(\Phi(\Psi)) =_{\text{def}} \Phi(\Psi)(Y) \cap f^{-1}(\Phi) \subset \Psi(Y) \cap f^{-1}(\Phi) =_{\text{def}} \Phi(\Psi).$$

Conversely, $\Phi(\Psi) \subset \Phi(\Psi)(Y)$ by 1, and $\Phi(\Psi) \subset f^{-1}(\Phi)$ by definition, whence $\Phi(\Psi) \subset \Phi(\Psi)(Y) \cap f^{-1}(\Phi) = \Phi(\Phi(\Psi))$.

[11] Note that $f|K : K \to Y$ being proper implies that $f(K)$ is closed in Y.

For 8, note that

$$\begin{aligned}\Phi(\Psi)|A^\bullet &= \{K \in \Psi(Y) \mid \overline{f(K)} \in \Phi \text{ and } K \subset A^\bullet\} \\ &= \{K \in \Psi(Y) \mid \overline{f(K)} \in \Phi \text{ and } f(K) \subset A\}.\end{aligned}$$

Now suppose that $K \in \Phi(\Psi)|A^\bullet$, and let $y \in \overline{f(K)}$. Then by the definition of $\Psi(Y)$, there exists a neighborhood N of y and a $K' \in \Psi$ such that $K \cap N^\bullet = K' \cap N^\bullet$. Therefore $f(K) \cap N = f(K') \cap N$. Thus $y \in \overline{f(K')} = f(K')$ since f is Ψ-closed. Hence $y \in f(K)$. Thus $f(K)$ is closed in Y, and so $\overline{f(K)} \subset A$. Also, $K' \cap A^\bullet \cap N^\bullet = K \cap N^\bullet$ since $K \subset A^\bullet$. Hence

$$\begin{aligned}\Phi(\Psi)|A^\bullet &= \{K \in \Psi(Y) \mid \overline{f(K)} \in \Phi|A\} = (\Phi|A)(\Psi) \\ \text{(also)} &= \{K \in (\Psi \cap A^\bullet)(Y) \mid \overline{f(K)} \in \Phi|A\} = (\Phi|A)(\Psi \cap A^\bullet)\end{aligned}$$

as claimed. Items 7 and 9 are obvious. □

5.5. Proposition. *If $f : X \to Y$ with Ψ paracompactifying on X and Φ paracompactifying on Y, then $\Phi(\Psi)$ is paracompactifying on X.*

Proof. Since every member of Φ has a neighborhood in Φ, it suffices to consider the case in which $Y \in \Phi$. Thus we can assume that Y is paracompact and $\Phi = cld$, whence $\Phi(\Psi) = \Psi(Y)$. It is also clear that we can assume that $E(\Psi) = X$ and hence that X is regular. Now, if $K \in \Psi(Y)$, then there is a locally finite open covering $\{U_\alpha\}$ and a shrinking $\{V_\alpha\}$ of $\{U_\alpha\}$ such that each $K \cap f^{-1}(\overline{V}_\alpha) \in \Psi$. Therefore $K = \bigcup_\alpha K_\alpha$, where $K_\alpha \in \Psi$ and $K_\alpha \subset f^{-1}(\overline{V}_\alpha)$, for some such covering. Conversely, any set K that can be written this way is in $\Psi(Y)$. Now, any such set K is paracompact since a locally finite union of closed paracompact subspaces of a regular space is paracompact; see [34, p. 178]. Also, by considering sets W_α with $\overline{V}_\alpha \subset W_\alpha$ and $\overline{W}_\alpha \subset U_\alpha$, it is clear that such a set K has a neighborhood K' of this form. □

6 The Leray spectral sequence of a map

Let $A \subset X$, let $f : X \to Y$, and let Ψ and Φ be families of supports on X and Y respectively. In this section we shall assume that at least one of the following two conditions holds:

(A) $\Psi = cld$.

(B) Φ *is paracompactifying.*

Let $\mathscr{A}$ be a sheaf on X and let $\mathscr{A}^* = \mathscr{C}^*(X, A; \mathscr{A})$, or as in (16) in general. Then each $f_\Psi \mathscr{A}^p$ is Φ-acyclic by 3.3 or II-5.7. Thus from 2.1 we have a spectral sequence, derived from the double complex $C^p_\Phi(Y; f_\Psi\mathscr{A}^q)$, in which $E_2^{p,q} = H^p_\Phi(Y; \mathscr{H}^q_\Psi(f, f|A; \mathscr{A}))$ and E_∞ is the graded group associated with some filtration of $H^*(\Gamma_\Phi(f_\Psi\mathscr{A}^*)) = H^*(\Gamma_{\Phi(\Psi)}(\mathscr{A}^*)) = H^*_{\Phi(\Psi)}(X, A; \mathscr{A})$.

From this and 4.2 we obtain the following basic result, which is one of the most important consequences of sheaf-theoretic cohomology:

6.1. Theorem. *If* (A) *or* (B) *holds, then there is a spectral sequence (the "Leray spectral sequence") in which*

$$\boxed{E_2^{p,q} = H^p_\Phi(Y; \mathscr{H}^q_\Psi(f, f|A; \mathscr{A})) \Longrightarrow H^{p+q}_{\Phi(\Psi)}(X, A; \mathscr{A}).}$$

Moreover, if f is Ψ-closed and $\Psi \cap A$-closed and if each $y^\bullet = f^{-1}(y)$ is Ψ-taut and each $y^\bullet \cap A$ is $\Psi \cap A$-taut in X, then the canonical homomorphisms

$$r_y^* : \mathscr{H}^*_\Psi(f, f|A; \mathscr{A})_y \to H^*_{\Psi \cap y^\bullet}(y^\bullet, y^\bullet \cap A; \mathscr{A})$$

are isomorphisms. □

This is clearly a far-reaching extension of the general Vietoris mapping theorem II-11.1; see 8.21. Most of the remainder of this chapter is concerned with applications of this spectral sequence.

6.2. Let us briefly discuss the naturality of the Leray spectral sequence. Consider the situation and notation of 3.5 and 4.3. Let Φ_i be a family of supports on Y_i satisfying (A) or (B) and with $h^{-1}(\Phi_2) \subset \Phi_1$. The h-cohomomorphism $f_{2,\Psi_2}\mathscr{A}_2^* \rightsquigarrow f_{1,\Psi_1}\mathscr{A}_1^*$ induces an h-cohomomorphism $\mathscr{C}^*(Y_2; f_{2,\Psi_2}\mathscr{A}_2^*) \rightsquigarrow \mathscr{C}^*(Y_1; f_{1,\Psi_1}\mathscr{A}_1^*)$ and hence a map of double complexes

$$C^*_{\Phi_2}(Y_2; f_{2,\Psi_2}\mathscr{A}_2^*) \to C^*_{\Phi_1}(Y_1; f_{1,\Psi_1}\mathscr{A}_1^*).$$

Therefore there is an induced map of spectral sequences

$${}^2E_r^{p,q} \to {}^1E_r^{p,q}.$$

The homomorphism ${}^2E_2^{p,q} \to {}^1E_2^{p,q}$, i.e.,

$$H^p_{\Phi_2}(Y_2; \mathscr{H}^q_{\Psi_2}(f_2, f_2|A_2; \mathscr{A}_2)) \to H^p_{\Phi_1}(Y_1; \mathscr{H}^q_{\Psi_1}(f_1, f_1|A_1; \mathscr{A}_1)),$$

is induced by the h-cohomomorphism (19) of Leray sheaves. Moreover, the homomorphism ${}^2E_\infty^{p,q} \to {}^1E_\infty^{p,q}$ is the graded map associated to the homomorphism

$$H^*_{\Phi_2(\Psi_2)}(X_2, A_2; \mathscr{A}_2) \to H^*_{\Phi_1(\Psi_1)}(X_1, A_1; \mathscr{A}_1)$$

induced by $k : \mathscr{A}_2 \rightsquigarrow \mathscr{A}_1$ [note that $g^{-1}(\Phi_2(\Psi_2)) \subset \Phi_1(\Psi_1)$].

6.3. In the situation of 6.1 again, let $f^\diamond$ denote the edge homomorphism

$$\boxed{f^\diamond : H^p_\Phi(Y; f_\Psi\mathscr{A}) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \rightarrowtail H^p_{\Phi(\Psi)}(X; \mathscr{A})}$$

of the Leray spectral sequence. We wish to relate this to more familiar maps. To do this, apply 6.2 to the diagram

$$\begin{array}{ccc} X, cld & \xrightarrow{1} & X, \Psi \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ X, f^{-1}\Phi & \xrightarrow{f} & Y, \Phi \end{array}$$

of maps and support families. There is a map of the spectral sequence of $f : (X, \Psi) \to (Y, \Phi)$ to that of $(X, cld) \to (X, f^{-1}\Phi)$. This gives the commutative diagram

$$\begin{array}{ccc} H^p_\Phi(Y; f_\Psi\mathscr{A}) & \xrightarrow{f^\diamond} & H^p_{\Phi(\Psi)}(X;\mathscr{A}) \\ \downarrow{\scriptstyle f^\dagger} & & \downarrow{\scriptstyle \iota^*} \\ H^p_{f^{-1}\Phi}(X;\mathscr{A}) & \xrightarrow{1^\diamond} & H^p_{f^{-1}\Phi}(X;\mathscr{A}) \end{array}$$

since $(f^{-1}\Phi)(cld) = f^{-1}\Phi$. The edge homomorphism $1^\diamond$ is the identity,[12] and ι^* is induced by inclusion of supports. By the general discussion of 6.2 the map $f^\dagger$ is induced by the f-cohomomorphism $f_\Psi\mathscr{A} \rightsquigarrow 1_{cld}\mathscr{A} = \mathscr{A}$, which is equivalent to the inclusion of supports homomorphism $f_\Psi\mathscr{A} \to f\mathscr{A}$, i.e., it is the composition $f_\Psi\mathscr{A} \to f\mathscr{A} \rightsquigarrow \mathscr{A}$. Therefore, we have the general relationship

$$\boxed{\iota^* \circ f^\diamond = f^\dagger : H^p_\Phi(Y; f_\Psi\mathscr{A}) \to H^p_{f^{-1}\Phi}(X;\mathscr{A}),}$$

where $f^\dagger$ is induced by the f-cohomomorphism $f_\Psi\mathscr{A} \rightsquigarrow \mathscr{A}$ and ι^* is induced by the inclusion of supports $\Phi(\Psi) \hookrightarrow f^{-1}\Phi$.

Now suppose that $\Psi = cld$. Then $f^\dagger : H^p_\Phi(Y; f\mathscr{A}) \to H^p_{f^{-1}\Phi}(X;\mathscr{A})$ is the map of that name in II-8. Also, ι^* is the identity. Thus the edge homomorphism $f^\diamond$ is just the canonical map

$$\boxed{f^\diamond = f^\dagger : H^p_\Phi(Y; f\mathscr{A}) \to H^p_{f^{-1}\Phi}(Y;\mathscr{A})}$$

of II-8 induced by the f-cohomomorphism $f\mathscr{A} \rightsquigarrow \mathscr{A}$.

Now also suppose that $\mathscr{A} = f^*\mathscr{B}$ for some sheaf $\mathscr{B}$ on Y. Then by (18) on page 63, $f^* = f^\dagger \circ \beta^*$, where $\beta : \mathscr{B} \to ff^*\mathscr{B}$ is the canonical homomorphism. Therefore the composition

$$\boxed{f^\diamond \circ \beta^* = f^* : H^p_\Phi(Y;\mathscr{B}) \to H^p_{f^{-1}\Phi}(X; f^*\mathscr{B})}$$

is the usual homomorphism induced by f.

See Exercise 5 regarding the other edge homomorphism.

6.4. Example. Consider the map $f : X \to Y = [0,1]$ of 4.8. The Leray sheaf $\mathscr{H}^1(f;\mathbb{Z})$, as computed there, is a subsheaf of the sheaf $\mathbb{Z}_{(0,1]}$ with quotient sheaf $\mathscr{Q}$ having stalk $\mathbb{Z}_2$ at 1 and being zero elsewhere. We compute $H^p(\mathbb{I};\mathbb{Z}_{(0,1]}) = H^p(\mathbb{I},\{0\};\mathbb{Z}) = 0$ for all p by II-12.3. Therefore, the cohomology sequence of the exact coefficient sequence $\mathscr{H}^1(f;\mathbb{Z}) \rightarrowtail \mathbb{Z}_{(0,1]} \twoheadrightarrow \mathscr{Q}$ gives that $H^p(\mathbb{I};\mathscr{H}^1(f;\mathbb{Z})) \approx H^{p-1}(\mathbb{I};\mathscr{Q}) \approx H^{p-1}(\{1\};\mathbb{Z}_2) \approx \mathbb{Z}_2$ (by II-10.2) for $p = 1$ and is zero otherwise. Thus the Leray spectral sequence of f has $E_2^{1,1} = \mathbb{Z}_2$, $E_2^{0,0} = \mathbb{Z}$, and $E_2^{p,q} = 0$ otherwise. This spectral sequence degenerates, i.e., $E_\infty^{p,q} = E_2^{p,q}$, and gives, of course, the usual cohomology groups of the projective plane. ◇

[12] See Exercise 29.

6.5. Example. This is an example of the Leray spectral sequence in a case in which the map f is *not* closed and the r_y^* are *not* all isomorphisms. Let $X = \mathbb{S}^2 - \{x_0\}$, where x_0 is the north pole; let Y be the unit disk in the plane; and let $f : X \to Y$ be the projection taking $x_0 \mapsto y_0$, the center of the disk Y. Use closed supports. Then at points $y \neq y_0$ in Y the Leray sheaf $\mathscr{H}^q(f;\mathbb{Z})$ is the same as that for the projection map from $\mathbb{S}^2$, and so it vanishes for $q > 0$. However, the stalk at y_0 is

$$\mathscr{H}^q(f;\mathbb{Z})_{y_0} = \varinjlim H^q(U^\bullet;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & \text{for } q = 0,\\ \mathbb{Z}, & \text{for } q = 1,\\ 0, & \text{for } q \neq 0,1,\end{cases}$$

where U ranges over the open disk neighborhoods of y_0, since such $U^\bullet$ are disjoint unions of open disks and open annuli (i.e., are homotopy equivalent to $\star + \mathbb{S}^1$). It is fairly clear that $\mathscr{H}^0(f;\mathbb{Z}) = f\mathbb{Z} \approx \mathbb{Z}\oplus\mathbb{Z}_{Y-B}$, where $B = \partial Y$. Also, $\mathscr{H}^1(f;\mathbb{Z}) \approx \mathbb{Z}_{\{y_0\}}$, the sheaf that has stalk $\mathbb{Z}$ at y_0 and is zero elsewhere. Thus the Leray spectral sequence has

$$E_2^{p,0} = H^p(Y;\mathscr{H}^0(f;\mathbb{Z})) \approx H^p(Y;\mathbb{Z})\oplus H_c^p(Y-B;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } p = 0,2,\\ 0, & \text{otherwise}\end{cases}$$

and

$$E_2^{p,1} = H^p(Y;\mathscr{H}^1(f;\mathbb{Z})) \approx H^p(Y;\mathbb{Z}_{\{y_0\}}) \approx H^p(\{y_0\};\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } p = 0,\\ 0, & \text{for } p \neq 0.\end{cases}$$

Since this converges to the cohomology of the contractible space X, we must have that the only possible nonzero differential,

$$\mathbb{Z} \approx E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \approx \mathbb{Z},$$

is an isomorphism.

It is of interest to consider the same example but with supports in c on X. Here, all of the hypotheses of 6.1 are valid, so that the Leray sheaf $\mathscr{H}_c^q(f;\mathbb{Z})$ vanishes for $q \neq 0$, and $\mathscr{H}_c^0(f;\mathbb{Z}) = f_c\mathbb{Z}$ has stalks

$$\mathscr{H}_c^0(f;\mathbb{Z})_y \approx \begin{cases} \mathbb{Z}, & \text{for } y \in B,\\ \mathbb{Z}\oplus\mathbb{Z}, & \text{for } y \in Y - B - \{y_0\},\\ \mathbb{Z}, & \text{for } y = y_0.\end{cases}$$

Here, of course, the Leray spectral sequence degenerates to the isomorphism

$$H^p(Y;\mathscr{H}_c^0(f;\mathbb{Z})) \approx H_c^p(X;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } p = 2,\\ 0, & \text{for } p \neq 2.\end{cases}$$

From the stalks, one might think that $\mathscr{H}_c^0(f;\mathbb{Z}) \approx \mathbb{Z}\oplus\mathbb{Z}_{Y-B-\{y_0\}}$, but this cannot hold since $\Gamma(\mathscr{H}_c^0(f;\mathbb{Z})) = H^0(Y;\mathscr{H}_c^0(f;\mathbb{Z})) = 0$, so that $\mathscr{H}_c^0(f;\mathbb{Z})$ has no nonzero global sections. ◇

6.6. Example. Let $U_1 \subset U_2 \subset \cdots$ be an increasing sequence of open subsets of a space X with union X. Let $\mathbb{N}$ be as in II-Exercise 27. Define $\pi : X \to \mathbb{N}$ by $\pi(x) = \min\{n \,|\, x \in U_n\}$. Then π is continuous (but not, of course, closed) since $\pi^{-1}\{1, \ldots, n\} = U_n$. Then the Leray spectral sequence of π with closed supports is identical to the spectral sequence of 2.11, as the reader may detail. ◇

As an application of the Leray spectral sequence, suppose that $f : X \to Y$ is a map between locally compact Hausdorff spaces. We say that f is a "c-Vietoris map" if $H_c^p(f^{-1}(y); L) = 0$ for $p > 0$ and any constant coefficients L. We say that f is a "locally c-Vietoris map" if $f|U : U \to Y$ is c-Vietoris for all open $U \subset X$, e.g., f finite-to-one or "discrete-to-one" such as any covering map. [Note that this implies that $\dim_L f^{-1}(y) = 0$.]

6.7. Theorem. *If $f : X \to Y$ is a locally c-Vietoris map between locally compact Hausdorff spaces and if $\mathscr{L}$ is a c-fine sheaf on Y, then $f^*\mathscr{L}$ is a c-fine sheaf on X.*

Proof. In the Leray spectral sequence

$$E_2^{p,q} = H_c^p(Y; \mathscr{H}_c^q(f; f^*\mathscr{L})) \Longrightarrow H_c^{p+q}(X; f^*\mathscr{L})$$

we have that the coefficient sheaf $\mathscr{H}_c^q(f; f^*\mathscr{L})$ has stalks $H_c^q(f^{-1}(y); \mathscr{L}_y) = 0$ for $q > 0$ by 6.1 and our assumption. Also, $\mathscr{H}_c^0(f; f^*\mathscr{L}) = f_c f^*\mathscr{L}$ is c-fine by Exercise 8. Hence $H_c^n(X; f^*\mathscr{L}) = 0$ for $n > 0$, meaning that $f^*\mathscr{L}$ is c-acyclic. This applies to the map $f|U$ for $U \subset X$ open, and so $(f^*\mathscr{L})|U$ is c-acyclic. Therefore $f^*\mathscr{L}$ is c-soft by II-16.1. If $\mathscr{L}$ is a module over a c-soft (hence c-fine) sheaf $\mathscr{R}$ of rings, then $f^*\mathscr{L}$ is a module over the c-soft sheaf $f^*\mathscr{R}$ of rings, so that $f^*\mathscr{L}$ is c-fine. □

6.8. We conclude this section with a discussion of the cup product in the Leray spectral sequence. Recall that the Leray spectral sequence of $f : X \to Y$ mod $A \subset X$ may be identified with the "first" spectral sequence of the double complex

$$F_\Phi^p(Y; f_\Psi \mathscr{A}^q), \quad \text{where} \quad \mathscr{A}^q = \mathscr{F}^q(X, A; \mathscr{A}).$$

Now, for $i = 1, 2$ let $\mathscr{A}_i$ be a sheaf on X, Ψ_i a family of supports on X, and Φ_i a family of supports on Y. Let $\mathscr{A}_1^q = \mathscr{F}^q(X; \mathscr{A}_1)$, $\mathscr{A}_2^q = \mathscr{F}^q(X, A; \mathscr{A}_2)$, and $\mathscr{A}_{1,2}^q = \mathscr{F}^q(X, A; \mathscr{A}_1 \otimes \mathscr{A}_2)$. The cup product (25) on page 93 provides a chain map $\mathscr{A}_1^q \otimes \mathscr{A}_2^t \to \mathscr{A}_{1,2}^{q+t}$. Similarly, we obtain the following cup product of double complexes:

$$\begin{aligned} F_{\Phi_1}^p(Y; f_{\Psi_1}\mathscr{A}_1^q) \otimes F_{\Phi_2}^s(Y; f_{\Psi_2}\mathscr{A}_2^t) &\xrightarrow{(-1)^{sq}\cup} F_{\Phi_1\cap\Phi_2}^{p+s}(Y; f_{\Psi_1}(\mathscr{A}_1^q) \otimes f_{\Psi_2}(\mathscr{A}_2^t)) \\ &\longrightarrow F_{\Phi_1\cap\Phi_2}^{p+s}(Y; f_{\Psi_1\cap\Psi_2}(\mathscr{A}_1^q \otimes \mathscr{A}_2^t)) \\ &\longrightarrow F_{\Phi_1\cap\Phi_2}^{p+s}(Y; f_{\Psi_1\cap\Psi_2}\mathscr{A}_{1,2}^{q+t}) \end{aligned}$$

(see 4.7). This induces the cup product of the corresponding spectral sequences

$$\cup : {}^1E_r^{p,q} \otimes {}^2E_r^{s,t} \to {}^{1,2}E_r^{p+s,q+t}$$

with the usual properties. On the E_2 terms this coincides with the composition

$$\begin{aligned} &H^p_{\Phi_1}(Y; \mathscr{H}^q_{\Psi_1}(f;\mathscr{A}_1)) \otimes H^s_{\Phi_2}(Y;\mathscr{H}^t_{\Psi_2}(f,f|A;\mathscr{A}_1)) \\ &\xrightarrow{(-1)^{sq}\cup} H^{p+s}_{\Phi_1\cap\Phi_2}(Y;\mathscr{H}^q_{\Psi_1}(f;\mathscr{A}_1)\otimes\mathscr{H}^t_{\Psi_2}(f,f|A;\mathscr{A}_2)) \\ &\longrightarrow H^{p+s}_{\Phi_1\cap\Phi_2}(Y;\mathscr{H}^{q+t}_{\Psi_1\cap\Psi_2}(f,f|A;\mathscr{A}_1\otimes\mathscr{A}_2)), \end{aligned}$$

and on E_∞ it is compatible with the cup product

$$\cup : H^n_{\Phi_1(\Psi_1)}(X;\mathscr{A}_1)\otimes H^m_{\Phi_2(\Psi_2)}(X,A;\mathscr{A}_2) \to H^{n+m}_{\Theta}(X,A;\mathscr{A}_1\otimes\mathscr{A}_2)$$

on the "total" cohomology, where $\Theta = \Phi_1(\Psi_1)\cap\Phi_2(\Psi_2) = (\Phi_1\cap\Phi_2)(\Psi_1\cap\Psi_2)$. [To see this, consider the *second* spectral sequence of the double complex $F^p_\Phi(Y; f_\Psi\mathscr{A}^q)$.]

The differential d_r satisfies the identity

$$\boxed{d_r(\alpha\cup\beta) = d_r\alpha\cup\beta + (-1)^{p+q}\alpha\cup d_r\beta,}$$

where $\alpha \in {}^1E_r^{p,q}$ (see Appendix A).

6.9. Let us specialize to the case in which $f : X \to Y$ is Ψ-closed and each $f^{-1}(y)$ is Ψ-taut and Φ is paracompactifying on X. Also assume that the Leray sheaf $\mathscr{H}^*_\Psi(f;L)$ is constant with stalks $H^*_\Psi(F;L)$, where L is some base ring with unit. There are the canonical homomorphisms $H^q_\Psi(F) \to H^0_\Phi(Y; H^q_\Psi(F))$ (the constant sections) and $H^p_\Phi(Y;L) \to H^p_\Phi(Y;\mathscr{H}^0(f))$ induced by the canonical homomorphism $L \to \mathscr{H}^0(f)$, which is often an isomorphism [e.g., when Y is locally connected and the $f^{-1}(y)$ are connected; but we need not assume this or even that $\mathscr{H}^0(f)$ is constant or that it has stalks $H^0(f^{-1}(y))$]. Thus we have the maps (coefficients in L)

$$H^p_\Phi(Y)\otimes_L H^q_\Psi(F) \to H^p_\Phi(Y;\mathscr{H}^0(f))\otimes_L H^0(Y;H^q_\Psi(F)) \xrightarrow{\cup} H^p_\Phi(Y;H^q_\Psi(F)) = E_2^{p,q},$$

and the preceding remarks imply that this is an *algebra* homomorphism with the usual cup product

$$(a\otimes b)\cup(c\otimes d) = (-1)^{\deg(b)\deg(c)}(a\cup c)\otimes(b\cup d)$$

on $H^p_\Phi(Y;L)\otimes_L H^q_\Psi(F;L)$. [The cup product on $H^q_\Psi(F;L)$ is that of the stalk $H^q_\Psi(f^{-1}(y);L)$. We are *assuming* that this is independent of $y\in Y$. The reader should show that this is always the case if Y is connected.] In particular, if L is a *field* and $\Phi = c = \Psi$ on locally compact Hausdorff spaces, then

$$H^p_c(Y;L)\otimes_L H^q_c(F;L) \xrightarrow{\approx} H^p_c(Y;H^q_c(F;L)) = E_2^{p,q}$$

is an isomorphism of algebras. More generally, this holds when L is a principal ideal domain and $H^q_c(F;L)$ is torsion-free over L.

7 Fiber bundles

Clearly it is important to understand the Leray sheaf in particular cases. We begin by looking at the case of a projection in a product space:

7.1. Theorem. *Let $f : Y \times F \to Y$ be the projection. Let Θ be a family of supports on F and put $\Psi = Y \times \Theta$. Let $\mathscr{B}$ be a sheaf on F and put $\mathscr{A} = Y \times \mathscr{B}$. Let $H^n_\Theta(F;\mathscr{B})$ denote the constant sheaf on Y with stalks $H^n_\Theta(F;\mathscr{B})$. Then there is a canonical monomorphism*

$$\pi^* : H^n_\Theta(F;\mathscr{B}) \rightarrowtail \mathscr{H}^n_\Psi(f;\mathscr{A}).$$

Moreover, if the map r^*_y of 4.2 is an isomorphism for all $y \in Y$, then π^* is an isomorphism.

Proof. Let $\pi : Y \times F \to F$ be the projection. Then for open sets $U \subset Y$, $\pi : U^\bullet = U \times F \to F$ induces the homomorphism $\pi^* : H^n_\Theta(F;\mathscr{B}) \to H^n_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A})$, which is a homomorphism of presheaves, the source being regarded as the constant presheaf. This then induces a homomorphism of the generated sheaves as required. On the stalks at $y \in Y$ this is a homomorphism

$$\pi^*_y : H^n_\Theta(F;\mathscr{B}) \to \varinjlim_{y\in U} H^n_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}),$$

and it is clear that $r^*_y \circ \pi^*_y = 1$. Thus π^* is a monomorphism, as claimed. Moreover, if r^*_y is an isomorphism, then $\pi^*_y = (r^*_y)^{-1}$, so that π^* is an isomorphism if this holds for all $y \in Y$. □

Note that the condition that the r^*_y be isomorphisms holds for fiberwise compact supports on locally compact Hausdorff spaces. More generally:

7.2. Theorem. *Let Y be an arbitrary space and let F be a locally compact Hausdorff space. Let $\mathscr{A}$ be any sheaf on $Y \times F$ and let $\pi: Y \times F \to Y$ be the projection. Then for any point $y \in Y$, the homomorphism*

$$r^*_y : \mathscr{H}^n_{Y\times c}(\pi;\mathscr{A})_y \to H^n_c(F;\mathscr{A})$$

of 4.2 is an isomorphism. Thus, for any sheaf $\mathscr{B}$ on F, the Leray sheaf $\mathscr{H}^n_{Y\times c}(\pi; Y \times \mathscr{B})$ is the constant sheaf $H^n_c(F;\mathscr{B})$.

Proof. Since π is $(Y \times c)$-closed, it suffices, by II-10.6, to show that each $\pi^{-1}(y) = \{y\} \times F$ is $(Y \times c)$-taut. This is trivial if Y is locally paracompact (see item (d) below II-10.5), but we wish to prove this for completely general Y, not necessarily even Hausdorff.

Thus let $\mathscr{F}$ be a flabby sheaf on $Y \times F$ and let $s \in \Gamma_c(\mathscr{F}|\{y\} \times F)$ with $|s| = \{y\} \times K$. Let $W = Y \times F^+$, where F^+ is the one-point compactification of F. By II-9.8 there is an extension $t \in \Gamma(\mathscr{F}^W|U \times F^+)$ of s, where U

is some neighborhood of y in Y. By cutting U down, we can assume that $|t| \subset U \times K'$ for some compact K' that is a neighborhood of K. Then t extends by 0 to $(U \times F^+) \cup (Y \times (F^+ - K'))$. Restrict this extension to $Y \times F$ and then extend it (using that $\mathscr{F}$ is flabby) to a section $t' \in \mathscr{F}(Y \times F)$ with $|t'| \subset Y \times K' \in Y \times c$. This shows that $\Gamma_{Y\times c}(\mathscr{F}) \to \Gamma_c(\mathscr{F}|\{y\} \times F)$ is surjective. By II-9.22, $\mathscr{F}|\{y\} \times K$ is soft for all compact $K \subset F$. By II-9.3, $\mathscr{F}|\{y\} \times F$ is c-soft. Therefore, by Definition II-10.5, $\{y\} \times F$ is $(Y \times c)$-taut. □

This result has an obvious generalization to (locally trivial) fiber bundles. A "fiber bundle" is a map $f : X \to Y$ such that each point $y \in Y$ has a neighborhood U such that there is a homeomorphism

$$h : U \times F \to f^{-1}(U) = U^\bullet \quad \text{with} \quad fh(u,t) = u \quad \text{for} \quad u \in Y,\, t \in F.$$

Often, h is restricted to belong to a smaller class of "admissible" homeomorphisms forming what is called the *structure group.*

Then let Ψ be a family of supports on X such that for sufficiently small open $U \subset Y$ and any admissible representation of $U^\bullet$ as a product $U \times F$, we have that $\Psi \cap U^\bullet = U \times \Theta$, where Θ is a given family of supports on F. (For locally compact Hausdorff spaces the families of closed, compact, fiberwise compact, or basewise compact supports all are examples.)

Suppose also that $\mathscr{A}$ is a sheaf on X that for any admissible representation $U^\bullet \approx U \times F$ has the form $U \times \mathscr{B}$ for a given sheaf $\mathscr{B}$ on F. Then we deduce:

7.3. Corollary. *In the preceding situation, assume that each r_y^* is an isomorphism (which is the case, for example, for compact or fiberwise compact supports on locally compact Hausdorff spaces). Then the Leray sheaf $\mathscr{H}^n_\Psi(f;\mathscr{A})$ is locally constant with stalks $H^n_\Theta(F;\mathscr{B})$. In this case, we usually denote the Leray sheaf by $\mathscr{H}^n_\Theta(F;\mathscr{B})$.* □

Analogous results obviously can be formulated in the relative case of $(X, A) \to Y$, where for any admissible $h : U \times F \xrightarrow{\approx} U^\bullet$ we have that $h^{-1}(U^\bullet \cap A) = U \times B$ for a given subspace $B \subset F$.

7.4. Example. Consider the solenoid Σ^1, which is the inverse limit of the covering maps

$$\mathbb{S}^1 \xleftarrow{2} \mathbb{S}^1 \xleftarrow{3} \mathbb{S}^1 \xleftarrow{4} \cdots$$

There is the projection $\pi : \Sigma^1 \to \mathbb{S}^1$ to the first factor. This is a bundle whose fiber F is the inverse limit of the sequence

$$1 \leftarrow \mathbb{Z}_{2!} \leftarrow \mathbb{Z}_{3!} \leftarrow \mathbb{Z}_{4!} \leftarrow \cdots$$

of epimorphisms, which is homeomorphic to the Cantor set. By continuity II-14.6, F has cohomology (integer coefficients) only in dimension zero, where

$$H = H^0(F) = \varinjlim \mathbb{Z}^{n!},$$

which is free abelian on a countably infinite set of generators. (This also follows from II-Exercise 34.) Also by continuity,

$$H^1(\Sigma^1) \approx \mathbb{Q},$$

and of course $H^0(\Sigma^1) \approx \mathbb{Z}$ since Σ^1 is connected. The Leray sheaf $\mathscr{H} = \mathscr{H}^0(\pi)$ is locally constant with stalks H by the preceding results. The Leray spectral sequence degenerates into the isomorphisms $H^0(\Sigma^1) \approx \Gamma(\mathscr{H})$ and $H^1(\Sigma^1) \approx H^1(\mathbb{S}^1; \mathscr{H})$. Since $H^0(\Sigma^1) \approx \mathbb{Z} \not\approx H$, the sheaf $\mathscr{H}$ is not constant (which is also clear from its definition). Of course $\mathscr{H} = \pi\mathbb{Z}$, and so the isomorphism $H^1(\mathbb{S}^1; \mathscr{H}) = H^1(\mathbb{S}^1; \pi\mathbb{Z}) \approx H^1(\Sigma^1)$ is also a consequence of II-11.1; also see 8.21. Let $y \in \mathbb{S}^1$ be given. The exact sequence of the pair $(\mathbb{S}^1, y)$ has the segment

$$0 \to H^0(\mathbb{S}^1; \mathscr{H}) \to H^0(y; H) \to H^1(\mathbb{S}^1, y; \mathscr{H}) \to H^1(\mathbb{S}^1; \mathscr{H}) \to 0,$$

which has the form

$$0 \to \mathbb{Z} \to H \xrightarrow{g} H \to \mathbb{Q} \to 0.$$

It is fairly clear that $g = 1 - \eta$, where $\eta : H \to H$ is the monodromy automorphism. There is the canonical homomorphism

$$\beta : \mathbb{Z} \to \pi\pi^*\mathbb{Z} = \mathscr{H}.$$

On the stalks at y, this is just the inclusion

$$\mathbb{Z} \hookrightarrow H^0(\pi^{-1}(y); \mathbb{Z}) = \{\text{continuous functions } \pi^{-1}(y) \to \mathbb{Z}\}$$

of the set of constant functions into that of continuous functions. Thus the quotient sheaf $\mathscr{G} = \mathscr{H}/\mathbb{Z}$ is torsion-free. (In fact, the stalk of $\mathscr{G}$ at y can be identified with the image of g, and so it is free abelian.) Since $\pi^\dagger : H^1(\mathbb{S}^1; \pi\pi^*\mathbb{Z}) \to H^1(\Sigma^1; \pi^*\mathbb{Z})$ is an isomorphism by II-11.1, $\beta^* : H^1(\mathbb{S}^1; \mathbb{Z}) \to H^1(\mathbb{S}^1; \mathscr{H})$ can be identified, by (18) on page 63, with $\pi^* : H^1(\mathbb{S}^1; \mathbb{Z}) \to H^1(\Sigma^1; \mathbb{Z})$, which is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. The exact sequence $0 \to \mathbb{Z} \to \mathscr{H} \to \mathscr{G} \to 0$ induces the exact sequence

$$0 \to \Gamma(\mathbb{Z}) \to \Gamma(\mathscr{H}) \to \Gamma(\mathscr{G}) \to H^1(\mathbb{S}^1; \mathbb{Z}) \to H^1(\mathbb{S}^1; \mathscr{H}) \to H^1(\mathbb{S}^1; \mathscr{G}) \to 0,$$

which by the preceding remarks has the form

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \Gamma(\mathscr{G}) \to \mathbb{Z} \hookrightarrow \mathbb{Q} \to H^1(\mathbb{S}^1; \mathscr{G}) \to 0,$$

and so $\Gamma(\mathscr{G}) = 0$ (since it is torsion-free) and $H^1(\mathbb{S}^1; \mathscr{G}) \approx \mathbb{Q}/\mathbb{Z}$. ◇

We wish to find other conditions on the spaces of projections $X \times F \to Y$ and the support families for which the r_y^* are isomorphisms. One useful such result is the following:

7.5. Theorem. *Let X be an arbitrary space and let Y be a locally compact Hausdorff space that is clc_L^k, where L is a given base ring that is a principal ideal domain. Let Θ be a family of supports on X and let $\mathscr{A}$ be a sheaf of L-modules on X. Let $\pi : X \times Y \to Y$ be the projection. Then the Leray sheaf $\mathscr{H}^n_{\Theta\times Y}(\pi; \mathscr{A} \times Y)$ of π is the constant sheaf $H^n_\Theta(X; \mathscr{A})$ for all $n < k$, and similarly for $\pi \bmod \pi|A \times Y$ for any subspace $A \subset X$.*

Proof. Let $y \in Y$ and the integer $n < k$ be fixed once and for all. If K_1 is any compact neighborhood of y in Y, the property clc_L^k implies that there exists a neighborhood $K_2 \subset K_1$ of y such that the restriction $H^q(K_1, y; L) \to H^q(K_2, y; L)$ is trivial for all $q \le k$. By II-17.3 and II-15.3 it follows that K_2 may be so chosen that

$$H^q(K_1, y; M) \to H^q(K_2, y; M)$$

is trivial for every L-module M and all $q \le n$. Similarly, choose $K_1 \supset K_2 \supset \cdots \supset K_m$ with the same property at each stage.

Consider the Leray spectral sequence ${}_iE_r^{p,q}$ of the projection $\eta : X \times K_i \to X$ mod$(X \times \{y\})$ with supports in Θ on X and closed on $X \times K_i$ and with coefficients in $\mathscr{A} \times K_i$. Thus

$${}_iE_2^{p,q} = H_\Theta^p(X; \mathscr{H}^q(\eta; \eta|X\times\{y\}; \mathscr{A}\times K_i)) \Longrightarrow H_{\Theta\times K_i}^{p+q}(X\times K_i, X\times\{y\}; \mathscr{A}\times K_i).$$

Since K_i and $\{y\}$ are compact, the stalk at $x \in X$ of the Leray sheaf of η is $H^q(K_i, y; \mathscr{A}_x)$.

The inclusion $K_{i+1} \hookrightarrow K_i$ induces a homomorphism of the spectral sequence ${}_iE_r^{p,q}$ into ${}_{i+1}E_r^{p,q}$, which for $r = 2$ reduces to the homomorphism attached to the coefficient homomorphism of Leray sheaves; see 4.3. But this coefficient homomorphism is trivial for $q \le n$ and for all i by the choice of the K_i.

If follows that the homomorphism ${}_iE_\infty^{p,q} \to {}_{i+1}E_\infty^{p,q}$ is trivial for all $q \le n$ and all $i = 1, 2, ..., m-1$. Now the "total terms" are filtered as follows:

$$H_{\Theta\times K_i}^n(X \times K_i, X \times \{y\}; \mathscr{A} \times K_i) = J_i^{0,n} \supset J_i^{1,n-1} \supset \cdots \supset J_i^{n,0} \supset 0,$$

where $J_i^{p,q}/J_i^{p+1,q-1} \approx {}_iE_\infty^{p,q}$. This filtration is preserved by $K_{i+1} \hookrightarrow K_i$ and is compatible with the homomorphism ${}_iE_\infty^{p,q} \to {}_{i+1}E_\infty^{p,q}$ (which is trivial for $q \le n$). Thus the induced homomorphism from $K_{i+1} \hookrightarrow K_i$ maps $J_i^{p,q}$ into $J_{i+1}^{p+1,q-1}$ for all i and all $q \le n$. It follows that the restriction map

$$H_{\Theta\times K_1}^n(X \times K_1, X \times \{y\}, \mathscr{A} \times K_1) \to H_{\Theta\times K_m}^n(X \times K_m, X \times \{y\}, \mathscr{A} \times K_m)$$

is trivial for all $m \ge n+2$.

The cohomology sequences of the pairs $(X \times K_i, X \times \{y\})$ show that any element $\alpha \in H_{\Theta\times K_1}^n(X \times K_1; \mathscr{A} \times K_1)$ that restricts to zero in $X \times \{y\}$ must also restrict to zero in $X \times K_m$ for $m \ge n+2$.

Now let $i_y : X \to X \times K$ be the inclusion $x \mapsto (x, y)$ and consider the following commutative diagram:

$$\begin{array}{ccccc} H_\Theta^n(X;\mathscr{A}) & \xrightarrow{\pi^*} & H_{\Theta\times K}^n(X\times K;\mathscr{A}\times K) & \xrightarrow{i_y^*} & H_\Theta^n(X;\mathscr{A}) \\ & \searrow & \Big\downarrow f \quad \varinjlim_K & \nearrow r_y^* & \\ & & \mathscr{H}_{\Theta\times Y}^n(\pi;\mathscr{A}\times Y)_y & & \end{array}$$

in which the composition across the top is the identity and $r_y^* = \varinjlim i_y^*$. The vertical map f is an isomorphism in the limit by the definition of the Leray sheaf of π. We have shown that if $\alpha \in H^n_{\Theta \times K}(X \times K; \mathscr{A} \times K)$ and if $i_y^*(\alpha) = 0$, then also $f(\alpha) = 0$. It follows that r_y^* is a monomorphism, and hence it must also be an isomorphism. The theorem now follows from 7.1. □

Remark: Theorem 7.5 remains true if we take Y to be locally contractible (i.e., every neighborhood of $y \in Y$ contains a smaller neighborhood of y contracting through it), but not necessarily locally compact. The proof is essentially contained in the last paragraph above, with the preceding spectral sequence argument replaced by a more elementary argument using homotopy invariance in the form of II-11.8.

We shall now use 7.5 to prove a result of the Künneth type.

7.6. Theorem. *Let $A \subset X$, where X is any space, and let the base ring L be a principal ideal domain. Let $\mathscr{A}$ be a sheaf of L-modules on X and let Φ be any family of supports on X. Let Y be a locally compact Hausdorff and clc_L^∞ space. Then there is a natural exact sequence*

$$\bigoplus_{p+q=n} H^p_\Phi(X,A;\mathscr{A}) \otimes H^q_c(Y;L) \rightarrowtail H^n_{\Phi \times c}((X,A) \times Y; \mathscr{A} \times Y) \twoheadrightarrow \bigoplus_{p+q=n+1} H^p_\Phi(X,A;\mathscr{A}) * H^q_c(Y;L)$$

which splits.[13]

Proof. Let $\pi : X \times Y \to Y$ and $\eta : X \times Y \to X$ be the projections. Let $\mathscr{B}^* = \mathscr{C}^*(Y;L)$, $\mathscr{A}^* = \mathscr{C}^*(X,A;\mathscr{A})$, and $\mathscr{L}^* = \mathscr{C}^*(X \times Y, A \times Y; \eta^*\mathscr{A})$ [or use $\mathscr{F}^*(\bullet;\bullet)$]. The η-cohomomorphism $\mathscr{A}^* \rightsquigarrow \mathscr{L}^*$ induces a homomorphism

$$\Gamma_\Phi(\mathscr{A}^*) \to \Gamma_{\eta^{-1}(\Phi)}(\mathscr{L}^*) \hookrightarrow \Gamma_{cld_Y(\Phi \times Y)}(\mathscr{L}^*) = \Gamma(\pi_{\Phi \times Y}\mathscr{L}^*)$$

by 5.2, since $\eta^{-1}(\Phi) = \Phi \times Y \subset (\Phi \times Y)(Y) = cld_Y(\Phi \times Y)$ by 5.4. Thus the cup product on Y induces the following homomorphism of double complexes:

$$\begin{aligned} K^{p,q} &= \Gamma_c(\mathscr{B}^p) \otimes \Gamma_\Phi(\mathscr{A}^q) \\ &\to \Gamma_c(\mathscr{B}^p) \otimes \Gamma(\pi_{\Phi \times Y}\mathscr{L}^q) \\ &\to \Gamma_c(\mathscr{B}^p \otimes \pi_{\Phi \times Y}\mathscr{L}^q) = L^{p,q}. \end{aligned}$$

The "second" spectral sequence of the double complex $L^{p,q}$ shows that the homology of the total complex L^* of $L^{*,*}$ is isomorphic to

$$H^*(\Gamma_c(L \otimes \pi_{\Phi \times Y}\mathscr{L}^*)) = H^*(\Gamma_{\Phi \times c}(\mathscr{L}^*)) = H^*_{\Phi \times c}((X,A) \times Y; \mathscr{A} \times Y).$$

In the "first" spectral sequences of $K^{*,*}$ and $L^{*,*}$, the homomorphism on the E_1 terms induced by $K^{p,q} \to L^{p,q}$ is

$$\Gamma_c(\mathscr{B}^p) \otimes H^q_\Phi(X,A;\mathscr{A}) \xrightarrow{\cup} \Gamma_c(\mathscr{B}^p \otimes \mathscr{H}^q_\Phi(X,A;\mathscr{A})),$$

[13] Also see Exercises 18–19, and V-Exercise 25.

since $\mathscr{B}^*$ is torsion free, where $\mathscr{H}^q_\Phi(X, A; \mathscr{A}) = \mathscr{H}^q_{\Phi\times Y}(\pi, \pi|A \times Y; \eta^*\mathscr{A})$ is constant by 7.1. Since $\mathscr{B}^*$ is torsion free and flabby, the universal coefficient theorem II-15.3 in degree zero implies that this homomorphism on the E_1 terms is an isomorphism and hence that the induced homomorphism $H^n(K^*) \to H^n(L^*)$ on the "total" terms is also an isomorphism. The result now follows from the algebraic Künneth theorem applied to the double complex $K^{*,*}$. □

7.7. Example. We shall show by example that the condition clc^∞_L in 7.6 (and hence in 7.5) is necessary, as is taking c to be the support family on Y. Moreover, the coefficient sheaf on Y cannot generally be taken to be any torsion-free sheaf (even constant).

Let $\mathscr{A} = L = \mathbb{Z}$. Let X be the disjoint union $\bigcup X_n$ where X_n is a connected polyhedron with $H^2(X_n; \mathbb{Z}) \approx \mathbb{Z}_n$ and the other reduced cohomology groups being zero. Let Y be a compact 1-dimensional space with $H^1(Y; \mathbb{Z}) \approx \mathbb{Q}$ (the rationals), or let Y be a 2-dimensional locally compact HLC space that has the singular homology groups $H_1(Y; \mathbb{Z}) \approx \mathbb{Q}$ and $H_2(Y; \mathbb{Z}) = 0$ (so that $H^2(Y; \mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$, which is a nonzero rational vector space; see V-14.8), or let $\mathbb{Q}$ be the coefficient sheaf on Y.[14] The contentions then follow from the fact that

$$\left(\prod_n \mathbb{Z}_n\right) \otimes \mathbb{Q} \not\approx \prod_n (\mathbb{Z}_n \otimes \mathbb{Q}) = 0,$$

since $\prod_n \mathbb{Z}_n$ is not all torsion.

Note, however, that the condition that Y be clc^∞_L was used in 7.6 only to ensure that the Leray sheaf of $\pi : X \times Y \to Y$ modulo $A \times Y$ is the constant sheaf $H^*_\Phi(X, A; \mathscr{A})$. ◇

7.8. Example. Here is another example, due to E. G. Skljarenko, showing that the clc^k_L assumption in 7.5 is needed. It is an instructive example about the Leray sheaf. For $n = 1, 2, \ldots$, let Y_n be the union of the circles in the upper half plane tangent to the x-axis at the origin and of radii $1/k$ for integers $k \geq n$. Put $Y = Y_1$ and let $y_0 \in Y$ be the origin. Let X be the same as Y but with the weak topology, making X a CW-complex. Let $\pi : X \times Y \to Y$ be the projection. Throughout the example, coefficients will be taken in the integers $\mathbb{Z}$ and will be suppressed from the notation. We wish to study the Leray sheaf $\mathscr{H}^2(\pi)$. Since Y is locally contractible, and hence clc, near all points except y_0, it is clear that $\mathscr{H}^2(\pi)$ is zero at all points other than y_0. We claim that it is not zero at y_0. From continuity, II-10.7, it is clear that

$$H^1(Y_n) \approx \bigoplus_{i=n}^{\infty} \mathbb{Z},$$

and if $a_n, a_{n+1}, \ldots$ is the obvious basis, the restriction $H^1(Y_n) \to H^1(Y_{n+1})$ is the map that kills a_n and retains the others.

[14]The construction of such spaces is left to the reader.

Since Y_n is compact, the Leray sheaf of the projection $\eta : X \times Y_n \to X$ is constant with stalks $H^*(Y_n)$ by 7.2, and so the Leray spectral sequence of η has

$$E_2^{p,q} = H^p(X; H^q(Y_n)) \Longrightarrow H^{p+q}(X \times Y_n),$$

which vanishes for $p > 1$ or $q > 1$. Thus $H^2(X \times Y_n) \approx H^1(X; H^1(Y_n))$. Since X is locally contractible, we have

$$H^1(X; H^1(Y_n)) \approx {}_{\Delta}H^1(X; H^1(Y_n)) \approx \operatorname{Hom}(H_1(X), H^1(Y_n)),$$

where the homology group is ordinary singular homology. Since X is a CW-complex with one 0-cell and countably many 1-cells, we have $H_1(X) = \bigoplus_{i=1}^{\infty} \mathbb{Z}$. Thus

$$H^2(X \times Y_n) \approx \operatorname{Hom}\left(\bigoplus_{i=1}^{\infty} \mathbb{Z}, \bigoplus_{j=n}^{\infty} \mathbb{Z} \right),$$

where the restriction to $H^2(X \times Y_{n+1})$ is induced by the map killing the basis element a_n in the second argument. Let $b_1, b_2, \ldots$ be the obvious basis of the first argument. Let $f : \bigoplus_{i=1}^{\infty} \mathbb{Z} \to \bigoplus_{j=n}^{\infty} \mathbb{Z}$ be defined by $f(b_i) = a_i$, this being regarded as zero if $i < n$. Then f survives in the direct limit over n, and so it defines a nonzero element of $\mathscr{H}^2(\pi)_{y_0} = \varinjlim H^2(X \times Y_n)$, as claimed. ◇

7.9. We shall now deal with the important special case of vector bundles at some length. Note that

$$H^p(\mathbb{R}^n, \mathbb{R}^n - \{0\}; L) \approx \begin{cases} L, & p = n, \\ 0, & p \neq n. \end{cases}$$

We shall take the base ring L to be a principal ideal domain. Note that comparison with $\mathbb{Z}$ using the universal coefficient theorem gives a preferred generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; L)$ well-defined up to sign.

Let ξ be an n-plane bundle over the arbitrary space Y, and let U be the total space of ξ. We shall identify Y with the zero section of ξ, and we put $U^\circ = U - Y$.

Let $\pi : U \to Y$ be the projection in ξ and let $\pi^\circ = \pi | U^\circ$. Let $\pi^{-1}(V) \approx V \times F$ be an admissible representation as a product of the restriction of ξ to some open set $V \subset Y$, where $F = \mathbb{R}^n$. Let D be the unit disk in F and ∂D its boundary. By homotopy invariance we have that over V, $\mathscr{H}^*(\pi; L) \approx \mathscr{H}^*(\pi | V \times D; L)$ and $\mathscr{H}^*(\pi, \pi^\circ; L) \approx \mathscr{H}^*(\pi | V \times D, \pi | V \times \partial D; L)$. It follows from 4.2 and 7.1 that globally, $\mathscr{H}^q(\pi; L)$ is the *constant* sheaf

$$\mathscr{H}^q(\pi; L) = \mathscr{H}^q(F; L) = \begin{cases} L, & q = 0, \\ 0, & q \neq 0 \end{cases}$$

and that $\mathscr{H}^q(\pi, \pi^\circ; L)$ is the *locally constant* sheaf $\mathscr{H}^q(F, F^\circ; L)$ (where $F^\circ = F - \{0\}$) with stalks

$$H^q(F,F^\circ;L)=\begin{cases} L, & q=n,\\ 0, & q\neq n,\end{cases}$$

and this, for $q=n$, has a generator well defined up to sign. [We remark that $\mathscr{H}^q(\pi,\pi^\circ;L)\approx\mathscr{H}^q_\Psi(\pi;L)$, where Ψ is the family of subsets of U that are "closed in the associated n-sphere bundle" (obtained by compactifying the fibers). For Y locally compact, Ψ consists of the fiberwise compact sets. However, Y would have to be assumed to be paracompact in the discussion below if we used this approach in place of the relative one.]

Let $\mathscr{O}=\mathscr{H}^n(F,F^\circ;L)$, the *orientation sheaf*, which is locally constant with stalks isomorphic to L. Each stalk $\mathscr{O}_y$ has a generator ε_y well-defined up to sign. Thus $\mathscr{O}\otimes\mathscr{H}^n(F,F^\circ;L)=\mathscr{O}\otimes\mathscr{O}$ is constant and has a canonical section given by $\varepsilon_y\otimes\varepsilon_y=-\varepsilon_y\otimes-\varepsilon_y$ in the fiber at $y\in Y$. Let this canonical section be $\sigma\in H^0(Y;\mathscr{O}\otimes\mathscr{H}^n(F,F^\circ;L))$. Alternatively stated, there is a *canonical* isomorphism

$$\boxed{\mathscr{O}\otimes\mathscr{H}^n(F,F^\circ;L)=\mathscr{O}\otimes\mathscr{O}\approx L.}$$

Let $\mathscr{B}$ be any sheaf on Y. Again, a usage of homotopy invariance II-11.8 together with 4.6 and (17) on page 214 ensures that the canonical maps

$$\mathscr{H}^q(\pi;L)\otimes\mathscr{B}\to\mathscr{H}^q(\pi;\pi^*\mathscr{B})$$

and

$$\mathscr{H}^q(\pi,\pi^\circ;L)\otimes\mathscr{B}\to\mathscr{H}^q(\pi,\pi^\circ;\pi^*\mathscr{B})$$

of (21) on page 215 are isomorphisms. Thus we have

$$\mathscr{H}^q(\pi;\pi^*\mathscr{B})\approx\begin{cases}\mathscr{B}, & q=0,\\ 0, & q\neq 0\end{cases}$$

and

$$\mathscr{H}^q(\pi,\pi^\circ;\pi^*\mathscr{B})\approx\begin{cases}\mathscr{O}\otimes\mathscr{B}, & q=n,\\ 0, & q\neq n.\end{cases}$$

Substitution of $\mathscr{O}\otimes\mathscr{B}$ for $\mathscr{B}$ gives

$$\mathscr{O}\otimes\mathscr{H}^q(\pi,\pi^\circ;\pi^*\mathscr{B})\approx\mathscr{H}^q(\pi,\pi^\circ;\pi^*(\mathscr{O}\otimes\mathscr{B}))\approx\begin{cases}\mathscr{B}, & q=n,\\ 0, & q\neq n.\end{cases}$$

Now let Φ be any family of supports on Y and let $\Theta=\pi^{-1}(\Phi)$. Consider the Leray spectral sequences $'E$, $''E$, and $'''E$ with

$$'E_2^{p,q}=H^p(Y;\mathscr{O}\otimes\mathscr{H}^q(F,F^\circ;L))\Longrightarrow H^{p+q}(U,U^\circ;\pi^*\mathscr{O})$$

[since $\mathscr{O}\otimes\mathscr{H}^q(F,F^\circ;L)\approx\mathscr{H}^q(\pi,\pi^\circ;\pi^*\mathscr{O})$ canonically as noted above],

$$''E_2^{p,q}=H^p_\Phi(Y;\mathscr{H}^q(\pi;\pi^*\mathscr{B}))\Longrightarrow H^{p+q}_\Theta(U;\pi^*\mathscr{B}),$$

and

$$'''E_2^{p,q} = H_\Phi^p(Y; \mathscr{O} \otimes \mathscr{H}^q(\pi, \pi^\circ; \pi^*\mathscr{B})) \Longrightarrow H_\Theta^{p+q}(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B})).$$

Since these spectral sequences have only one nontrivial fiber degree, they provide the canonical isomorphisms

$$H^p(Y; L) \approx H^p(Y; \mathscr{O} \otimes \mathscr{H}^n(F, F^\circ; L)) \approx H^{p+n}(U, U^\circ; \pi^*\mathscr{O}),$$

$$H_\Phi^p(Y; \mathscr{B}) \approx H_\Theta^p(U; \pi^*\mathscr{B}) \quad \text{via } \pi^*$$

by 6.3, and, using that $\mathscr{O} \otimes \mathscr{O} \otimes \mathscr{B} \approx \mathscr{B}$ canonically,

$$H_\Phi^p(Y; \mathscr{B}) \approx H_\Theta^{p+n}(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B})),$$

respectively. We have the cup product from 6.8:

$$\cup : {}'E_r^{p,q} \otimes {}''E_r^{s,t} \to {}'''E_r^{p+s,q+t}.$$

The homomorphism ${}''E_2^{p,0} \to {}'''E_2^{p,n}$ defined by $\alpha \mapsto \sigma \cup \alpha$ is $(-1)^{pn}$ times the map

$$H_\Phi^p(Y; \mathscr{H}^0(\pi; \pi^*\mathscr{B})) \to H_\Phi^p(Y; \mathscr{O} \otimes \mathscr{H}^n(\pi, \pi^\circ; \pi^*\mathscr{B}))$$

induced by the coefficient homomorphism

$$\sigma \cup (\bullet) : \mathscr{H}^0(\pi; \pi^*\mathscr{B}) \to \mathscr{O} \otimes \mathscr{H}^n(\pi, \pi^\circ; \pi^*\mathscr{B})$$

[from $\mathscr{O} \otimes \mathscr{H}^n(\pi, \pi^\circ; L) \otimes \mathscr{H}^0(\pi; \pi^*\mathscr{B}) \xrightarrow{\cup} \mathscr{O} \otimes \mathscr{H}^n(\pi, \pi^\circ; \pi^*\mathscr{B})$]. On the typical stalk at $y \in Y$ this is equivalent to the isomorphism

$$\sigma(y) \cup (\bullet) : H^0(F; L) \otimes \mathscr{B}_y \xrightarrow{\approx} H^n(F, F^\circ; L) \otimes \mathscr{B}_y.$$

Thus the map

$$\sigma \cup (\bullet) : {}''E_r^{p,0} \to {}'''E_r^{p,n}$$

is an isomorphism for $r = 2$, and hence for any $2 \le r \le \infty$ and for the "total" terms. That is, σ defines an element $\tau \in H^n(U, U^\circ; \pi^*\mathscr{O}) \approx {}'E_\infty^{0,n}$, and the map

$$\tau \cup (\bullet) : H_\Theta^p(U; \pi^*\mathscr{B}) \to H_\Theta^{p+n}(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B}))$$

is an isomorphism.

As remarked above, $\pi^* : H_\Phi^p(Y; \mathscr{B}) \to H_\Theta^p(U; \pi^*\mathscr{B})$ is also an isomorphism, and hence the map

$$\boxed{\tau \cup \pi^*(\bullet) : H_\Phi^p(Y; \mathscr{B}) \xrightarrow{\approx} H_\Theta^{p+n}(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B}))} \tag{22}$$

is an isomorphism. It is called the *Thom isomorphism*, and the class $\tau \in H^n(U, U^\circ; \pi^*\mathscr{O})$ is called the *Thom class* of ξ. [More precisely, if the bundle

is *orientable,* then there is an isomorphism, by definition, $\mathscr{O} \approx L$ well defined only up to sign in each fiber. An *orientation* is a choice of such an isomorphism. Given such a choice the element τ produces a class in $H^n(U, U^\circ; L)$, and the latter class is called a Thom class. It depends on a choice of orientation, unlike the class $\tau \in H^n(U, U^\circ; \pi^*\mathscr{O})$.]

Let j^* denote the canonical map $H^*_\Psi(U, U^\circ) \to H^*_\Psi(U)$ for any Ψ and any coefficients. The diagram

$$\begin{array}{ccc} H^p(U, U^\circ; \pi^*\mathscr{O}) \otimes H^q_\Theta(U; \pi^*\mathscr{B}) & \xrightarrow{\cup} & H^{p+q}_\Theta(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B})) \\ \downarrow{\scriptstyle j^*\otimes 1} & & \downarrow{\scriptstyle j^*} \\ H^p(U; \pi^*\mathscr{O}) \otimes H^q_\Theta(U; \pi^*\mathscr{B}) & \xrightarrow{\cup} & H^{p+q}_\Theta(U; \pi^*(\mathscr{O} \otimes \mathscr{B})) \end{array}$$

commutes; that is, $j^*(\alpha \cup \beta) = j^*(\alpha) \cup \beta$.

Let $\omega = i^*j^*(\tau) \in H^n(Y; \mathscr{O})$, where $i : Y \hookrightarrow U$ is the inclusion. Then $i^*j^*(\tau \cup \pi^*(\beta)) = i^*(j^*(\tau) \cup \pi^*(\beta)) = \omega \cup i^*\pi^*(\beta) = \omega \cup \beta$. That is, the following diagram commutes:

$$\begin{array}{ccc} H^p_\Phi(Y; \mathscr{B}) & \xrightarrow{\omega\cup(\bullet)} & H^{p+n}_\Phi(Y; \mathscr{O} \otimes \mathscr{B}) \\ \approx \downarrow{\scriptstyle \tau\cup\pi^*(\bullet)} & & \approx \downarrow{\scriptstyle \pi^*=(i^*)^{-1}} \\ H^{p+n}_\Theta(U, U^\circ; \pi^*(\mathscr{O} \otimes \mathscr{B})) & \xrightarrow{j^*} & H^{p+n}_\Theta(U; \pi^*(\mathscr{O} \otimes \mathscr{B})). \end{array} \tag{23}$$

The class ω is called the *Euler class* of ξ.

If there exists[15] an associated n-disk bundle N to ξ, then U and U° may be replaced by N and ∂N. Note that ∂N can then be any $(n-1)$-sphere bundle on Y, ξ being the associated n-plane bundle.

Thus using (23), we see that if $\pi : X \to Y$ is an $(n-1)$-sphere bundle, then there is the exact *Gysin sequence*:

$$\boxed{\begin{array}{r} \cdots \to H^p_\Phi(Y; \mathscr{B}) \xrightarrow{\omega\cup(\bullet)} H^{p+n}_\Phi(Y; \mathscr{O} \otimes \mathscr{B}) \xrightarrow{\pi^*} H^{p+n}_{\pi^{-1}(\Phi)}(X; \pi^*(\mathscr{O} \otimes \mathscr{B})) \\ \to H^{p+1}_\Phi(Y; \mathscr{B}) \to \cdots \end{array}} \tag{24}$$

which results from the exact sequence of the pair (U, U°) and the fact that U° has X as a deformation retract preserving fibers. (This sequence can also be derived directly from the spectral sequence of $\pi : X \to Y$ in a similar manner to the derivation of the Wang sequence in 7.10.) Also, see Section 13.

7.10. Let $\pi : X \to \mathbb{S}^n$ be a bundle projection with fiber F. Let Θ be any family of supports on F invariant under the structure group. By 7.5, the Leray sheaf is locally constant with stalks $H^*_\Theta(F; L)$. Thus it is constant for $n > 1$, and will be *assumed* here to be constant for $n = 1$. In the Leray spectral sequence of π we have

$$E_2^{p,q} = H^p(\mathbb{S}^n; H^q_\Theta(F; L)) \Longrightarrow H^{p+q}_\Psi(X; L),$$

[15] Always when Y is paracompact.

where Ψ is as in 7.3. Thus $E_2^{p,q} = \cdots = E_n^{p,q}$ and $E_{n+1}^{p,q} = \cdots = E_\infty^{p,q}$. There are also the exact sequences

$$0 \to E_\infty^{n,k-n} \to H_\Psi^k(X) \to E_\infty^{0,k} \to 0,$$

$$0 \to E_{n+1}^{0,k} \to E_n^{0,k} \xrightarrow{d_n} d_n(E_n^{0,k}) \to 0,$$

and

$$0 \to d_n(E_n^{0,k}) \to E_n^{n,k-n+1} \to E_{n+1}^{n,k-n+1} \to 0.$$

Putting these together, we obtain the exact sequence

$$\cdots \to H_\Psi^k(X) \to E_2^{0,k} \xrightarrow{d_n} E_2^{n,k-n+1} \to H_\Psi^{k+1}(X) \to \cdots$$

and hence the exact *Wang sequence*

$$\boxed{\cdots \to H_\Psi^k(X) \xrightarrow{i^*} H_\Theta^k(F) \xrightarrow{\Delta} H_\Theta^{k-n+1}(X) \to H_\Psi^{k+1}(X) \to \cdots} \tag{25}$$

where $i : F \hookrightarrow X$ is the inclusion. The map Δ results from

$$H^0(\mathbb{S}^n; H_\Theta^k(F)) = E_n^{0,k} \xrightarrow{d_n} E_n^{n,k-n+1} = H^n(\mathbb{S}^n; H_\Theta^{k-n+1}(F)).$$

If Θ_1 and Θ_2 are two such support families on F and if we consider the cup product $H_{\Theta_1}^k(F) \otimes H_{\Theta_2}^s(F) \to H_{\Theta_1 \cap \Theta_2}^{k+s}(F)$, then it follows from the general results in Appendix A that the maps Δ satisfy the relation

$$\boxed{\Delta(\alpha \cup \beta) = \Delta\alpha \cup \beta + (-1)^{k(n-1)}\alpha \cup \Delta\beta,} \tag{26}$$

where $\alpha \in H_{\Theta_1}^k(F)$ and $\beta \in H_{\Theta_2}^s(F)$.

8 Dimension

In this section we apply the Leray spectral sequence and other results of this chapter to the theory of cohomological dimension. As before, for a map $f : X \to Y$ and $B \subset Y$, we use the shorthand notation $B^\bullet = f^{-1}(B)$.

If Φ is a paracompactifying family of supports on Y and $A \subset Y$, we define the *relative dimension* of A in Y to be

$$\boxed{\underline{\dim}_{\Phi,L}^Y A = \sup\{\dim_L K \mid K \subset A,\ \varnothing \neq K \in \Phi\}.}$$

Note that $\underline{\dim}_{\Phi,L}^Y \varnothing = -\infty$. Also note that $\underline{\dim}_{\Phi,L}^Y A \leq \dim_{\Phi,L} A$ by II-16.9. This inequality may be strict as in the case for which $Y = \mathbb{R}^2$, $\Psi = cld$, and A is the Knaster explosion set less the explosion point, since a subset K of A that is closed in Y is compact and totally disconnected, whence $\dim_L K = 0$, while $\dim_L A = 1$.

Note that if A is locally closed in Y and Φ is paracompactifying on Y, then $\Phi|A$ is paracompactifying on A, and so $\underline{\dim}_{\Phi,L}^Y A = \dim_{\Phi|A,L} A$ in this case by II-16.7.

8.1. Lemma. *Let Φ be paracompactifying on Y. If $A \subset Y$ and $\mathscr{A}$ is a sheaf of L-modules on Y that is concentrated on A, then $H^p_\Phi(Y;\mathscr{A}) = 0$ for $p > \underline{\dim}^Y_{\Phi,L} A$.*

Proof. By II-14.13 we have that $H^p_\Phi(Y;\mathscr{A}) \approx \varinjlim H^p_\Phi(Y;\mathscr{S})$, where $\mathscr{S}$ ranges over sheaves concentrated on sets $K \in \Phi|A$. But if $\mathscr{S}$ is concentrated on $K \in \Phi|A$, then $H^p_\Phi(Y;\mathscr{S}) = H^p_\Phi(Y;\mathscr{S}_K) \approx H^p(K;\mathscr{S}|K) = 0$, by II-10.2 for $p > \dim_L K$, whence for $p > \underline{\dim}^Y_{\Phi,L} A$. □

8.2. Theorem. (E. G. Skljarenko.) *Let Ψ and Φ be paracompactifying families of supports on X and Y respectively and let $f : X \to Y$ be a Ψ-closed map. Let L be an arbitrary base ring. Let*

$$M_k = \{y \in Y \mid \dim_{\Psi,L} y^\bullet \geq k\}$$

and put $d_k = \underline{\dim}^Y_{\Phi,L} M_k$. Then

$$\boxed{\dim_{\Phi(\Psi),L} X \leq \sup\{k + d_k\} \leq \dim_{\Phi,L} Y + \sup_{y\in Y} \dim_{\Psi,L} y^\bullet.}$$

Proof. Let $\mathscr{A}$ be a sheaf on X. Then $\mathscr{H}^k_\Psi(f;\mathscr{A})_y \approx H^k_\Psi(y^\bullet;\mathscr{A}) = 0$ if $y \notin M_k$, so that $\mathscr{H}^k_\Psi(f;\mathscr{A})$ is concentrated on M_k. By 8.1 we have that $H^p_\Phi(Y;\mathscr{H}^k_\Psi(f;\mathscr{A})) = 0$ for $p > d_k$. Therefore, in the Leray spectral sequence of f we have $E_2^{p,q} = 0$ if $p+q > \sup\{k+d_k\}$, and so $H^n_{\Phi(\Psi)}(X;\mathscr{A}) = 0$ for $n > \sup\{k + d_k\}$. The second inequality is obvious. □

8.3. Example. Given X there is generally no upper bound for $\dim_L Y$ in the situation of 8.2, as is shown by letting Y be the Hilbert cube, $X = Y$ but with the discrete topology, $f : X \to Y$ the identity, $\Phi = cld$, and Ψ the family of finite subsets of X. Then f is Ψ-closed but not closed. Of course, $\dim_L X = 0$ and $\dim_L Y = \infty$. See, however, 8.12 and its succeeding results. ◇

When X and Y are both locally compact Hausdorff and $\Psi = c = \Phi$, the conditions of 8.2 are all satisfied and so:

8.4. Corollary. *If $f : X \to Y$ is a map of locally compact Hausdorff spaces, then for any base ring L,*

$$\boxed{\dim_L X \leq \sup\{k + d_k\} \leq \dim_L Y + \sup_{y\in Y} \dim_L y^\bullet,}$$

where $d_k = \sup_K\{\dim K \mid \varnothing \neq K \in c,\ y \in K \Rightarrow \dim_L y^\bullet \geq k\}$. □

Now suppose that X is locally compact Hausdorff and that Y is locally paracompact. Since the product of a compact space with a paracompact space is paracompact, $X \times Y$ is locally paracompact, and so $\dim_L X \times Y$ makes sense.

8.5. Corollary. *If X is locally compact Hausdorff and Y is locally paracompact, then*[16]

$$\boxed{\dim_L X \times Y \le \dim_L X + \dim_L Y}$$

for any base ring L. Moreover, if L is a principal ideal domain and there is an open set $U \subset X$ such that $H_c^{\dim X}(U; L)$ has L as a direct summand, then equality holds. In particular, equality always holds when L is a field.

Proof. It follows from the local nature of dimension II-16.8 on locally paracompact spaces that it suffices to consider the case for which X is compact and Y is paracompact. Then 8.2 applies to the projection $X \times Y \to Y$ with closed supports and immediately yields the desired inequality.

For the second statement, let $n = \dim_L X$ and $m = \dim_L Y$ and recall from II-16.32 that there is an open set $W \subset Y$ with $H^m_{cld|W}(W; L) \neq 0$. Consider the Leray spectral sequence of the projection $\pi_W : U \times W \to W$ with $\Phi = cld_Y|W$ on W and $\Psi = c \times \Phi$ on $U \times W$. This has

$$E_2^{p,q} = H^p_\Phi(W; \mathscr{H}^q_c(U; L)) \Longrightarrow H^{p+q}_{\Phi(\Psi)}(U \times W; L),$$

and the Leray sheaf $\mathscr{H}^q_c(U; L)$ on W is constant with stalks $H^q_c(U; L)$ by 7.2. [The Leray sheaf for $\Psi = c \times W$ as in 7.2 is clearly the same as that for $\Psi = c \times \Phi$ as here.] Therefore $\mathscr{H}^n_c(U; L)$ has the constant sheaf L as a direct summand, and so $E_\infty^{m,n} \approx E_2^{m,n} = H^m_\Phi(W; H^n_c(U; L)) \neq 0$ since $E_2^{p,q} = 0$ for $p > m$ or $q > n$. It follows that $H^{n+m}_{\Phi(\Psi)}(U \times W; L) \neq 0$ and hence that $\dim_L X \times Y \ge n + m$, since $\Phi(\Psi)$ is paracompactifying by 5.5. [It can be seen that $\Phi(\Psi) = cld_{X \times Y}|U \times W$, but that is not needed here.] □

8.6. Corollary. *If Y is a separable metric space and X is a metric space, then*

$$\boxed{\dim_L X \times Y \le \dim_L X + \operatorname{Ind} Y}$$

for any base ring L with unit.

Proof. According to [49, p. 65], Y can be embedded in a compact metric space K with $\operatorname{Ind} K = \operatorname{Ind} Y$. Since $X \times K$ is hereditarily paracompact, we have

$$\begin{aligned}
\dim_L X \times Y &\le \dim_L X \times K && \text{by II-16.8}\\
&\le \dim_L X + \dim_L K && \text{by 8.5}\\
&\le \dim_L X + \operatorname{Ind} K && \text{by II-16.39}\\
&= \dim_L X + \operatorname{Ind} Y.
\end{aligned}$$

□

[16]The inequality is also valid for X locally contractible and $X \times Y$ locally paracompact.

8.7. Example. This example shows that the sets M_k in 8.2 need not be locally closed. Let $Y = [0,1]$ and let X be the union of Y with the vertical intervals $\{x\} \times [0, 1/q]$, where x ranges over the rational numbers $x = p/q$ in lowest terms. Let $f : X \to Y$ be the projection. Then M_1 is the set of rationals in Y. Both X and Y are compact, whence f is closed. In this example, $d_0 = 0 = d_1$. By 8.2, $\dim_L X = 1$. ◇

8.8. Example. This example shows that the condition that the map f be Ψ-closed is essential to 8.2. Let X be the Knaster explosion set with the explosion point removed; see II-16.22. Let Y be the Cantor set and $f : X \to Y$ the obvious map. The fiber $y^\bullet$ is the set of rational points in an interval for some y and the set of irrational points for the other y. Thus $\dim_L y^\bullet = 0$ for all y and $\dim_L Y = 0$. Therefore $\sup\{d_k + k\} = 0$, while $\dim_L X = 1$. The map f is open but not closed. ◇

8.9. Theorem. *Let Y be paracompact and $X \subsetneq Y$ a dense paracompact proper subspace. Then for a principal ideal domain L we have*

$$\boxed{\dim_L Y \le \dim_L X + \underline{\dim}_L^Y (Y - X) + 1.}$$

Proof. Let $\mathscr{A}$ be a sheaf on X and let $i : X \hookrightarrow Y$. The Leray spectral sequence of i has

$$E_2^{p,q} = H^p(Y; \mathscr{H}^q(i; \mathscr{A})) \Longrightarrow H^{p+q}(X; \mathscr{A}).$$

Now, for $x \in X$ and U ranging over the neighborhoods of x, we have $\mathscr{H}^q(i;\mathscr{A})_x = \varinjlim H^q(U \cap X; \mathscr{A}) = 0$ for $q > 0$ by II-10.6, since a point is taut. Thus $\mathscr{H}^q(i;\mathscr{A})$ is concentrated on $Y - X$ for $q > 0$, whence $E_2^{p,q} = 0$ for $p > \underline{\dim}_L^Y (Y - X)$ and $q > 0$, by 8.1. Also, $\mathscr{H}^q(i;\mathscr{A}) = 0$ for $q > \dim_L X$ since it is generated by the presheaf $V \cap X \mapsto H^q(\overline{V} \cap X; \mathscr{A}) = 0$ for $q > \dim_L X$, V open in Y. Thus all the terms $E_2^{p,0}$ survive to $E_\infty^{p,0}$ for $p > \dim_L X + \underline{\dim}_L^Y (Y - X) + 1$. Consequently, $H^p(Y; i\mathscr{A}) = E_2^{p,0} \approx E_\infty^{p,0} = H^p(X;\mathscr{A}) = 0$ for $p > \dim_L X + \underline{\dim}_L^Y (Y - X) + 1$. Now let $F \subset Y$ be an arbitrary closed subspace, put $U = Y - F$, and specialize to the case in which $\mathscr{A} = L_U|X$. The canonical homomorphism $L_U \to i(L_U|X)$ is a monomorphism by I-Exercise 18. Let $\mathscr{Q}$ be its cokernel, so that there is the exact sequence

$$0 \to L_U \to i(L_U|X) \to \mathscr{Q} \to 0.$$

Now, $i(L_U|X)|X = L_U|X$ by I-Exercise 2, so that $\mathscr{Q}$ is concentrated on $Y - X$. Consequently, $H^k(Y;\mathscr{Q}) = 0$ for $k > \underline{\dim}_L^Y (Y - X)$ by 8.1. The cohomology sequence induced by the displayed coefficient sequence has the segment

$$H^{p-1}(Y;\mathscr{Q}) \to H^p(Y; L_U) \to H^p(Y; i(L_U|X)).$$

The two end terms vanish for $p > \dim_L X + \underline{\dim}_L^Y (Y - X) + 1$. Therefore $H^p(Y, F; L) \approx H^p(Y; L_U) = 0$ for $p > \dim_L X + \underline{\dim}_L^Y (Y - X) + 1$, and the result now follows from II-16.33. □

8.10. Corollary. *If X, Y, and $X \cup Y$ are paracompact, $Y \not\subset X$, and L is a principal ideal domain, then*[17]

$$\boxed{\dim_L X \cup Y \le \dim_L X + \underline{\dim}_L^{X\cup Y}(Y - X) + 1 \le \dim_L X + \dim_L Y + 1.}$$

Proof. If $\overline{X}$ is the closure of X in $X \cup Y$, then

$$\begin{aligned}
\dim_L X \cup Y &\le \max\{\dim_L \overline{X},\ \dim_L(Y - \overline{X})\}\\
&\le \max\{\dim_L \overline{X},\ \underline{\dim}_L^{X\cup Y}(Y - X)\}\\
&\le \max\{\dim_L X + \underline{\dim}_L^{\overline{X}}(\overline{X} - X) + 1,\ \underline{\dim}_L^{X\cup Y}(Y - X)\}\\
&\le \max\{\dim_L X + \underline{\dim}_L^{X\cup Y}(Y - X) + 1,\ \underline{\dim}_L^{X\cup Y}(Y - X)\}\\
&= \dim_L X + \underline{\dim}_L^{X\cup Y}(Y - X) + 1 \le \dim_L X + \dim_L Y + 1
\end{aligned}$$

by II-Exercise 11, II-16.8, and the theorem.[18] □

This corollary is half of what is called the "decomposition theorem" in the classical dimension theory of metric spaces. That result says that a metric space X has $\operatorname{Ind} X \le n \Leftrightarrow X$ is the union of $n + 1$ subspaces A_i with $\operatorname{Ind} A_i \le 0$. The "decomposition" half of this cannot hold for $\dim_L$ in place of Ind because that would imply that $\dim_L X = \operatorname{Ind} X$ whenever $\dim_L X < \infty$ by II-16.35, and that is generally false even for $L = \mathbb{Z}$ and X compact metric.

8.11. Example. Let X be the closure of the Knaster explosion set K in the plane; see II-16.22. Let $A = X - K$. Then A is the union of a countable number of sets closed in A and zero-dimensional since each of them is homeomorphic to a totally disconnected subset of an interval (the set of points of irrational height on the ray to a given end point of a complementary interval of the Cantor set, or the set of points of given rational height on all rays to the non-end points). Thus $\dim_L A = 0$ by II-16.40. Also, $\underline{\dim}_L^X K = 0$ even though $\dim_L K = 1$, since a compact subset C of K is totally disconnected and so has $\dim_L C = 0$. For this example the first inequality in 8.10, for A and K in place of X and Y, reads $\dim_L X \le 0 + 0 + 1$. ◇

We turn now to inequalities of the opposite type to that in 8.2.

8.12. Theorem. *Let $f : X \twoheadrightarrow Y$ be a proper closed surjection between locally paracompact spaces such that the Leray sheaf $\mathscr{H}^*(f; L)$ is locally constant with stalks H^* that are finitely generated over the principal ideal domain L. Let $n = \max\{p \mid \operatorname{rank} H^p > 0\}$. Then $\dim_L Y \le \dim_L X - n$.*

[17] The case $X \cup Y$ metric and $L = \mathbb{Z}$ of the outside inequality is due to Rubin [71].
[18] Note that the proof of the first inequality does not use that Y is paracompact.

Proof. By II-16.8 we may assume that $\mathscr{H}^*(f;L)$ is constant. Since each $y^\bullet$ is taut in X and $H^n(y^\bullet;L)$ is finitely generated, we may also assume that the restriction $H^n(X;L) \to H^n(y^\bullet;L)$ is surjective for any given $y \in Y$. By Exercise 5, this restriction is the composition of the edge homomorphism $H^n(X;L) \to E_2^{0,n} = H^0(Y;\mathscr{H}^n(f;L)) = \Gamma(\mathscr{H}^n(f;L))$ with the map $i_y^* : \Gamma(\mathscr{H}^n(f;L)) \to H^n(y^\bullet;L)$. Let $\beta \in \Gamma(\mathscr{H}^n(f;L))$ be in the image of the edge homomorphism and such that $i_y^*(\beta)$ is a generator of a direct summand isomorphic to L. Since $\mathscr{H}^n(f;L)$ is constant, we can pass to a neighborhood of y over which β is a constant section. Let $0 \neq \alpha \in H_\Phi^k(Y;L)$ for any paracompactifying family Φ of supports. The augmentation $L \rightarrowtail \mathscr{H}^0(f;L)$ is to a direct summand, and so it induces a canonical monomorphism $H_\Phi^k(Y;L) \rightarrowtail H_\Phi^k(Y;\mathscr{H}^0(f;L))$, whence we may regard α as a class in $E_2^{k,0}(\Phi) = H_\Phi^k(Y;\mathscr{H}^0(f;L))$ of the Leray spectral sequence. Also, $\beta \in E_2^{0,n} = H^0(Y;\mathscr{H}^n(f;L)) = \Gamma(\mathscr{H}^n(f;L))$. By the discussion in II-7.4, $\alpha \cup \beta \in E_2^{k,n}(\Phi) = H_\Phi^k(Y;\mathscr{H}^n(f;L))$ is the image of α under the map induced by the coefficient homomorphism $L \rightarrowtail \mathscr{H}^0(f;L) \to \mathscr{H}^n(f;L)$ given by $1 \mapsto 1 \cdot \beta(y) = \beta(y)$ in the stalks at y. Since the latter is a monomorphism onto a direct summand, we have that $\alpha \cup \beta \neq 0$. Since β is in the image of the edge homomorphism, it is a permanent cocycle. Also, α is automatically a permanent cocycle. It follows that $\alpha \cup \beta$ is a permanent cocycle, since $d_r(\alpha \cup \beta) = d_r\alpha \cup \beta \pm \alpha \cup d_r\beta = 0$.

Now, if L were a *field*, then n would be the largest degree in which $\mathscr{H}^*(f;L)$ would be nonzero. Then $\alpha \cup \beta$ could not be killed, and so $E_\infty^{k,n}(\Phi) \neq 0$, giving $H_{f^{-1}\Phi}^{k+n}(X) \neq 0$, whence $k + n \leq \dim_L X$. It would follow that $H_\Phi^k(Y;L) = 0$ for $k > \dim_L X - n$ and any paracompactifying Φ, and so $\dim_L Y \leq \dim_L X - n$ by II-16.33.

In the general case, let p be a prime of L and let $L_p = L/pL$. The appropriate value of n for the field L_p is at least that for L, and so we know that $\dim_{L_p} Y \leq \dim_{L_p} X - n \leq \dim_L X - n$. Let $k > \dim_L X - n$. Then the exact coefficient sequences[19] $0 \to L \xrightarrow{p} L \to L_p \to 0$ show that $H_\Phi^k(Y;L)$ is divisible, meaning that given any $0 \neq m \in L$, there is a class $\alpha' \in H_\Phi^k(Y;L)$ with $\alpha = m\alpha'$. As with α, $\alpha' \cup \beta$ is a permanent cocycle. Now, for any $q > n$, $\mathscr{H}^q(f;L)$ is killed by multiplication by some nonzero $m \in L$, and so the same is true of each $E_r^{s,q}$. Therefore, if $\alpha \cup \beta$ survives nonzero to $E_r^{k,n}$ and $E_r^{k-r,n+r-1}$ is killed by m and $\gamma \in E_r^{k-r,n+r-1}$ has $d_r\gamma = \alpha' \cup \beta$, where $m\alpha' = \alpha$, then $0 = d_r(m\gamma) = m\alpha' \cup \beta = \alpha \cup \beta \neq 0$, a contradiction. Thus such an $\alpha' \cup \beta$ survives nonzero to $E_{r+1}^{k,n}$. An inductive argument of this type shows that some element of $E_2^{k,n}$ must survive nonzero to $E_\infty^{k,n}$ and so $H_{f^{-1}\Phi}^{k+n}(X;L) \neq 0$, contrary to our assumption that $k+n > \dim_L X$. The result follows as before. □

8.13. Corollary. (Dydak and Walsh [37]) *Let $f : X \twoheadrightarrow Y$ be a proper closed surjection between separable metric spaces. Let L be a principal*

[19] This is also what shows that $\dim_{L_p} X \leq \dim_L X$.

ideal domain and let H^* be a finitely generated graded L-module such that $H^*(y^\bullet; L) \approx H^*$ for all $y \in Y$. Suppose that n is maximal such that $\operatorname{rank} H^n > 0$. *Then* $\dim_L Y \le \dim_L X - n$.

Proof. By 4.14, Y is the union of countably many closed subsets K over which $\mathscr{H}^*(f; L)$ is constant. By 8.12, $\dim_L K \le \dim_L X - n$, and the result follows from the sum theorem II-16.40. □

8.14. Corollary. (Dydak and Walsh [37])[20] *Let $f : X \twoheadrightarrow Y$ be a proper closed surjection between separable metric spaces. For a given principal ideal domain L, assume that each $H^*(y^\bullet; L)$ is finitely generated. Let m be a maximal integer such that there is a sequence*[21] *$G_1 \dashv G_2 \dashv \cdots \dashv G_m$ of graded groups $G_i \approx H^*(y_i^\bullet; L)$ for some points $y_i \in Y$. Let n be such that for each $y \in Y$,* $\operatorname{rank} H^s(y^\bullet; L) > 0$ *for some $s \ge n$. Then*

$$\boxed{\dim_L Y \le m(\dim_L X - n) + m - 1.}$$

Proof. Let Y_i be the set of all points $y \in Y$ such that there exists a chain of *maximum* length of the form $H^*(y^\bullet; L) = G_i \dashv \cdots \dashv G_m$ as in the statement. By 4.14, Y_i is covered by a countable collection of relatively closed sets K such that $\mathscr{H}^*(f; L)|K$ is constant. By 8.13, $\dim_L K \le \dim_L X - n$. By II-16.40, $\dim_L Y_i \le \dim_L X - n$. Since $Y = Y_1 \cup \cdots \cup Y_m$, an inductive use of 8.10 gives the desired formula. □

Remark: Nowhere in the proofs of 8.12, 8.13, or 8.14 did we use any information, such as the finite generation, about the sheaves $\mathscr{H}^q(f; L)$ for $0 < q < n$. Thus all of these results can be sharpened.

Perhaps the item of greatest interest among all of our results in this direction is the following cohomological analogue and strengthening of a well-known fact from classical dimension theory:

8.15. Theorem. *Let $f : X \twoheadrightarrow Y$ be a finite-to-one closed map between separable metric spaces. Suppose that $n_1 < \cdots < n_m$ are natural numbers such that each $y \in Y$ has $\#y^\bullet = n_i$ for some i. Then*

$$\boxed{\dim_L X \le \dim_L Y \le \dim_L X + m - 1}$$

for any principal ideal domain L.[22] *Moreover, if* $\dim_L Y = \dim_L X + m - 1$ *and if we put $Y_p = \{y \in Y \mid \#y^\bullet \ge n_p\}$ and $X_p = f^{-1}(Y_p)$, then* $\dim_L X_p = \dim_L X$ *and* $\dim_L Y_p = \dim_L X + m - p$ *for each $1 \le p \le m$.*

[20] Both the hypotheses and conclusion are slightly weaker in [37].

[21] Here $G \dashv H$ means that G is a graded proper direct summand of H.

[22] This also holds for covering dimension because of the classical result implying that $\operatorname{covdim} Y \le \operatorname{covdim} X + n_m - 1$ and the fact that covdim coincides with $\dim_{\mathbb{Z}}$ for spaces of finite covering dimension.

Proof. The first inequality follows from 8.2. Put

$$W_p = \{y \in Y \mid \#y^\bullet = n_p\} \text{ so that } Y_p = W_p \cup \cdots \cup W_m.$$

Then the theorem is the case $p = 1$ of the formula

$$\dim_L Y_p \le \dim_L X + m - p.$$

The proof of this formula will be by downwards induction on p. Suppose it is true for $p > r$ (a vacuous assumption if $r = m$) and that $Y = Y_r$. Let $r \le p \le m$ and put $k = \dim_L X + m - p$. By 4.14, W_p is covered by a countable number of sets K, closed in Y, such that the constant sheaf L^p on K is a subsheaf of $\mathscr{H}^0(f;L)|K \approx gL$, where $g = f|K^\bullet$, with quotient sheaf $\mathscr{Q}$ concentrated on $K \cap Y_{p+1}$. (Note that $\mathscr{Q} = 0$ if $p = m$.) Let Φ be a paracompactifying family of supports on K. The exact sequence $0 \to L^p \to gL \to \mathscr{Q} \to 0$ of sheaves on K gives the exact sequence

$$H^k_\Phi(K;\mathscr{Q}) \to H^{k+1}_\Phi(K;L^p) \to H^{k+1}_\Phi(K;gL).$$

The term on the right is $H^{k+1}_\Phi(K;gL) \approx H^{k+1}_{f^{-1}\Phi}(K^\bullet;L) = 0$ by II-11.1 and since $k+1 > \dim_L X$. The term on the left is zero since $\mathscr{Q}$ is concentrated on $K \cap Y_{p+1}$, which has dimension at most $\dim_L X + m - (p+1) = k - 1$ by the inductive assumption. Thus $H^{k+1}_\Phi(K;L) = 0$ and so $\dim_L K \le k = \dim_L X + m - p$. (Also, if $p < m$ and $\dim_L Y_{p+1} < k - 1$, we would deduce that $\dim_L K < k$.) Doing this for each p, $r \le p \le m$, gives a countable closed covering of Y by sets K with $\dim_L K \le \dim_L X + m - r$. By the sum theorem II-16.40, $\dim_L Y \le \dim_L X + m - r$ (with strict inequality if $\dim_L Y_p < k-1$ for any $r < p \le m$), completing the induction. If for some p, $\dim_L X_p < \dim_L X$, then we would have that $\dim_L Y_p \le \dim_L X_p + m - p < \dim_L X + m - p$, whence $\dim_L Y < \dim_L X + m - r$. □

It is worthwhile stating some immediate special cases of 8.15. If the n_i form the segment of integers between n and m, then we get:

8.16. Corollary. *Let $f : X \twoheadrightarrow Y$ be a finite-to-one closed map between separable metric spaces and let $1 \le n \le m$. Assume that $n \le \#y^\bullet \le m$ for all $y \in Y$. Then*

$$\boxed{\dim_L X \le \dim_L Y \le \dim_L X + m - n}$$

for any principal ideal domain L. Moreover, if $\dim_L Y = \dim_L X + m - n$ and if we put $Y_p = \{y \in Y \mid p \le \#y^\bullet \le m\}$ and $X_p = f^{-1}(Y_p)$, then $\dim_L X_p = \dim_L X$ and $\dim_L Y_p = \dim_L X + m - p$ for each $n \le p \le m$. □

The cases $n = 1$ and $n = m$ give the following two corollaries:

8.17. Corollary. *If $f : X \twoheadrightarrow Y$ is a closed surjection between separable metric spaces that is at most m-to-1, then $\dim_L Y \le \dim_L X - m + 1$.* □

8.18. Corollary. *If $f : X \twoheadrightarrow Y$ is a closed map between separable metric spaces that is exactly m-to-1, then $\dim_L Y = \dim_L X$.* □

Remark: Examples exist, for any m, of closed and at most m-to-1 maps $f : X \to Y$ of separable metric spaces for which $\dim_{\mathbb{Z}} Y = \dim_{\mathbb{Z}} X + m - 1$; see, for example, [62, p. 78]. Thus the last part of 8.16 gives that $\dim_{\mathbb{Z}} Y_p = \dim_{\mathbb{Z}} X - m + p$ and $\dim_{\mathbb{Z}} X_p = \dim_{\mathbb{Z}} X$ for all $1 \le p \le m$. This shows that the inequalities in 8.16 are best possible.

Finally, we present Skljarenko's improvement of the Vietoris mapping theorem. Note that 8.2 is an immediate consequence of the following theorem, but the direct proof is just as easy.

8.19. Theorem. *Let $f : X \to Y$ be a Ψ-closed map, where Ψ is a family of supports on X, and suppose that each $y^\bullet$ is Ψ-taut in X. Let $\mathscr{A}$ be a sheaf on X and Φ a paracompactifying family of supports on Y. Let*

$$S_k = \{y \in Y \mid H^k_{\Psi \cap y^\bullet}(y^\bullet; \mathscr{A}) \neq 0\}$$

for $k > 0$ and put

$$b_k = \underline{\dim}^Y_{\Phi,L} S_k.$$

For a given integer (or ∞) N let

$$n = 1 + \sup\{k + b_k \mid 0 < k < N\}.$$

Then the edge homomorphism

$$f^\diamond : H^p_\Phi(Y; f_\Psi \mathscr{A}) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \rightarrowtail H^p_{\Phi(\Psi)}(X; \mathscr{A})$$

in the Leray spectral sequence is an isomorphism for $n < p < N$, an epimorphism for $p = n$, and a monomorphism for $p = N$.

Proof. As in the proof of 8.2 we have that $E_2^{p,q} = 0$ for $\sup\{k + b_k \mid 0 < k < N\} < p + q < N$, and the result follows immediately. □

8.20. Proposition. *Let $f : X \twoheadrightarrow Y$ be a closed surjection with each $y^\bullet$ taut in X. Let $\mathscr{B}$ be a sheaf on Y and Φ a paracompactifying family of supports on Y. Let*

$$S_0 = \{y \in Y \mid y^\bullet \text{ is not connected and } \mathscr{B}_y \neq 0\}$$

and

$$b_0 = \underline{\dim}^Y_{\Phi,L} S_0.$$

Then

$$\beta^* : H^p_\Phi(Y; \mathscr{B}) \to H^p_\Phi(Y; ff^* \mathscr{B})$$

is an isomorphism for $p > 1 + b_0$ and an epimorphism for $p = 1 + b_0$.

Proof. By the proof of II-11.7 it follows that the monomorphism $\beta : \mathscr{B} \rightarrowtail ff^*\mathscr{B}$ is an isomorphism on the complement of S_0. If $\mathscr{C}$ is the cokernel of β, then $\mathscr{C}$ is concentrated on S_0, and hence $H^p_\Phi(Y; \mathscr{C}) = 0$ for $p > b_0$. Thus the result follows from the cohomology sequence of the coefficient sequence $0 \to \mathscr{B} \to ff^*\mathscr{B} \to \mathscr{C} \to 0$. □

8.21. Corollary. (E. G. Skljarenko.) *Let $f : X \twoheadrightarrow Y$ be a closed surjection with each $y^\bullet$ taut in X. Let $\mathscr{B}$ be a sheaf on Y and Φ a paracompactifying family of supports on Y. Let S_k and b_k, $k \geq 0$, be as in* 8.19 (*with $\Psi = cld$ and $\mathscr{A} = f^*\mathscr{B}$*) *and* 8.20. *For a given integer* (*or* ∞) *N let*

$$n = 1 + \sup\{k + b_k \mid 0 \leq k < N\}.$$

Then

$$f^* : H^p_\Phi(Y; \mathscr{B}) \to H^p_{f^{-1}\Phi}(X; f^*\mathscr{B})$$

is an isomorphism for $n < p < N$, an epimorphism for $p = n$, and a monomorphism for $p = N$.

Proof. Putting $\mathscr{A} = f^*\mathscr{B}$, the result follows from 8.19, 8.20, and the fact that $f^* = f^\circ \circ \beta^*$ when $\Psi = cld$; see 6.3. □

9 The spectral sequences of Borel and Cartan

In this section G will denote a compact Lie group (perhaps finite), and X will denote a space upon which G acts as a topological transformation group.[23] We *assume* that the orbits of G are relatively Hausdorff in X, e.g., X Hausdorff. E_G denotes a compact N-universal G-bundle; that is, for our purposes, E_G is a compact Hausdorff space upon which G acts freely[24] and such that $\tilde{H}^p(E_G; L) = 0$ for $p < N$. An N-universal bundle for G exists for all N and can be taken to be the join of sufficiently many copies of G. Thus we may assume that E_G and $B_G = E_G/G$ are locally contractible (in fact, finite polyhedra). The quotient space B_G is called an *N-classifying space for G.*

Let G act on the product $X \times E_G$ by the diagonal action, that is, $g(x, y) = (g(x), g(y))$, and denote by $X_G = X \times_G E_G$ the quotient space of $X \times E_G$ under this action of G.

The projections $X \leftarrow X \times E_G \to E_G$ are G-equivariant, and hence they induce maps

$$\boxed{X/G \xleftarrow{\eta} X_G \xrightarrow{\pi} B_G.}$$

[23] G can be any compact group if X is locally compact Hausdorff and if we use compact supports on it throughout.

[24] That is, for any $y \in E_G$, $g(y) = y \Rightarrow g = e$.

For $x \in X$, let $\breve{x} = G(x) = \{g(x) \mid g \in G\}$, which is called the *orbit* of x. The *isotropy subgroup* of G at x is defined to be $G_x = \{g \in G \mid g(x) = x\}$. It is easy to see that the natural map $G/G_x \to G(x)$, taking gG_x into $g(x)$, is a homeomorphism, since G is compact. Let f denote the canonical map $X \to X/G$ sending x into its orbit $\breve{x}$, and note that f is a closed map.

The set $\eta^{-1}(\breve{x}) = G(x) \times_G E_G \subset X_G$ can be identified with $B_{G_x} = E_G/G_x$ as follows: Map E_G/G_x into $G(x) \times_G E_G$ by taking the orbit $G_x(y)$, $y \in E_G$, into the orbit of (x, y) under the diagonal action. It is immediately verified that this map is continuous and bijective and hence is a homeomorphism since the spaces involved are compact Hausdorff. It is also easily seen that each $\eta^{-1}(\breve{x})$ is relatively Hausdorff in X_G and that η is a closed map.

Similarly, since G acts *freely* on E_G, the fibers of π are homeomorphic to X. It is not difficult to see that π is actually a bundle projection with fiber X and group G. In fact, if $U \subset B_G$ and $h : G \times U \to E_G$ is an admissible homeomorphism (in the sense of Section 7), we see that the composition of the canonical maps

$$X \times U \to X \times_G (G \times U) \xrightarrow{1 \times h} X \times_G E_G = X_G \tag{27}$$

is an admissible homeomorphism onto $\pi^{-1}(U)$.

Let Φ be a family of supports on X/G and let $\Psi = \eta^{-1}\Phi$ be the corresponding family of supports on X_G. Note that on $X \times U$, as above, $\Psi \cap (X \times U)$ has the form $\Theta \times U$, where $\Theta = f^{-1}\Phi$. (That is, locally in B_G, $\Psi = \Theta \times B_G$.)

Let $\mathscr{A}$ be a sheaf on X/G, and note that $(\eta^*\mathscr{A})|X = f^*\mathscr{A}$, where X is any fiber of π. Consider the Leray spectral sequence ${}'E_r^{p,q}$ of η with coefficients in $\eta^*\mathscr{A}$, with closed supports on X_G, and with Φ-supports on X/G. Then, since $\Phi(cld) = \eta^{-1}\Phi = \Psi$,

$${}'E_2^{p,q} = H_\Phi^p(X/G; \mathscr{H}^q(\eta; \eta^*\mathscr{A})) \Longrightarrow H_\Psi^{p+q}(X_G; \eta^*\mathscr{A}). \tag{28}$$

Also consider the Leray spectral sequence ${}''E_r^{p,q}$ of π with coefficients in $\eta^*\mathscr{A}$, with Ψ-supports on X_G, and with closed supports on B_G. Then

$${}''E_2^{p,q} = H^p(B_G; \mathscr{H}_\Psi^q(\pi; \eta^*\mathscr{A})) \Longrightarrow H_\Psi^{p+q}(X_G; \eta^*\mathscr{A}), \tag{29}$$

since $cld(\Psi) = \Psi(B_G) = \Psi$, because B_G is compact.

We have

$$\mathscr{H}^q(\eta; \eta^*\mathscr{A})_{\breve{x}} \approx H^q(B_{G_x}; \mathscr{A}_{\breve{x}}), \tag{30}$$

since η is closed, $\eta^{-1}(\breve{x}) \approx B_{G_x}$ is compact and relatively Hausdorff, and $\eta^*\mathscr{A}|\eta^{-1}(\breve{x})$ is the constant sheaf with stalk $\mathscr{A}_{\breve{x}}$. [Thus the coefficients $\mathscr{A}_{\breve{x}}$ on the right side of (30) are *constant*.]

Note that on $\pi^{-1}(U) \approx X \times U$ [as in (27)] $\eta^*\mathscr{A}$ has the form $f^*\mathscr{A} \times U$. This fact, together with the fact that B_G is compact and locally contractible, implies by 7.5 that $\mathscr{H}_\Psi^q(\pi; \eta^*\mathscr{A})$ is *locally constant* with stalks

$H^q_\Theta(X; f^*\mathscr{A})$ (and structure group G). Thus, as in Section 7, we use the notation

$$\mathscr{H}^q_\Theta(X; f^*\mathscr{A}) = \mathscr{H}^q_\Psi(\pi; \eta^*\mathscr{A}). \tag{31}$$

If G is allowed to be any compact group, then this also follows from 7.3 when X is locally compact Hausdorff and the support families Θ and Ψ are c or c restricted to an open set.

9.1. Lemma. *Let $V \subset X/G$ be an open set containing $E(\Phi)$; see* I-6. *Suppose that $G_x = \{e\}$ for all x with $\breve{x} \in V$. Then*

$$H^*_\Phi(X/G; \mathscr{H}^q(\eta; \eta^*\mathscr{A})) = 0$$

for $0 < q < N$. [More generally, this is true when $H^q(B_{G_x}; \mathscr{A}_{\breve{x}}) = 0$ for $0 < q < N$ and all $\breve{x} \in V$.]

Proof. Since $V \supset E(\Phi)$ it follows directly from the definition of cohomology that $H^*_\Phi(X/G; \mathscr{B}) = H^*_{\Phi|V}(V; \mathscr{B})$ for any sheaf $\mathscr{B}$, since

$$\mathscr{C}^*(X/G; \mathscr{B})|V = \mathscr{C}^*(V; \mathscr{B}).$$

In our situation, the coefficient sheaf vanishes on V, by (30), since $B_{G_x} = E_G$ is N-acyclic for $\breve{x} \in V$, and the result follows. □

Since each fiber B_{G_x} of η is compact and connected, it follows from II-11 that

$$\mathscr{A} \approx \eta\eta^*\mathscr{A} = \mathscr{H}^0(\eta; \eta^*\mathscr{A}).$$

Thus, *under the assumptions of* 9.1, we see that the first spectral sequence (28) and 6.3 imply that

$$\eta^* : H^n_\Phi(X/G; \mathscr{A}) \xrightarrow{\approx} H^n_\Psi(X_G; \eta^*\mathscr{A}) \quad \text{for} \quad 0 \le n < N. \tag{32}$$

If $N_1 < N_2$, universal bundles may be so chosen that there is a *canonical* G-equivariant map from the N_1-universal bundle to the N_2-universal bundle. This induces a map of spectral sequences that is an isomorphism in total degrees less than N_1. Thus, it is permissible to pass to a limit on N and think of E_G as being ∞-universal. With this in mind, the second spectral sequence (29) in the situation of 9.1 yields the following result:

9.2. Theorem. *With the notation above, suppose that $H^q(B_{G_x}; \mathscr{A}_{\breve{x}}) = 0$ for all $q > 0$ and all x with $\breve{x}$ in some neighborhood of $E(\Phi)$. Then there is a spectral sequence* (Borel) *with*

$$\boxed{E_2^{p,q} = H^p(B_G; \mathscr{H}^q_\Theta(X; f^*\mathscr{A})) \Longrightarrow H^{p+q}_\Phi(X/G; \mathscr{A}).}$$

This also holds for arbitrary compact groups G when X is locally compact Hausdorff and $\Phi = c = \Theta$. □

For $G = \mathbb{Z}_p$, p prime, B_G is an infinite lens space (real projective space for $p = 2$) and so

$$H^k(B_{\mathbb{Z}_p}; \mathbb{Z}) \approx \begin{cases} \mathbb{Z}_p, & \text{for } k > 0 \text{ even}, \\ 0, & \text{for } k > 0 \text{ odd}. \end{cases}$$

Also, for $G = \mathbb{S}^1$, B_G is an infinite complex projective space, so that

$$H^k(B_{\mathbb{S}^1}; \mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } k \geq 0 \text{ even}, \\ 0, & \text{for } k > 0 \text{ odd}. \end{cases}$$

For a *finite* group G, the transfer map μ for the covering map $\kappa : E_G \to B_G$ satisfies: $\mu\kappa^* : H^k(B_G; L) \to H^k(E_G; L) \to H^k(B_G; L)$ is multiplication by the order $|G|$ of G. Thus $|G|\gamma = 0$ for all $\gamma \in H^k(B_G; L)$, $k > 0$. In particular,

$$H^k(B_G; \mathbb{Q}) = 0 \text{ for } G \text{ finite, } k > 0. \tag{33}$$

9.3. Corollary. *Suppose that* $\dim_L X = n < \infty$ *and that the compact Lie group* G *acts on* X. *Assume that* $\Theta = f^{-1}\Phi$ *is paracompactifying on* X. *Suppose that* $F \subset X$ *is closed with* $G(F) = F$. *Assume either that* $G_x = \{e\}$ *for all* $x \in X - F$ *or that* G_x *is finite for all* $x \in X - F$ *and* $\mathscr{A}$ *is a sheaf of* $\mathbb{Q}$*-modules. Then the restriction*

$$H^k_\Psi(X_G; \eta^*\mathscr{A}) \to H^k_{\Psi|F}(F_G; \eta^*\mathscr{A})$$

is an isomorphism for $n < k < N$.

Proof. Since $\Psi = B_G \times \Theta$ locally in B_G, it follows that Ψ is paracompactifying. If $U = X - F$, then

$$H^k_{\Psi|U_G}(U_G; \eta^*\mathscr{A}) \approx H^k_{\Phi|(U/G)}(U/G; \mathscr{A}) \text{ for } k < N$$

by 9.2. This is zero for $k > n$ since $\dim_L U/G \leq \dim_L X = n$ by II-Exercise 54. Thus the result follows from the cohomology sequence of the pair (X_G, F_G). □

In most cases of interest F is the fixed-point set of G on X. In that case $F_G = F \times B_G$, so that the isomorphism of 9.3 gives a strong connection between the cohomology of X and that of F. The reader will find many applications of this in the references [6], [2], and [15]. We shall be content with the following application of 9.3 and (32).

9.4. Theorem. *Let* L *be a principal ideal domain. Let the circle group* G *act on a space* X *with fixed-point set* F. *Let* $\mathscr{A}$ *be a sheaf of* L*-modules on* X/G. *Assume that either* G *acts freely on* $X - F$,[25] *or* $L = \mathbb{Q}$. *Let* Φ *be a family of supports on* X/G *with* $f^{-1}\Phi$ *paracompactifying on* X *and with* $\dim_{f^{-1}\Phi, L} X < \infty$.

Then, if $H^p_{f^{-1}\Phi}(X; f^*\mathscr{A}) = 0$ *for all* $p \neq 0$, *we also must have that* $H^p_{\Phi|F}(F; \mathscr{A}|F) = 0$, *for* $p \neq 0$, *and that the restriction map* $\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi|F}(\mathscr{A}|F)$ *is an isomorphism.*

[25] This type of action on X is called "semi-free."

Proof. Let $U = X - F$. By (33) we have that $H^p(B_H; \mathbb{Q}) = 0$ for $p > 0$, where H is any *finite* group. It follows that $H^p(B_{G_x}; \mathscr{A}_{\breve{x}}) = 0$ when $p > 0$ and $\breve{x} \in U/G$. By 9.3 the restriction

$$H^n_\Psi(X_G; \eta^*\mathscr{A}) \to H^n_{\Psi|F_G}(F_G; \eta^*\mathscr{A}) \tag{34}$$

is an *isomorphism* for large n (and larger N). Now $F_G = F \times B_G$ and $\eta^*\mathscr{A}|F_G = \mathscr{A}|F \times B_G$ (where F is regarded as a subspace of X/G). Also, $\Psi|F_G = (\Phi|F) \times B_G$.

Consider the Leray spectral sequence $E_r^{p,q}$ of π. By (29) and (31) we have

$$E_2^{p,q} = H^p(B_G; \mathscr{H}^q_\Theta(X; f^*\mathscr{A})) \Longrightarrow H^{p+q}_\Psi(X_G; \eta^*\mathscr{A}).$$

Since by assumption $H^q_\Theta(X; f^*\mathscr{A}) = 0$ for $q > 0$, the spectral sequence of π degenerates and provides the isomorphism

$$H^n_\Psi(X_G; \eta^*\mathscr{A}) \approx H^n(B_G; \mathscr{H}^0_\Theta(X; f^*\mathscr{A})).$$

We also have that $H^0_\Theta(X; f^*\mathscr{A}) = \Gamma_\Theta(f^*\mathscr{A}) = \Gamma_\Phi(ff^*\mathscr{A}) = \Gamma_\Phi(\mathscr{A})$ by II-11, since f has compact, *connected* fibers and since $\Phi(cld) = f^{-1}\Phi = \Theta$. Now, G acts trivially on $\Gamma_\Phi(\mathscr{A})$, and it follows that the locally constant sheaf $\mathscr{H}^0_\Theta(X; f^*\mathscr{A})$ is actually *constant* with stalks $\Gamma_\Phi(\mathscr{A})$. By the universal coefficient formula,

$$H^n(B_G; \Gamma_\Phi(\mathscr{A})) = H^n(B_G; L) \otimes \Gamma_\Phi(\mathscr{A}).$$

There is a natural map of the spectral sequence of $\pi : X_G \to B_G$ into that of $F_G \to B_G$. The differentials of the spectral sequence of the latter map are trivial since $F_G = F \times B_G$. The restriction homomorphism (34) on the total spaces is an isomorphism for large degrees. It follows that the $E_2^{p,q}$ term of the spectral sequence of $F_G \to B_G$ must be zero for p large and $q > 0$; that is,

$$H^p(B_G; H^q_{\Phi|F}(F; \mathscr{A}|F)) = 0$$

for $q > 0$ and for p large.[26] The universal coefficient formula gives that $H^q_{\Phi|F}(F; \mathscr{A}|F) = 0$ for $q > 0$. Similarly, for $q = 0$, we see that we must have an isomorphism

$$H^n(B_G; \Gamma_\Phi(\mathscr{A})) \xrightarrow{\approx} H^n(B_G; \Gamma_{\Phi|F}(\mathscr{A}|F))$$

implying again that $\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi|F}(\mathscr{A}|F)$ is an isomorphism. □

The reader is invited to prove an analogous result for the cyclic group of prime order. The fact that the circle group is connected was used to show that $\Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(ff^*\mathscr{A}) = \Gamma_\Theta(f^*\mathscr{A})$ is an isomorphism, so that in the analogous case this will have to be assumed.

[26]Recall that $\dim_{L,\Phi|F} F \le \dim_{L,\Phi} X/G < \infty$.

It should be noted that 9.4 and its generalization to orientable sphere fibrations with singularities also follows easily from the "Smith-Gysin" sequence in Section 13. Moreover, by using the cup product structure (or the $H^*(B_G; L)$-module structure in the present treatment), one can show that the finite dimensionality assumption can be dropped when X is *locally compact Hausdorff and* $\Phi = c$ (see Exercise 21).

When G is finite, $H^p(B_G; M)$ is denoted by $H^p(G; M)$. (Here M is a G-module and can be regarded as a locally constant sheaf on B_G in a canonical manner: As a topological space it is $E_G \times_G M$, and its map to B_G is the induced projection to $E_G \times_G \{0\} = E_G/G = B_G$.) Thus 9.2 translates to:

9.5. Theorem. *Let $f : X \to Y$ be a finite regular*[27] *covering map with group G of deck transformations. Let Φ be a family of supports on Y and put $\Theta = f^{-1}\Phi$. Then for a sheaf $\mathscr{A}$ on Y, there is a spectral sequence* (Cartan) *with*

$$\boxed{E_2^{p,q} = H^p(G; H^q_\Theta(X; f^*\mathscr{A})) \Longrightarrow H^{p+q}_\Phi(Y; \mathscr{A}),}$$

where G operates on the cohomology of X in the canonical manner. □

As to applications, we content ourselves with the following well-known result:

9.6. Corollary. *If G is a finite group that can act freely on $\mathbb{S}^n$, then $H^*(G; \mathbb{Z})$ is periodic with period $n+1$. That is, $H^k(G; \mathbb{Z}) \approx H^{k+n+1}(G; \mathbb{Z})$ for all $k > 0$.*

Proof. Let $X = \mathbb{S}^n$ and $Y = X/G$. If any element of G reverses orientation on X, then it is easy to see that n is even and $G = \mathbb{Z}_2$. Thus we can assume that G preserves orientation. Now, $H^i(Y; \mathbb{Z}) = 0$ for $i > n$ since $\dim_{\mathbb{Z}} Y = n$. Also, the spectral sequence of 9.5 has $E_2^{p,q} = 0$ for $q \neq 0, n$. It follows that the differential $d_{n+1} : E_2^{k,n} \to E_2^{k+n+1,0}$ must be onto for $k = 0$ and an isomorphism for $k > 0$. Thus

$$H^k(G; \mathbb{Z}) \approx H^k(G; H^n(X; \mathbb{Z})) = E_2^{k,n} \approx E_2^{k+n+1,0} = H^{k+n+1}(G; \mathbb{Z})$$

for $k > 0$, as claimed. □

For much more on this topic see [24].

10 Characteristic classes

In this section we take Y to be any connected space and ξ to be an n-plane bundle over Y in either the real orthogonal, complex unitary, or

[27] Regularity of f means that G is simply transitive on the fibers.

quaternionic symplectic sense. Let $d = 1, 2$, or 4, respectively, in these three cases, so that nd is the real dimension of ξ. We shall denote by G the multiplicative group of scalars of norm one; that is, G is the real unit $(d-1)$-sphere.

We shall take the coefficient domain L to be $\mathbb{Z}_2$ in the real case and $\mathbb{Z}$ in the other two cases. Recall that $H^*(B_G; L) \approx L[t]$, where $\deg t = d$. We shall make the choice of the generator $t \in H^d(B_G; L)$ precise later.

As in 7.9, we let U and F be the total space and fiber of ξ, respectively, and put $U^\circ = U - Y$ and $F^\circ = F - \{0\}$. The group G operates on U in the canonical way, freely outside the zero section Y. With the notation of Section 9, U_G° is a subspace of U_G. Let $f : U_G \to B_G$ be the projection, as in Section 9, and consider the Leray spectral sequence $E_r^{p,q}$ of f mod $f|U_G^\circ$. By 7.5 the Leray sheaf $\mathscr{H}^q(f, f|U_G^\circ; L)$ is the locally constant sheaf $\mathscr{H}^q(U, U^\circ; L)$, and we have that

$$E_2^{p,q} = H^p(B_G; \mathscr{H}^q(U, U^\circ; L)) \Longrightarrow H^{p+q}(U_G, U_G^\circ; L).$$

By 7.9 we see that $E_r^{p,q} = 0$ for $q < nd$ and moreover, the locally constant sheaf $\mathscr{H}^{nd}(U, U^\circ; L)$ with stalks L must actually be *constant*, since $L = \mathbb{Z}_2$ in the real case and since B_G is simply connected in the other two cases. Thus we have the natural isomorphisms

$$H^{nd}(U_G, U_G^\circ; L) \approx H^0(B_G; \mathscr{H}^{nd}(U, U^\circ; L)) \approx H^{nd}(U, U^\circ; L) \approx L.$$

The right-hand side of this equation is generated by the Thom class[28] τ, and we shall denote the corresponding generator of $H^{nd}(U_G, U_G^\circ; L)$ by $\tau_G = \tau_G(\xi)$. Then τ_G is the unique class that restricts to the orientation class in $H^{nd}(F, F^\circ; L)$ under the restriction

$$H^{nd}(U_G, U_G^\circ; L) \to H^{nd}(U, U^\circ L) \to H^{nd}(F, F^\circ; L),$$

where $U \hookrightarrow U_G$ is the inclusion of the fiber in the fibration $U_G \to B_G$. Note that $F = \mathbb{R}^{nd}$ has a canonical orientation when $d = 2, 4$.

The inclusion $r : Y \times B_G = Y_G \hookrightarrow U_G$ induces the homomorphism

$$r^* : H^*(U_G, U_G^\circ; L) \to H^*(Y \times B_G; L) \approx H^*(Y; L) \otimes H^*(B_G; L),$$

where we have used 7.6 and the fact that $H^*(B_G; L)$ is torsion-free in the complex and quaternionic cases.

We define the characteristic classes $\chi_i = \chi_i(\xi) \in H^{id}(Y; L)$ by the equation

$$\boxed{r^*(\tau_G) = \textstyle\sum_{i=0}^n \chi_i \otimes t^{n-i}.} \tag{35}$$

We also put $\chi = \sum \chi_i \in H^*(Y; L)$.

10.1. Theorem. (Whitney duality.) *If ξ_1 and ξ_2 are vector bundles (over the same field) on Y, then $\chi(\xi_1 \oplus \xi_2) = \chi(\xi_1) \cdot \chi(\xi_2)$.*

[28] See 7.9.

Proof. Let ξ_i be an n_i-plane bundle and U_i its total space. Then $U_1 \times U_2$ is a bundle over $Y \times Y$ and $\xi_1 \oplus \xi_2$ is, by definition, the restriction of this to the diagonal. We shall denote the total space of $\xi_1 \oplus \xi_2$ by $U_1 \Delta U_2$. Consider the map $(U_1 \Delta U_2) \times E_G \to U_1 \times E_G \times U_2 \times E_G$ defined by $\langle x, y, z \rangle \mapsto \langle x, z, y, z \rangle$. This induces a map

$$k : (U_1 \Delta U_2)_G \to U_{1,G} \times U_{2,G},$$

and the diagram

$$\begin{array}{ccc} & F_1 \times F_2 & \\ \swarrow & & \searrow \\ U_1 \Delta U_2 & \longrightarrow & U_1 \times U_2 \\ \downarrow & & \downarrow \\ (U_1 \Delta U_2)_G & \xrightarrow{k} & U_{1,G} \times U_{2,G} \end{array}$$

commutes. It follows immediately that

$$\tau_G(\xi_1 \oplus \xi_2) = k^*(\tau_G(\xi_1) \times \tau_G(\xi_2)),$$

since both sides of this equation restrict to the orientation class of $F_1 \times F_2$.

The diagram

$$\begin{array}{ccc} Y \times B_G = Y_G & \xrightarrow{r} & (U_1 \Delta U_2)_G \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle k} \\ Y \times B_G \times Y \times B_G = Y_G \times Y_G & \xrightarrow{r_1 \times r_2} & U_{1,G} \times U_{2,G} \end{array}$$

also commutes. Consequently, we compute

$$\begin{aligned} r^*\tau_G(\xi_1 \oplus \xi_2) &= r^*k^*(\tau_G(\xi_1) \times \tau_G(\xi_2)) = \Delta^*(r_1^*\tau_G(\xi_1) \times r_2^*\tau_G(\xi_2)) \\ &= r_1^*\tau_G(\xi_1) \cup r_2^*\tau_G(\xi_2). \end{aligned}$$

This implies, by definition, that

$$\begin{aligned} \sum_k \chi_k(\xi_1 \oplus \xi_2) \otimes t^{n_1+n_2-k} &= \left(\sum_i \chi_i(\xi_1) \otimes t^{n_1-i}\right)\left(\sum_j \chi_j(\xi_2) \otimes t^{n_2-j}\right) \\ &= \sum_k \left(\sum_i \chi_i(\xi_1)\chi_{k-i}(\xi_2)\right) \otimes t^{n_1+n_2-k} \end{aligned}$$

(since $\deg t$ is even or $L = \mathbb{Z}_2$) and the theorem follows. □

If $f : X \to Y$ is a map and ξ is a vector bundle on Y, then it is clear that

$$\boxed{\chi_i(f^*\xi) = f^*(\chi_i(\xi)),} \tag{36}$$

where $f^*\xi$ is the induced bundle on X. That is, the χ_i are *natural.*

Now consider the exact cohomology sequence of the pair (U_G, U_G°) and note that by (32) we have $H^*(U_G^\circ; L) \approx H^*(U^\circ/G; L)$. Thus, using the isomorphism $H^p(U_G; L) \approx H^p(Y_G; L)$, we have the exact sequence

$$\cdots \to H^p(U_G, U_G^\circ; L) \xrightarrow{r^*} H^p(Y_G; L) \to H^p(U^\circ/G; L) \to \cdots \tag{37}$$

Let T be the unit $(nd-1)$-sphere bundle associated with ξ. Then T/G is a (fiber) deformation retract of U°/G and is the projective space bundle associated with ξ. The exact sequence (37) may be rewritten as

$$\cdots \to H^{p-1}(T/G) \to H^p(U_G, U_G^\circ) \xrightarrow{r^*} H^p(Y_G) \to H^p(T/G) \to \cdots \tag{38}$$

If Y is a point $\star$ and ξ is a line bundle, we have $T/G \approx Y = \star$ and $Y_G = B_G$, so that the following sequence, resulting from (38), shows that r^* is an isomorphism for $p = d$:

$$0 \to H^{d-1}(B_G) \to H^{d-1}(\star) \to H^d(U_G, U_G^\circ) \xrightarrow{r^*} H^d(B_G) \to 0.$$

Thus, since $r^*(\tau_G) = \chi_0 \otimes t \in H^0(\star) \otimes H^d(B_G)$, χ_0 is a generator of $H^0(\star)$. Since we have not yet made any definite choice of t as a generator of $H^d(B_G)$, we are free to choose t such that $\chi_0 = 1 \in H^0(\star)$. Then, by Whitney duality 10.1 and (36), it follows that

$$\boxed{\chi_0(\xi) = 1 \in H^0(Y; L)}$$

for any ξ; for if $f : \star \to Y$, then $f^*(\chi_0(\xi)) = \chi_0(f^*\xi) = 1$.

Note that the diagram

$$\begin{array}{ccc} (Y, \varnothing) & \xrightarrow{i} & (U, U^\circ) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ (Y_G, \varnothing) & \xrightarrow{r} & (U_G, U_G^\circ) \end{array}$$

commutes, so that for an n-plane bundle ξ, we have $\chi_n(\xi) = f^*(\sum \chi_i \otimes t^{n-i}) = f^*r^*(\tau_G) = i^*g^*(\tau_G) = i^*(\tau) = \omega(\xi)$, where $\omega(\xi)$ is the Euler class of ξ; see Section 7. Thus

$$\boxed{\chi_n(\xi) = \omega(\xi) \quad \text{for any } n\text{-plane bundle } \xi \text{ on } Y.} \tag{39}$$

From the properties we have proved for the classes $\chi_i(\xi)$ it can be shown that up to a possible sign of $(-1)^i$, $\chi_i(\xi)$ is the ith Stiefel-Whitney, Chern, or symplectic Pontryagin class w_i, c_i, or e_i for real, complex, or quaternionic bundles, respectively. [Note that it follows from (39) that $w_n(\xi)$ is the reduction mod two of the *integral* Euler class of the n-plane bundle ξ.]

Now let ξ be a complex n-plane bundle over Y, and let $\xi_{\mathbb{R}}$ denote the underlying real $2n$-plane bundle. Let $\mathbb{R}^*$ and $\mathbb{C}^*$ be the sets of real and

complex numbers of norm one, respectively. Since $\mathbb{R}^* \subset \mathbb{C}^*$ we have a natural diagram (taking $E_{\mathbb{R}^*} = E_{\mathbb{C}^*}$)

$$\begin{array}{ccc} & & U_{\mathbb{R}^*} \\ & \nearrow & \downarrow \\ U & & \\ & \searrow & \\ & & U_{\mathbb{C}^*} \end{array}$$

and it follows that $\rho(\tau_{\mathbb{C}^*}) = \tau_{\mathbb{R}^*}$, where ρ is induced by $U_{\mathbb{R}^*} \to U_{\mathbb{C}^*}$ and by reduction of coefficients modulo two: $\mathbb{Z} \to \mathbb{Z}_2$. The diagram

$$\begin{array}{ccc} H^*(U_{\mathbb{C}^*}, U^\circ_{\mathbb{C}^*}; \mathbb{Z}) & \xrightarrow{r^*} & H^*(Y;\mathbb{Z}) \otimes H^*(B_{\mathbb{C}^*};\mathbb{Z}) \\ \downarrow \rho & & \downarrow \rho \\ H^*(U_{\mathbb{R}^*}, U^\circ_{\mathbb{R}^*}; \mathbb{Z}_2) & \xrightarrow{r^*} & H^*(Y;\mathbb{Z}_2) \otimes H^*(B_{\mathbb{R}^*};\mathbb{Z}_2) \end{array}$$

also commutes, where the map $H^*(B_{\mathbb{C}^*};\mathbb{Z}) \to H^*(B_{\mathbb{R}^*};\mathbb{Z}_2)$ is induced by the canonical map $B_{\mathbb{R}^*} = E_{\mathbb{C}^*}/\mathbb{R}^* \to E_{\mathbb{C}^*}/\mathbb{C}^* = B_{\mathbb{C}^*}$ and by reduction mod two. This map is a fiber bundle projection with circle fiber, and it is easily seen that the generator $t_{\mathbb{C}} \in H^2(B_{\mathbb{C}^*};\mathbb{Z})$ must be taken into the nonzero element $t^2_{\mathbb{R}} \in H^2(B_{\mathbb{R}^*};\mathbb{Z}_2)$. Therefore

$$\boxed{w_{2i+1}(\xi_{\mathbb{R}}) = 0 \quad \text{and} \quad w_{2i}(\xi_{\mathbb{R}}) = \rho(c_i(\xi)).}$$

Similarly, if ξ is a quaternionic n-plane bundle with underlying complex bundle $\xi_{\mathbb{C}}$, then

$$\boxed{c_{2i+1}(\xi_{\mathbb{C}}) = 0 \quad \text{and} \quad c_{2i}(\xi_{\mathbb{C}}) = e_i(\xi).}$$

We shall now return to our general discussion. Let $\pi_G : U_G \to Y_G$ denote the map induced in the canonical way by the projection $\pi : U \to Y$, and note that $\pi_G r : Y_G \to Y_G$ is the identity. Let $h : Y_G = Y \times B_G \to Y$ be the projection.

Let Φ be any family of supports on Y and put $\Psi = (h\pi_G)^{-1}(\Phi)$ on U_G. Let $\mathscr{A}$ be any sheaf on Y and put $\mathscr{B} = \pi_G^* h^* \mathscr{A}$ on U_G. Note that for $\alpha \in H^*_\Phi(Y;\mathscr{A})$, we have

$$h^*(\alpha) = \alpha \otimes 1 \in H^*_\Phi(Y;\mathscr{A}) \otimes H^*(B_G) = H^*_\Psi(Y_G;\mathscr{B}),$$

using 7.6.

Let ζ be the composition

$$\zeta : H^p_\Phi(Y;\mathscr{A}) \xrightarrow{\pi_G^* h^*} H^p_\Psi(U_G;\mathscr{B}) \xrightarrow{\tau_G \cup (\bullet)} H^{p+nd}_\Psi(U_G, U^\circ_G;\mathscr{B}).$$

Then $r^*(\zeta(\alpha)) = r^*(\tau_G \cup \pi_G^* h^*(\alpha)) = r^*(\tau_G) \cup (\alpha \otimes 1)$, so that the map

$$r^*(\zeta(\bullet)) : H^*_\Phi(Y;\mathscr{A}) \to H^*_\Psi(Y_G;\mathscr{B}) = H^*_\Phi(Y;\mathscr{A}) \otimes H^*(B_G)$$

is given by

$$\boxed{r^*(\zeta(\alpha)) = \sum_{i=0}^{n} (\chi_i \cup \alpha) \otimes t^{n-i},}$$

which generalizes (35). Since $\chi_0 = 1$, the map $r^*(\zeta(\bullet))$ is a *monomorphism*.

10.2. The definition of characteristic classes that we have given is useful for studying the normal classes of the fixed-point set of a differentiable action of the group G (as above) on a differentiable manifold M. We shall illustrate this in the case of the circle group $G = \mathbb{C}^*$. Let M be $(2n+k)$-dimensional (real), and let Y be a k-dimensional component of the fixed-point set of some differentiable action of G on M. Assume, for simplicity, that G acts freely on the unit sphere in any normal disk to Y. Then the normal bundle ξ has a canonical complex structure in which the operation by $\mathbb{C}^*$ coincides with the given action of G. Thus, ξ is a complex n-plane bundle. Let $c_i \in H^{2i}(Y;\mathbb{Z})$ be the ith Chern class of ξ. Let U be an open tubular neighborhood of Y in M.

Let Φ be a family of supports on M/G with inverse image Θ on M. Let $\Psi = \eta^{-1}(\Phi)$, where $\eta : M_G \to M/G$ is the projection; see Section 9. We shall *assume* that $\Theta \cap U = \pi^{-1}(\Theta|Y)$, where $\pi : U \to Y$ is the canonical projection of the tubular neighborhood U onto Y. (It follows that $\Psi \cap U_G$ is the same type of support family as that considered in the preceding general discussion.) For example, Φ, and hence Θ and Ψ, could be taken to consist of all compact sets or of all closed sets. By excision, we have the natural isomorphism

$$H^*_{\Psi\cap U_G}(U_G, U^\circ_G) \approx H^*_\Psi(M_G, M^\circ_G),$$

where $M^\circ = M - Y$. Thus r^* factors as

$$\begin{array}{ccc} H^*_{\Psi\cap U_G}(U_G, U^\circ_G) & \xrightarrow{\quad r^* \quad} & H^*_{\Psi\cap Y_G}(Y_G) = H^*_\Theta(Y) \otimes H^*(B_G) \\ & {\scriptstyle j^*}\searrow \qquad \nearrow{\scriptstyle i^*} & \\ & H^*_\Psi(M_G). & \end{array}$$

10.3. Theorem. *In the situation of* 10.2, *suppose that* $H^i_\Theta(M;\mathbb{Z}) = 0$ *for all* $i \equiv p \pmod 2$ *with* $p + 2(n-j) < i \le p + 2n$ *and let* $\alpha \in H^p_\Theta(Y)$. *Then* $c_i \cup \alpha = 0$ *for* $i > n - j$, *where* c_i *is the ith normal Chern class of* Y *in* M.

Proof. Consider $j^*(\zeta(\alpha)) \in H^{2n+p}_\Psi(M_G)$. In the spectral sequence $E^{s,t}_r$ of the map $M_G \to B_G$ with supports in Ψ we have that $E^{s,t}_r = 0$ for all $t \equiv p \pmod 2$ with $p + 2(n-j) < t \le p + 2n$. Thus, the complementary degree of $j^*(\zeta(\alpha))$ is at most $p + 2(n-j)$. Since i^* preserves filtration with respect to the spectral sequences of $M_G \to B_G$ and of $Y_G \to B_G$, it follows that $r^*(\zeta(\alpha)) = \sum (c_i \cup \alpha) \otimes t^{n-i}$ has complementary degree at most $p + 2(n-j)$. Thus we must have that $c_i \cup \alpha = 0$ for $p + 2i > p + 2(n-j)$ as claimed. □

Taking $\alpha = 1$, we obtain:

10.4. Corollary. *If* $H^i(M) = 0$ *for* $2(n-j) < i \le 2n$, *then* $c_i = 0$ *for* $i > n - j$. □

In the real case (Stiefel-Whitney classes of normal bundles to fixed-point sets of involutions) these results are due to Conner and Floyd [28].

We shall illustrate the real case by proving a result of Conner [25]:

10.5. Theorem. *Let F be a component of the fixed-point set of a differentiable involution on an m-manifold M, and let Θ be any family of supports on M that is invariant under the involution. Suppose that the restriction $H^i_\Theta(M;\mathbb{Z}_2) \to H^i_\Theta(F;\mathbb{Z}_2)$ is a monomorphism for all $i < r$. Then every component $Y \neq F$ of the fixed-point set having some neighborhood in Θ has dimension at most $m - r$.*

Proof. Let Y, as above, have dimension p. Let $n = m - p$ and let ξ be the normal bundle to Y in M. Let U be an open tubular neighborhood of Y in M with closure in Θ. With the notation of 10.2 consider the class $\tau_G \in H^n(U_G, U_G^\circ) = H^n_{\Psi\cap U_G}(U_G, U_G^\circ)$.

Let i^* and j^* be as in 10.2 and let $k^* : H^n_\Psi(M_G) \to H^n_\Psi(F_G)$ be induced by the inclusion $k : F_G \hookrightarrow M_G$. Now, $k^*j^* = 0$, but $j^*(\tau_G) \neq 0$ since $i^*j^*(\tau_G) = r^*(\tau_G) = \sum w_i(\xi) \otimes t^{n-i} \neq 0$. Thus $k^* : H^n_\Psi(M_G) \to H^n_\Psi(F_G)$ is not a monomorphism. However, since the spectral sequence of $F_G \to B_G$ is degenerate, it follows from our hypotheses that

$$k^* : H^i_\Psi(M_G) \to H^i_\Psi(F_G)$$

is a monomorphism for $i < r$. Thus $n \ge r$, whence $p = m - n \le m - r$. □

11 The spectral sequence of a filtered differential sheaf

Let $\mathscr{L}^*$ be a differential sheaf on the space X, and assume that we are given a filtration $\{F_p\mathscr{L}^*\}$ of $\mathscr{L}^*$, that is, for each q we have a sequence

$$\cdots \supset F_p\mathscr{L}^q \supset F_{p+1}\mathscr{L}^q \supset \cdots$$

($p \in \mathbb{Z}$) of submodules of $\mathscr{L}^q$ with $\mathscr{L}^q = \bigcup_p F_p\mathscr{L}^q$, and such that the differential δ of $\mathscr{L}^*$ maps $F_p\mathscr{L}^q$ into $F_p\mathscr{L}^{q+1}$.

We denote the associated graded sheaf by $\{G_p\mathscr{L}^*\}$, where

$$\boxed{G_p\mathscr{L}^* = F_p\mathscr{L}^*/F_{p+1}\mathscr{L}^*.}$$

The differential δ induces a differential on each $G_p\mathscr{L}^*$, and the associated derived sheaf is denoted by

$$\mathscr{H}^q(G_p\mathscr{L}^*),$$

where q refers to the degree in $\mathscr{L}^*$. As with filtered differential groups, there is associated with the filtered differential sheaf $F\mathscr{L}^*$ a spectral sequence $\mathscr{E}_r^{p,q}$ of sheaves. (This is of secondary interest to us and is introduced for notational convenience.) Thus, we let

$$\boxed{\mathscr{Z}_r^p = \{x \in F_p\mathscr{L}^* \mid \delta x \in F_{p+r}\mathscr{L}^*\}}$$

and

$$\boxed{\mathscr{E}_r^p = \mathscr{Z}_r^p/(\delta\mathscr{Z}_{r-1}^{p-r+1} + \mathscr{Z}_{r-1}^{p+1}).}$$

We also let $\mathscr{Z}_r^{p,q}$ and $\mathscr{E}_r^{p,q}$ be the terms in $\mathscr{Z}_r^p$ and $\mathscr{E}_r^p$ respectively of total degree $p+q$. Then, as with spectral sequences of filtered differential groups, δ induces a differential δ_r on $\mathscr{E}_r^{p,q}$ of degree $(r, -r+1)$, and the resulting derived sheaf is $\mathscr{E}_{r+1}^{p,q}$. We have $\mathscr{E}_0^p = G_p\mathscr{L}^*$ and $\mathscr{E}_1^{p,q} = \mathscr{H}^{p+q}(G_p\mathscr{L}^*)$. In the application we will make of this in the next section, the differentials δ_r will vanish for $r \geq 1$. Note that the stalks $(\mathscr{E}_r^{p,q})_x$, for any $x \in X$, form the spectral sequence of the filtered differential group $F_p(\mathscr{L}_x^*) = (F_p\mathscr{L}^*)_x$.

We now turn to the spectral sequence of primary interest. Consider the double complex $L^{p,q} = \bigcup_t C_\Phi^p(X; F_t\mathscr{L}^q)$ with differentials d' and d'' [with d'' induced by $(-1)^p\delta$] and the associated total complex L^* with differential $d = d' + d''$, as in Section 1.

We shall define a new filtration of L^* and study the resulting spectral sequence. Let

$$\boxed{F_pL^* = \bigoplus_{s+t=p} C_\Phi^s(X; F_t\mathscr{L}^*),}$$

where we consider $C_\Phi^s(X; F_{t'}\mathscr{L}^*)$ to be a subgroup of $C_\Phi^s(X; F_t\mathscr{L}^*)$ for $t' > t$ and hence have that $F_pL^* \supset F_{p+1}L^*$. Both differentials d' and d'' preserve this filtration, and d', in fact, increases the filtration degree by one.

Let $\{E_r^{p,q}\}$ be the spectral sequence of this filtration of L^*. Thus

$$\boxed{Z_r^p = \{a \in F_pL^* \mid da = (d' + d'')(a) \in F_{p+r}L^*\}}$$

and

$$\boxed{E_r^p = Z_r^p/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})}$$

and as usual, we replace the upper index p by p, q when we refer to homogeneous terms of total degree $p+q$.

We have

$$E_0^{p,q} = G_pL^{p+q} = \bigoplus_t C_\Phi^{p-t}(X; G_t\mathscr{L}^{q+t}).$$

Since d' increases filtration degree, we see that d_0 is induced by d'', that is, by δ_0. We shall continue to use the notation d'' for d_0.

Consequently,

$$\boxed{E_1^{p,q} = \bigoplus_t C_\Phi^{p-t}(X; \mathscr{E}_1^{t,q}) = \bigoplus_t C_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*)).}$$

Clearly, any element $a \in C_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*))$ is represented by an element $\alpha \in C_\Phi^{p-t}(X; F_t\mathscr{L}^{q+t})$ with $d''\alpha \in C_\Phi^{p-t}(X; F_{t+1}\mathscr{L}^{q+t+1})$. Now $d'\alpha$ represents the element $d'a \in C_\Phi^{p+1-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*)) \subset E_1^{p+1,q}$, while $d''\alpha$ represents $(-1)^{p-t}\delta_1^* a \in C_\Phi^{p+1-(t+1)}(X; \mathscr{H}^{q+(t+1)}(G_{t+1}\mathscr{L}^*)) \subset E_1^{p+1,q}$. Thus $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is given by $d_1 = d' + \varepsilon\delta_1^*$, where ε denotes the sign $(-1)^{p-t}$ above.

We now make the assumption

(A) $\delta_1 : \mathscr{E}_1^{t,q} \to \mathscr{E}_1^{t+1,q}$ *is zero for all* t, q.

Under assumption (A), we have

$$\boxed{E_2^{p,q} = \bigoplus_t H_\Phi^{p-t}(X; \mathscr{E}_2^{t,q}) = \bigoplus_t H_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*))}$$

since $\mathscr{E}_2^{t,q} = \mathscr{E}_1^{t,q} = \mathscr{H}^{q+t}(G_t\mathscr{L}^*)$.

We shall now compute d_2 under assumption (A). Let

$$a \in H_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*)).$$

The class a is represented by a d'-cocycle in $C_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*))$ and hence by an element $b \in C_\Phi^{p-t}(X; G_t\mathscr{L}^{q+t})$ with $d''b = 0$,[29] and with $d'b \in C_\Phi^{p-t+1}(X; \operatorname{Im}\delta_0) = \operatorname{Im} d''$; that is, there is an element

$$c \in C_\Phi^{p-t+1}(X; G_t\mathscr{L}^{q+t-1})$$

with $d'b = -d''c$. Let $\beta \in C_\Phi^{p-t}(X; F_t\mathscr{L}^{q+t})$ and $\gamma \in C_\Phi^{p-t+1}(X; F_t\mathscr{L}^{q+t-1})$ represent b and c respectively. We have that

$$d'\beta + d''\gamma \in C_\Phi^{p-t+1}(X; F_{t+1}\mathscr{L}^{q+t}) \subset F_{p+2}L^*.$$

Note that $d''\beta \in C_\Phi^{p-t}(X; F_{t+1}\mathscr{L}^{q+t+1})$ represents $\pm\delta_1^* b = 0$ since $\delta_1 = 0$. Thus $d''\beta \in C_\Phi^{p-t}(X; F_{t+2}\mathscr{L}^{q+t+1}) \subset F_{p+2}L^*$. Also,

$$d'\gamma \in C_\Phi^{p-t+2}(X; F_t\mathscr{L}^{q+t-1}) \subset F_{p+2}L^*.$$

Hence $d(\beta+\gamma) = (d'+d'')(\beta+\gamma) = d'\gamma + (d'\beta + d''\gamma) + d''\beta \in F_{p+2}L^*$, and therefore $\beta + \gamma \in Z_2^{p,q}$ represents $a \in E_2^{p,q}$ (since it represents b in $E_0^{p,q}$). Consequently, $d(\beta+\gamma) = d'\gamma + (d'\beta + d''\gamma) + d''\beta$ represents $d_2 a \in E_2^{p+2,q-1}$.

We now make the further assumption that

(B) $\delta_2 : \mathscr{E}_2^{t,q} \to \mathscr{E}_2^{t+2,q-1}$ *is zero for all* t, q.

In case (A) and (B) both hold, $d''\beta$ must be in $C_\Phi^{p-t}(X; F_{t+3}\mathscr{L}^{q+t+1}) \subset F_{p+3}L^*$, for otherwise $\delta_2^* b$ would be nonzero. Thus

$$d(d''\beta) = d'd''\beta \in d'(F_{p+3}L^*) \subset F_{p+4}L^*,$$

[29] Recall that C_Φ^* is exact, so that $\operatorname{Ker} d'' = C_\Phi^*(X; \operatorname{Ker}\delta_0)$.

so that $d''\beta \in Z_2^{p+2}$. Also, $dd'\gamma = d''d'\gamma = -d'd''\gamma$ represents $d'(-d''c) = d'd'b = 0$ in $C_\Phi^{p-t+2}(X; G_t\mathscr{L}^{q+t})$, so that $d''(d'\gamma) \in C_\Phi^{p-t+2}(X; F_{t+1}\mathscr{L}^{q+t})$. The latter element represents $\pm\delta_1^*(d'c) = 0$ [by (A)] so that in fact,

$$dd'\gamma \in C_\Phi^{p-t+2}(X; F_{t+2}\mathscr{L}^{q+2}) \subset F_{p+4}L^*.$$

Thus $d'\gamma \in Z_2^{p+2}$. It follows that $d'\beta + d''\gamma = d(\beta+\gamma) - d'\gamma - d''\beta \in Z_2^{p+2}$ and that all three elements $d'\gamma$, $(d'\beta + d''\gamma)$, and $d''\beta$ represent classes in $E_2^{p+2,q-1}$ with sum d_2a. Clearly, $d''\beta$ represents $0 = \pm\delta_2^*a \in H_\Phi^{p-t}(X; \mathscr{E}_2^{t+2,q-1})$, by (B).

Now $d'\gamma$ represents a class $d_2^0a \in H_\Phi^{p-t+2}(X; \mathscr{H}^{q+t-1}(G_t\mathscr{L}^*))$, while $(d'\beta + d''\gamma)$ represents a class $d_2^1a \in H_\Phi^{p-t+1}(X; \mathscr{H}^{q+t}(G_{t+1}\mathscr{L}^*))$, where the superscript indicates the change of the degree "t." Note that both of these groups are direct summands of $E_2^{p+2,q-1}$ and that

$$\boxed{d_2 = d_2^0 + d_2^1}$$

under conditions (A) and (B). We shall now give an interpretation of the operators d_2^0 and d_1^1.

For t_0 fixed, the differential sheaf $G_{t_0}(\mathscr{L}^*)$ gives rise to a spectral sequence (1) on page 199 (which coincides with the present spectral sequence when $\mathscr{L}^*$ has the trivial filtration $F_{t_0}\mathscr{L}^* = \mathscr{L}^*$, $F_{t_0+1}\mathscr{L}^* = 0$). Let us denote this spectral sequence by ${}^{t_0}E_r^{p,q}$ and its differentials by ${}^{t_0}d_r$. Then

$${}^{t_0}E_2^{p-t_0,q+t_0} = H_\Phi^{p-t_0}(X; \mathscr{H}^{q+t_0}(G_{t_0}\mathscr{L}^*)).$$

By the construction of d_2^0 it is clear that

$$\boxed{d_2^0 = \bigoplus_t {}^td_2.}$$

(This can be seen by direct computation or by a more abstract approach. The details are left to the reader.)

Recall that δ_1 is the connecting homomorphism of the homology sequence of the short exact sequence

$$0 \to \frac{F_{t+1}\mathscr{L}^*}{F_{t+2}\mathscr{L}^*} \to \frac{F_t\mathscr{L}^*}{F_{t+2}\mathscr{L}^*} \to \frac{F_t\mathscr{L}^*}{F_{t+1}\mathscr{L}^*} \to 0,$$

and by (A), the resulting sequence

$$0 \to \mathscr{H}^{q+t}(G_{t+1}\mathscr{L}^*) \to \mathscr{H}^{q+t}(F_t\mathscr{L}^*/F_{t+2}\mathscr{L}^*) \to \mathscr{H}^{q+t}(G_t\mathscr{L}^*) \to 0 \quad (40)$$

is exact. This exact "coefficient sequence" gives rise to a connecting homomorphism

$${}^t\Delta : H_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*)) \to H_\Phi^{p-t+1}(X; \mathscr{H}^{q+t}(G_{t+1}\mathscr{L}^*)). \quad (41)$$

We claim that Δ coincides with d_2^1. This is merely a matter of tracing through the definition of Δ. With notation as above, β determines a class

$\beta' \in C_\Phi^{p-t}(X; F_t\mathscr{L}^{q+t}/F_{t+2}\mathscr{L}^{q+t})$ with $d''\beta' = 0$, since $d''\beta \in F_{p+2}L^*$. Thus β' determines a class $b' \in C_\Phi^{p-t}(X; \mathscr{H}^{q+t}(F_t\mathscr{L}^*/F_{t+2}\mathscr{L}^*))$. Now $d'b'$ represents $d'b = 0$ in $C_\Phi^{p-t+1}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*))$, and hence $d'b'$ may be regarded, by (40), to be in $C_\Phi^{p-t+1}(X; \mathscr{H}^{q+t}(G_{t+1}\mathscr{L}^*))$ and, by definition, represents ${}^t\Delta a$. Specifically, $d'\beta + d''\gamma \in C_\Phi^{p-t+1}(X; F_{t+1}\mathscr{L}^{q+t})$ represents an element of $C_\Phi^{p-t+1}(X; \mathscr{H}^{q+t}(G_{t+1}\mathscr{L}^*))$ that maps into $d'b'$. This gives

$$\boxed{d_2^1 a = {}^t\Delta a.}$$

We shall now consider the question of convergence. Recall that the filtration of L^* is called *regular* if $F_pL^n = 0$ for $p > f(n)$ for some function $f : \mathbb{Z} \to \mathbb{Z}$.

We shall now make two further assumptions:

(C) *The filtration of $\mathscr{L}^*$ is regular* [*that is, $F_p\mathscr{L}^n = 0$ for $p > g(n)$ for some function g*].

(D) *Either*:

(i) *$\mathscr{L}^*$ is bounded below (that is, $\mathscr{L}^n = 0$ for $n < n_0$), or*

(ii) $\dim_\Phi X < \infty$.

Of course, in case D(ii) we replace C_Φ^* by a canonical resolution of *finite* length.

We claim that under these assumptions the filtration of L^* is also regular.

Indeed, in case D(i) assume, for convenience, that $\mathscr{L}^q = 0$ for $q < 0$. We may also assume, in this case, that g is an increasing function. Note that

$$F_pL^n = \bigoplus_{s=0}^{n} C_\Phi^s(X; F_{p-s}\mathscr{L}^{n-s}).$$

Thus, if $p > n + g(n)$, we have $p - s \ge p - n > g(n) \ge g(n-s)$ and $F_pL^n = 0$.

In case D(ii), where $m = \dim_\Phi X$, we have

$$F_pL^n = \bigoplus_{s=0}^{m} C_\Phi^s(X; F_{p-s}\mathscr{L}^{n-s}),$$

which is zero for $p > m + \sup\{g(n-s) \mid 0 \le s \le m\}$.

Note that under assumption (D) we can show, as in Section 1, that

$$H^n(L^*) = H^n(\bigcup_s \Gamma_\Phi(F_s\mathscr{L}^*))$$

when $F\mathscr{L}^*$ consists of Φ-acyclic sheaves. More generally, the assumption

$$\varprojlim{}_t H^q(H_\Phi^p(X; F_t\mathscr{L}^*)) = 0 \quad \text{for} \quad p > 0$$

implies this result.

Thus, for example, we have proved the following:

11.1. Theorem. *Let $\mathscr{L}^*$ be a differential sheaf on X. Let $\{F_p\mathscr{L}^*\}$ be a filtration of $\mathscr{L}^*$ consisting of Φ-acyclic sheaves. Also assume that* (A), (C), *and* (D) *hold. Then there is a spectral sequence with*

$$\boxed{E_2^{p,q} = \bigoplus_t H_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*)) \Longrightarrow H^{p+q}(\bigcup_s \Gamma_\Phi(F_s\mathscr{L}^*)).}$$

If, moreover, (B) *holds, then on the summand $H_\Phi^{p-t}(X; \mathscr{H}^{q+t}(G_t\mathscr{L}^*))$ we have that*

$$\boxed{d_2 = {}^t d_2 + {}^t\Delta,}$$

where ${}^t d_2$ is the second differential of the spectral sequence (1) *on page* 199 *of the differential sheaf $G_t\mathscr{L}^*$, and ${}^t\Delta$ is the connecting homomorphism* (41) *of the coefficient sequence* (40). □

12 The Fary spectral sequence

We shall now apply the results of the previous section to a more specific situation.

Let $f : X \to Y$ be a map and let

$$Y = K_0 \supset K_1 \supset K_2 \supset \cdots$$

be a decreasing filtration of Y by *closed* sets. Let

$$U_p = Y - K_{1-p}.$$

Then

$$\cdots \supset U_{-1} \supset U_0 \supset U_1 = \varnothing,$$

so that $\{U_p\}$ forms a decreasing filtration of Y by open sets. Also, put

$$A_t = K_t - K_{t+1} = U_{-t} - U_{-t+1}.$$

Let $\mathscr{A}$ be a sheaf on X, and let Ψ be a family of supports on X. Consider the differential sheaf

$$\mathscr{L}^* = f_\Psi \mathscr{A}^*$$

on Y, where $\mathscr{A}^* = \mathscr{C}^*(X; \mathscr{A})$. Then $\mathscr{L}^*$ is filtered by the subsheaves

$$F_p\mathscr{L}^* = \mathscr{L}^*_{U_p},$$

and this filtration is bounded above by zero. The associated graded sheaf is

$$G_{-t}\mathscr{L}^* = \mathscr{L}^*_{A_t},$$

and clearly,

$$\mathscr{H}^q(G_{-t}\mathscr{L}^*) = \mathscr{H}^q(\mathscr{L}^*)_{A_t} = \mathscr{H}^q_\Psi(f; \mathscr{A})_{A_t}$$

since $\mathscr{B} \mapsto \mathscr{B}_A$ is an exact functor for $A \subset Y$ locally closed. Stalkwise, $\{F_p\mathscr{L}^*\}$ reduces to a filtration with only two terms, and it follows that $\delta_r = 0$ for $r \geq 1$, where δ_r is as in Section 11. Thus conditions (A) and (B) of Section 11 are satisfied. Condition (C) follows from the fact that $\{U_p\}$ is bounded above, and (D) is obviously satisfied. Note that the exact sequence (40) of Section 11 becomes

$$0 \to \mathscr{H}_\Psi^{q-t}(f;\mathscr{A})_{A_{t-1}} \to \mathscr{H}_\Psi^{q-t}(f;\mathscr{A})_{A_t \cup A_{t-1}} \to \mathscr{H}_\Psi^{q-t}(f;\mathscr{A})_{A_t} \to 0. \quad (42)$$

If Φ is *paracompactifying* on Y, then by II-10.2 we have

$$H_\Phi^*(Y;\mathscr{H}_\Psi^*(f;\mathscr{A})_{A_t}) \approx H_{\Phi|A_t}^*(A_t;\mathscr{H}_\Psi^*(f;\mathscr{A})|A_t).$$

Since $F_p\mathscr{L}^*$ is Φ-soft and $\bigcup_t \Gamma_\Phi(\mathscr{L}_{U_t}^*) = \Gamma_\Theta(\mathscr{L}^*) = \Gamma_{\Theta(\Psi)}(\mathscr{A}^*)$, where $\Theta = \bigcup_t(\Phi|U_t)$, the spectral sequence of 11.1, together with 2.3, yields:

12.1. Theorem. *Let $f : X \to Y$ be a map, let $Y = K_0 \supset K_1 \supset \cdots$ be a decreasing filtration of Y by closed subsets, and put $A_t = K_t - K_{t+1}$. Let Φ be a paracompactifying family of supports on Y, and let $\mathscr{A}$ be any sheaf on X. Then there is a spectral sequence* (Fary) *with*

$$\boxed{E_2^{p,q} = \bigoplus_t H_{\Phi|A_t}^{p+t}(A_t;\mathscr{H}_\Psi^{q-t}(f;\mathscr{A})|A_t) \Longrightarrow H_{\Theta(\Psi)}^{p+q}(X;\mathscr{A}),}$$

where $\Theta = \bigcup_t(\Phi|Y - K_t)$. Moreover,

$$\boxed{d_2 = \bigoplus_t ({}^td_2 + {}^t\Delta),}$$

where ${}^td_2$ is the second differential of the spectral sequence 2.1 *of the differential sheaf $(f_\Psi\mathscr{C}^*(X;\mathscr{A}))|A_t$ and ${}^t\Delta$ is the connecting homomorphism of the cohomology sequence associated with the coefficient sequence* (42) [*that is, the cohomology sequence of the pair $(K_{t-1} - K_{t+1}, A_t)$ with coefficients in $\mathscr{H}_\Psi^{q-t}(f;\mathscr{A})$*]. □

We note that with suitable restrictions, $\mathscr{H}_\Psi^{q-t}(f;\mathscr{A})|A_t$ is the Leray sheaf $\mathscr{H}_{\Psi\cap A_t^\bullet}^{q-t}(f|A_t^\bullet;\mathscr{A})$ of $f|A_t^\bullet$, where $A_t^\bullet = f^{-1}(A_t)$. We shall now show that under the same restrictions, the spectral sequence of the differential sheaf $(f_\Psi\mathscr{C}^*(X;\mathscr{A}))|A_t$ can be identified with the Leray spectral sequence of $f|A_t^\bullet$.

12.2. Proposition. *Let $f : X \to Y$ be Ψ-closed, where each $f^{-1}(y)$, $y \in Y$, is Ψ-taut in X. Let Φ be a paracompactifying family of supports on Y and let $A \subset Y$ be locally closed with $A^\bullet = f^{-1}(A)$. Then for any sheaf $\mathscr{A}$ on X, the spectral sequence* 2.1 *of the differential sheaf $(f_\Psi\mathscr{C}^*(X;\mathscr{A}))|A$ on A, with supports in $\Phi|A$, may be identified with the Leray spectral sequence*

$$\boxed{E_2^{p,q} = H_{\Phi|A}^p(A;\mathscr{H}_{\Psi\cap A^\bullet}^q(f|A^\bullet;\mathscr{A}|A^\bullet)) \Longrightarrow H_{\Phi(\Psi)|A^\bullet}^{p+q}(A^\bullet;\mathscr{A}|A^\bullet)}$$

of $f|A^\bullet$.

Proof. Note that $\Phi(\Psi)|A^\bullet = (\Phi|A)(\Psi \cap A^\bullet)$ by 5.4(8). We have the canonical homomorphism

$$(f_\Psi \mathscr{C}^*(X;\mathscr{A}))|A \to f_{\Psi\cap A^\bullet}(\mathscr{C}^*(A^\bullet;\mathscr{A}|A^\bullet)),$$

which induces (20) on page 214, whence there is the homomorphism of double complexes

$$C^*_{\Phi|A}(A; f_\Psi(\mathscr{C}^*(X;\mathscr{A}))|A) \to C^*_{\Phi|A}(A; f_{\Psi\cap A^\bullet}(\mathscr{C}^*(A^\bullet;\mathscr{A}))).$$

There results a homomorphism of spectral sequences 2.1, which reduces to

$$r^* : H^p_{\Phi|A}(A; \mathscr{H}^q_\Psi(f;\mathscr{A})|A) \to H^p_{\Phi|A}(A; \mathscr{H}^q_{\Psi\cap A^\bullet}(f|A^\bullet;\mathscr{A}|A^\bullet))$$

on the E_2 terms. By 4.4, this map is an isomorphism and the result follows. □

13 Sphere bundles with singularities

In this section we will apply the Fary spectral sequence to obtain an exact sequence for a sphere bundle with singularities that we call the Smith-Gysin sequence because it is a generalization of the Gysin sequence of a sphere bundle and an analogue of the Smith sequences of periodic maps of prime order.

Let $f : X \to Y$ be a *closed* map between Hausdorff spaces that is a k-sphere fibration with closed singular set $F \subset Y$. That is, $f : F^\bullet = f^{-1}(F) \to F$ is a homeomorphism and $f : X - F^\bullet \to Y - F$ is a fiber bundle projection with fiber $\mathbb{S}^k$; $k \geq 1$. (More generally, suppose the Leray sheaf of f has stalks as if this were the situation. For example, this applies to the orbit map of a circle group action with rational coefficients or a semi-free circle group action with integer coefficients.)

We have

$$\mathscr{H}^q(f;L)|Y - F = \begin{cases} L, & q = 0, \\ \mathscr{O}, & q = k, \\ 0, & q \neq 0, k, \end{cases}$$

where $\mathscr{O}$ is a locally constant sheaf on $Y - F$ with stalks L. The fibration f is said to be *orientable* if $\mathscr{O}$ is constant. This is always the case when f is the orbit map of a circle action.

If $\mathscr{B}$ is any sheaf on Y, it follows from 4.6 that

$$\mathscr{H}^q(f;f^*\mathscr{B})|Y - F = \begin{cases} \mathscr{B}|Y - F, & q = 0, \\ \mathscr{O} \otimes (\mathscr{B}|Y - F), & q = k, \\ 0, & q \neq 0, k. \end{cases}$$

Also, it is clear that on F,

$$\mathscr{H}^{q-k}(f|F^\bullet; f^*\mathscr{B}) = \begin{cases} \mathscr{B}|F, & q = k, \\ 0, & q \neq k. \end{cases}$$

Now let $K_0 = Y$; $K_1 = K_2 = \cdots = K_k = F$; and $K_p = \varnothing$ for $p > k$. The Fary spectral sequence with coefficients in $f^*\mathscr{B}$ and supports in the *paracompactifying* family Φ on Y has

$$E_2^{p,q} = H^p_{\Phi|Y-F}(Y-F; \mathscr{H}^q(f; f^*\mathscr{B})) \oplus H^{p+k}_{\Phi|F}(F; \mathscr{H}^{q-k}(f|F^\bullet; f^*\mathscr{B}))$$

and converges to $H^{p+q}_{f^{-1}\Phi}(X; f^*\mathscr{B})$. Thus

$$\begin{aligned} E_2^{p,0} &= H^p_{\Phi|Y-F}(Y-F; \mathscr{B}), \\ E_2^{p,k} &= H^p_{\Phi|Y-F}(Y-F; \mathscr{O}\otimes\mathscr{B}) \oplus H^{p+k}_{\Phi|F}(F; \mathscr{B}), \end{aligned}$$

and $E_2^{p,q} = 0$ for $q \neq 0, k$. [We have written $\mathscr{O}\otimes\mathscr{B}$ for $\mathscr{O}\otimes(\mathscr{B}|Y-F)$ for simplicity of notation.]

As with any spectral sequence in which $E_2^{p,q}$ is nonzero in only two complementary degrees $q = 0, k$, there results an exact sequence

$$\cdots \to E_2^{p,0} \to H^p_{f^{-1}\Phi}(X; f^*\mathscr{B}) \to E_2^{p-k,k} \xrightarrow{d_{k+1}} E_2^{p+1,0} \to \cdots,$$

which yields the exact *Smith-Gysin sequence*:

$$\boxed{\begin{aligned} &\cdots \to H^p_{\Phi|Y-F}(Y-F; \mathscr{B}) \to H^p_{f^{-1}\Phi}(X; f^*\mathscr{B}) \\ &\to H^{p-k}_{\Phi|Y-F}(Y-F; \mathscr{O}\otimes\mathscr{B}) \oplus H^p_{\Phi|F}(F; \mathscr{B}) \to H^{p+1}_{\Phi|Y-F}(Y-F; \mathscr{B}) \to \cdots \end{aligned}} \tag{43}$$

See [6] for some other applications of the Fary spectral sequence. See [15, III-10] for a more elementary derivation of the Smith-Gysin sequence.

Note that a somewhat less subtle "Smith-Gysin sequence" can be obtained directly from the Leray spectral sequence of f, namely, the exact sequence

$$\cdots \to H^p_\Phi(Y; \mathscr{B}) \to H^p_{f^{-1}\Phi}(X; f^*\mathscr{B}) \to H^{p-k}_{\Phi|Y-F}(Y-F; \mathscr{O}\otimes\mathscr{B}) \to \cdots$$

As an example of the use of the Smith-Gysin sequence (43), we have the following result:

13.1. Theorem. *Let $f : X \to Y$ be a closed map that is an orientable k-sphere fibration with closed singular set $F \subset Y$. Let Φ be a paracompactifying family of supports on Y. Let L be a field and suppose that $\dim_{\Phi,L} Y < \infty$. Then, with coefficients in L and for any p, we have the inequality*

$$\boxed{\dim H^{p-k}_{\Phi|Y-F}(Y-F) + \sum_{j=0}^{\infty} \dim H^{p+j(k+1)}_{\Phi|F}(F) \le \sum_{j=0}^{\infty} \dim H^{p+j(k+1)}_{f^{-1}\Phi}(X).}$$

Proof. Let

$$a_q = \dim H^q_{f^{-1}\Phi}(X),\ b_q = \dim H^q_{\Phi|F}(F), \text{ and } c_q = \dim H^q_{\Phi|Y-F}(Y-F).$$

Note that $\dim_{\Phi|F} F \leq \dim_\Phi Y$ by II-16.9, so that b_q and c_q are both zero for sufficiently large q. The sequence (43) implies that a_q is also zero for large q. The exact sequence

$$H^q_{f^{-1}\Phi}(X) \to H^{q-k}_{\Phi|Y-F}(Y-F) \oplus H^q_{\Phi|F}(F) \to H^{q+1}_{\Phi|Y-F}(Y-F)$$

shows that

$$b_q + c_{q-k} \leq a_q + c_{q+1},$$

which degenerates to $0 \leq 0$ for q sufficiently large. Adding these inequalities for $q = p + j(k+1)$, $j = 0, 1, 2, \ldots$, we see that the terms in c_i cancel out, except for c_{p-k}, and there remains the inequality

$$c_{p-k} + \sum_{j=0}^{\infty} b_{p+j(k+1)} \leq \sum_{j=0}^{\infty} a_{p+j(k+1)}$$

as claimed. □

13.2. Theorem. *Let $f : X \to Y$ be a closed map of Hausdorff spaces that is an orientable k-sphere fibration with closed singular set $F \subset Y$. Let L be a principal ideal domain. Assume either that X is paracompact with $\dim_L X < \infty$ or that X is compact. If X is acyclic over L, then Y and F are also acyclic over L.*[30]

Proof. Let $\Phi = cld|Y - F$. The Smith-Gysin sequence shows that

$$H^{p-k}_\Phi(Y-F;L) \oplus \tilde{H}^p(F;L) \xrightarrow{\approx} H^{p+1}_\Phi(Y-F;L).$$

If X, whence Y, is compact, then the map $H^{p-k}_\Phi(Y-F;L) \to H^{p+1}_\Phi(Y-F;L)$ is the cup product with the Euler class $\omega \in H^{k+1}(Y-F;L)$, whence it is zero for large p by II-Exercise 53. In the paracompact case, $\dim_L(Y-F) \leq \dim_L X - k$ by 8.12. Thus, in either case, a downwards induction shows that $H^*_\Phi(Y-F;L) = 0 = \tilde{H}^*(F;L)$. The result then follows from the cohomology sequence of the pair (Y,F). □

Some other applications of the Smith-Gysin sequence can be found in [11].

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[30] Also see Exercise 21.

14 The Oliver transfer and the Conner conjecture

In this section we study the transfer map for actions of compact Lie groups due to Oliver [65] and use it to give Oliver's solution to the Conner conjecture. The results in this section are due to Oliver except for those at the end credited to Conner.

Let G be a compact *connected* Lie group acting on a completely regular space X. In this situation, each orbit $\breve{x} = G(x)$ has an invariant neighborhood $U^\bullet$ (where $U \subset X/G$ is open) possessing an equivariant retraction $\sigma_{x,U} : U^\bullet \to G(x)$; see [15, II-5.4]. The neighborhood $U^\bullet$ together with the retraction $\sigma_{x,U}$ is called a "tube" about $G(x)$. Let $A \subset X$ be a closed subspace invariant under G and put $W = X - A$.

Let L be a given base ring. Let $f : X \to X/G$ be the orbit map. We wish to define a natural homomorphism

$$\pi : \mathscr{H}^n(f; L) \to H^n(G; L)$$

of the Leray sheaf of f to the constant sheaf on X/G with stalks $H^n(G; L)$, where n is arbitrary for the present. Since $\mathscr{H}^n(f, f|A; L) = \mathscr{H}^n(f; L)_{W/G}$, because W/G is open and $\mathscr{H}^n(f, f|A; L)_{\breve{y}} \approx H^n(\breve{y}, \breve{y} \cap A; L) = 0$ for $y \in A$ by 4.2, π will induce a sheaf homomorphism

$$\pi : \mathscr{H}^n(f, f|A; L) \to H^n(G; L)_{W/G} \subset H^n(G; L),$$

and hence, by II-10.2,

$$\begin{aligned} \pi^* : H^s(X/G; \mathscr{H}^n(f, f|A; L)) &\to H^s(X/G; H^n(G; L)_{W/G}) \\ &\approx H^s(X/G, A/G; H^n(G; L)). \end{aligned}$$

Because G is compact and, hence, f is closed, the canonical map $r^*_{\breve{x}} : \mathscr{H}^n(f; L)_{\breve{x}} \to H^n(G(x); L)$ of 4.2 is an isomorphism for all $\breve{x} \in X/G$.

For $x \in X$ let $\eta_x : G \to G(x)$ be the equivariant map $\eta_x(g) = g(x)$. Since G is connected, η_x depends on x, in its orbit, only up to homotopy. Thus $\eta^*_x : H^n(G(x); L) \to H^n(G; L)$ is well-defined, independent of x in its orbit $\breve{x}$. Let $\pi_{\breve{x}} = \eta^*_x r^*_{\breve{x}} : \mathscr{H}^n(f; L)_{\breve{x}} \to H^n(G; L)$. Then the $\pi_{\breve{x}}$ fit together to define a *function* π, and we need only show that this is continuous.

To show continuity, fix x and $\sigma_{x,U} : U^\bullet \to G(x)$ as above, and let $y \in U^\bullet$. It is no loss of generality to assume that $\sigma_{x,U}(y) = x$. Also, let $i_{U,y} : G(y) \hookrightarrow U^\bullet$ be the inclusion. Then $\sigma_{x,U}(i_{U,y}(\eta_y(g))) = \sigma_{x,U}(g(y)) = g(\sigma_{x,U}(y)) = g(x) = \eta_x(g)$; that is, the diagram

$$\begin{array}{ccc} U^\bullet & \xrightarrow{\sigma_{x,U}} & G(x) \\ {\scriptstyle i_{U,y}}\uparrow & & \uparrow{\scriptstyle \eta_x} \\ G(y) & \xleftarrow{\eta_y} & G \end{array}$$

commutes. Let $\theta_U : H^n(U^\bullet; L) \to \Gamma(\mathscr{H}^n(f;L)|U)$ be the canonical map from presheaf to generated sheaf and let

$$\gamma_{U,y} : \Gamma(\mathscr{H}^n(f;L)|U) \to \mathscr{H}^n(f;L)_{\breve{y}}$$

be the restriction. Then we have the diagram

$$\begin{array}{ccccc}
\Gamma(\mathscr{H}^n(f;L)|U) & \xleftarrow{\theta_U} & H^n(U^\bullet;L) & \xleftarrow{\sigma^*_{x,U}} & H^n(G(x);L) \\
\downarrow{\scriptstyle \gamma_{U,y}} & & \downarrow{\scriptstyle i^*_{U,y}} & & \downarrow{\scriptstyle \eta^*_x} \\
\mathscr{H}^n(f;L)_{\breve{y}} & \xrightarrow[\approx]{r^*_{\breve{y}}} & H^n(G(y);L) & \xrightarrow{\eta^*_y} & H^n(G;L)
\end{array}$$

in which the composition along the bottom is $\pi_{\breve{y}}$. The left-hand square commutes because it is just the definition of $r^*_{\breve{y}}$. The right-hand square commutes because it is induced by the preceding commutative diagram.

Let $\alpha \in H^n(G(x);L)$, and put $s = \theta_U\sigma^*_{x,U}(\alpha) \in \Gamma(\mathscr{H}^n(f;L)|U)$. Then the value of the section s at $\breve{y}$ is $s(\breve{y}) = \gamma_{U,y}(s)$ by the definition of γ. We compute:

$$\begin{aligned}
\pi_{\breve{y}}\, s(\breve{y}) &= \pi_{\breve{y}}\, \gamma_{U,y}(s) \\
&= \eta^*_y\, r^*_{\breve{y}}\, \gamma_{U,y}\, \theta_U\, \sigma^*_{x,U}(\alpha) \\
&= \eta^*_y\, i^*_{U,y}\, \sigma^*_{x,U}(\alpha) \\
&= \eta^*_x(\alpha) \\
&= \pi_{\breve{x}}\, s(\breve{x}),
\end{aligned}$$

the last equation holding by substitution of x for y in the preceding parts. This equation shows that π is constant on the image of the section s. The image under the isomorphism $r^*_{\breve{x}}$ of the value of the section s at $\breve{x}$ is $r^*_{\breve{x}}(s(\breve{x})) = r^*_{\breve{x}}\gamma_{U,x}(s) = r^*_{\breve{x}}\gamma_{U,x}\theta_U\sigma^*_{x,U}(\alpha) = i^*_{U,x}\sigma^*_{x,U}(\alpha) = 1^*(\alpha) = \alpha$. It follows that π is continuous at the *arbitrary* element $s(\breve{x}) = (r^*_{\breve{x}})^{-1}(\alpha) \in \mathscr{H}^n(f;L)_{\breve{x}}$ and hence is continuous in the large as claimed.

Now let us specialize to the case $n = \dim G$ and let $\lambda : H^n(G;L) \xrightarrow{\approx} L$ be an "orientation." The Leray spectral sequence of the map $f : (X,A) \to (X/G, A/G)$ has $E_2^{s,t} = 0$ for $t > n$, and so there is the canonical homomorphism

$$H^{s+n}(X,A;L) \twoheadrightarrow E_\infty^{s,n} \rightarrowtail E_2^{s,n} = H^s(X/G; \mathscr{H}^n(f, f|A; L)),$$

and the composition of this with $\lambda^* \circ \pi^*$ gives a natural homomorphism

$$\boxed{\tau_{X,A} : H^{s+n}(X,A;L) \to H^s(X/G, A/G; L),}$$

called the "Oliver transfer."

14.1. Proposition. *The Oliver transfer τ satisfies the relationship*

$$\boxed{\tau_{X,A}(f^*(\beta) \cup \alpha) = \beta \cup \tau_X(\alpha),}$$

where $\alpha \in H^{s+n}(X;L)$ and $\beta \in H^t(X/G, A/G; L)$.

Proof. Let ${}'E_r^{s,t}$ be the Leray spectral sequence of f and $E_r^{s,t}$ the Leray spectral sequence of $(f, f|A)$. The element α represents a class $\alpha_r \in {}'E_r^{s,n}$ for each $2 \le r \le \infty$ and $\tau_X(\alpha) = \lambda^*(\pi^*(\alpha_2))$. The element $f^*\beta$ represents a class $\beta_r \in E_r^{t,0}$ for each $2 \le r \le \infty$, and $\beta_2 \in H^t(X/G; \mathscr{H}^0(f, f|A; L))$ corresponds to $\beta \in H^t(X/G, A/G; L) \approx H^t(X/G; L_{W/G})$ under the canonical isomorphism $\mathscr{H}^0(f, f|A; L) \approx L_{W/G}$. By 6.2 and 6.8, the composition $H^{s+n+t}(X, A; L) \twoheadrightarrow E_\infty^{s+t,n} \rightarrowtail E_2^{s+t,n}$ takes $f^*(\beta) \cup \alpha$ to $\beta_2 \cup \alpha_2$, and hence $\lambda^* \circ \pi^*$ takes this to $\beta \cup \tau_X(\alpha)$. But by definition, this composition takes $f^*(\beta) \cup \alpha$ to $\tau_{X,A}(f^*(\beta) \cup \alpha)$. □

14.2. Corollary. *The following diagram commutes:*

$$\begin{array}{ccc} H^{s+n-1}(A; L) & \xrightarrow{\delta} & H^{s+n}(X, A; L) \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_{X,A}} \\ H^{s-1}(A/G; L) & \xrightarrow{\delta} & H^s(X/G, A/G; L). \end{array}$$

Proof. The proof is similar to Steenrod's proof of the similar relationship between the Steenrod squares and connecting homomorphisms; see [19, VI-15.2] or [79, 1.2]. We consider the G-space $I \times X$ and its various subspaces, where G acts trivially on the factor $I = [0,1]$. Let $p_X : I \times X \to X$, $p_I : I \times X \to I$, and $\breve{p}_I : I \times X/G \to I$ be the projections. Let $W = \{0\} \times X \cup I \times A$. Then the naturality of the Oliver transfer implies that it suffices to prove the result for the pair $(W, I \times A)$. Naturality then shows, in turn, that it suffices to prove it for the pair $(W, \{1\} \times A)$. Put $C = \{1\} \times A$ and $B = [0, \frac{1}{2}] \times A \cup \{0\} \times X \subset W$. Consider the commutative diagram (coefficients in L)

$$\begin{array}{ccccc} H^s(B \cup C) & \xrightarrow{r} & H^s(C) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ H^{s+1}(W, B \cup C) & \longrightarrow & H^{s+1}(W, C) & & \end{array}$$

with exact row. Again, naturality applied to this diagram implies that it suffices to prove the result for the pair $(W, B \cup C)$. By excision and homotopy invariance, it suffices to prove the result for the pair $([\frac{1}{2}, 1] \times A, \{\frac{1}{2}\} \times A \cup \{1\} \times A)$, which is equivalent to the pair $(I \times A, \partial I \times A)$. Now, $H^{s+n}(\partial I \times A)$ is generated by elements of the form $\bar{x} \cup \bar{y} = x \times y$, where $x \in H^0(\partial I)$, $y \in H^{s+n}(A)$, $\bar{x} = p_I^*(x)$, and $\bar{y} = p_X^*(y)$. Let $\breve{x} = \breve{p}_I^*(x)$, so that $f^*(\breve{x}) = \bar{x}$. We compute

$$\begin{aligned} \tau_{I\times A, \partial I \times A}(\delta(\bar{x} \cup \bar{y})) &= \tau_{I\times A, \partial I \times A}(\delta(f^*(\breve{x})) \cup \bar{y}) \\ &= \tau_{I\times A, \partial I \times A}(f^*(\delta(\breve{x})) \cup \bar{y}) \\ &= \delta(\breve{x}) \cup \tau_{I \times A}(\bar{y}) && \text{by 14.1} \\ &= \delta(\breve{x} \cup \tau_{I\times A}(\bar{y})) && \text{by II-7.1(b)} \\ &= \delta(\tau_{I\times A}(f^*(\breve{x}) \cup \bar{y})) && \text{by 14.1} \\ &= \delta(\tau_{I\times A}(\bar{x} \cup \bar{y})), \end{aligned}$$

and so $\tau_{I\times A,\partial I\times A}\circ\delta=\delta\circ\tau_{I\times A}$. $\square$

This transfer map was the main (unknown at the time) tool in Oliver's solution of the Conner conjecture. We shall now go on to give his proof. The following result sums up the main usage of the transfer:

14.3. Theorem. *Let G be a compact, connected, nontrivial Lie group acting on a paracompact space X. Let p be a prime number and let $P\subset X$ be the union of the fixed-point sets of all order p subgroups of G. Then the transfer $\tau_P : H^{s+n}(P;\mathbb{Z}_p)\to H^s(P/G;\mathbb{Z}_p)$ is zero for all s. Assume further that $H^t(X/G,P/G;\mathbb{Z}_p)=0$ for $t>k$. Then the transfer $\tau_{X,P}: H^{k+n}(X,P;\mathbb{Z}_p)\to H^k(X/G,P/G;\mathbb{Z}_p)$ is an isomorphism and the canonical map $H^{k+n}(X,P;\mathbb{Z}_p)\to H^{k+n}(X;\mathbb{Z}_p)$ is a monomorphism.*

Proof. If $w\in W=X-P$ and if u is sufficiently near w, then G_u is conjugate to a subgroup of G_w, as follows from the existence of a tube about $G(w)$. It follows that $u\in W$. Thus P is closed and it is obviously invariant.

As before, let f be the orbit map. Let $x\in P$. Then the isotropy group G_x contains a subgroup of order p, and so G_x is either of positive dimension or it is finite of order a multiple of p. In both cases the map $\eta_x^*: H^n(G(x);\mathbb{Z}_p)\approx H^n(G/G_x;\mathbb{Z}_p)\to H^n(G;\mathbb{Z}_p)$ is zero. It follows that $\pi:\mathscr{H}^n(f|P;\mathbb{Z}_p)\to H^n(G;\mathbb{Z}_p)$ is zero, and so the Oliver transfer τ_P : $H^{s+n}(P;\mathbb{Z}_p)\to H^s(P/G;\mathbb{Z}_p)$ is zero for all s, proving the first assertion.

Now, for $x\in W=X-P$, we have that G_x is finite of order prime to p, so that the map $\eta_x^*: H^*(G(x);\mathbb{Z}_p)\approx H^*(G/G_x;\mathbb{Z}_p)\to H^*(G;\mathbb{Z}_p)$ is an isomorphism by II-19.2 since G is connected. It follows that the map $\pi:\mathscr{H}^*(f,f|P;\mathbb{Z}_p)\to H^*(G;\mathbb{Z}_p)_{W/G}$ is an isomorphism, since it is an isomorphism on each stalk. Therefore, for the Leray spectral sequence of $(f,f|P)$, we have

$$\begin{aligned} E_2^{s,t} &= H^s(X/G;\mathscr{H}^t(f,f|P;\mathbb{Z}_p))\approx H^s(X/G;H^t(G;\mathbb{Z}_p)_{W/G})\\ &\approx H^s(X/G,P/G;H^t(G;\mathbb{Z}_p)).\end{aligned}$$

Thus $E_2^{s,t}=0$ for $s>k$ and for $t>n$, and so the Oliver transfer, which is the composition

$$\begin{aligned}\tau_{X,P}: H^{k+n}(X,P;\mathbb{Z}_p)\xrightarrow{\approx}E_\infty^{k,n}\xrightarrow{\approx}E_2^{k,n}\approx H^k(X/G,P/G;H^n(G;\mathbb{Z}_p))\\ \approx H^k(X/G,P/G;\mathbb{Z}_p)\end{aligned}$$

of isomorphisms, is an isomorphism, proving the second assertion. The commutative diagram

$$\begin{array}{ccc} H^{k+n-1}(P;\mathbb{Z}_p) & \xrightarrow{\delta} & H^{k+n}(X,P;\mathbb{Z}_p)\\ \downarrow{\scriptstyle \tau_P=0} & & \approx\downarrow{\scriptstyle \tau_{X,P}}\\ H^{k-1}(P/G;\mathbb{Z}_p) & \xrightarrow{\delta} & H^k(X/G,P/G;\mathbb{Z}_p)\end{array}$$

of 14.2 shows that $\delta : H^{k+n-1}(P;\mathbb{Z}_p) \to H^{k+n}(X,P;\mathbb{Z}_p)$ is zero, giving the last assertion by the exact sequence of the pair (X,P). □

14.4. Theorem. *Let G be a compact Lie group acting on the paracompact space X. For a prime p, let P be the union of the fixed-point sets of all order p subgroups of G. Suppose that X is $\mathbb{Z}_p$-acyclic and has $\dim_{\mathbb{Z}_p} X < \infty$. Then $H^*(X/G,P/G;\mathbb{Z}_p) = 0$.*[31]

Proof. By passing to the unreduced suspension of X, it can be assumed that G has a fixed point $x_\circ$ on X. First, we shall reduce this to the case in which G is connected. Let $G \subset K$ where K is a connected compact Lie group. (It is a standard result that this always exists, and in fact, K can be taken to be some unitary group.) Let $K \times_G X$ be the quotient space of $K \times X$ by the action $g\langle k,x\rangle = \langle kg^{-1}, g(x)\rangle$. If $[k,x]$ is the orbit of $\langle k,x\rangle$ under this action, then there is an action of K on Y given by $h[k,x] = [hk,x]$. The projection $K \times X \to K$ induces a map $\xi : K \times_G X \to K/G$, which is a fiber bundle projection with fiber X. Moreover, the set $K \times_G x_\circ$ is a cross section of ξ. Let $Y = (K \times_G X)/(K \times_G x_\circ)$. Since X is $\mathbb{Z}_p$-acyclic and the stalks of the Leray sheaf of ξ are $H^*(X;\mathbb{Z}_p)$ by 7.2, the map $H^*(K \times_G X;\mathbb{Z}_p) \to H^*(K \times_G x_\circ;\mathbb{Z}_p)$ is an isomorphism. It follows from the relative homeomorphism theorem II-12.11 that Y is $\mathbb{Z}_p$-acyclic. If Q is the union of the fixed point sets of order p subgroups of K on Y then we claim that $Q = (K \times_G P)/(K \times_G x_\circ)$. Indeed, suppose that $h \in K$ has order p and that $h[k,x] = [k,x]$. Then $[hk,x] = [k,x]$, which means that there is a $g \in G$ with $\langle hk,x\rangle = \langle kg^{-1}, g(x)\rangle$. Thus $g \in G_x$ and $h = kg^{-1}k^{-1}$. Then g has order p and fixes x, whence $x \in P$.

Now, $Y/K \approx X/G$ and $Q/K \approx P/G$, so that the result for Y and K would imply the result for X and G. Thus we may as well assume that G is connected.

By II-Exercise 54, $\dim_{\mathbb{Z}_p} X/G \le \dim_{\mathbb{Z}_p} X < \infty$. If the theorem is false, then there exists an integer k with $H^k(X/G,P/G;\mathbb{Z}_p) \neq 0$ and $H^t(X/G,P/G;\mathbb{Z}_p) = 0$ for $t > k$. But then, since $n = \dim G > 0$, the isomorphism $\tau_{X,P} : H^{k+n}(X,P;\mathbb{Z}_p) \approx H^k(X/G,P/G;\mathbb{Z}_p) \neq 0$ and the monomorphism $H^{k+n}(X,P;\mathbb{Z}_p) \rightarrowtail H^{k+n}(X;\mathbb{Z}_p)$ of 14.3 show that X is not $\mathbb{Z}_p$-acyclic, contrary to assumption. □

A compact Lie group S will be called a *p-torus* if S is an extension of a toral group by a p-group. The trivial group is regarded as a p-torus for all p. If T is a maximal torus of the compact Lie group H, then $\chi(H/N_H(T)) = 1$, as is well known. If $S \subset H$ is a p-torus, then Smith theory shows that $\chi((H/N_H(T))^S) \equiv 1 \pmod p$, whence $(H/N_H(T))^S \neq \varnothing$.[32] This implies that S is conjugate to a subgroup of $N_H(T)$. Assuming, then, that $S \subset N_H(T)$, it follows that ST is a p-toral subgroup of H. In particular, if S is a *maximal* p-torus in H, then $\operatorname{rank} S = \operatorname{rank} H$. Any two

[31] Also see Exercise 20.

[32] X^G denotes the set of fixed points of G on X.

maximal p-tori of H are conjugate in H, as follows immediately from the corresponding facts about maximal tori and about p-Sylow subgroups of a finite group.

For a given action of a compact Lie group G on a space X, let $\mathfrak{C}_p$ be the collection of conjugacy classes in G of maximal p-toral subgroups of isotropy subgroups of G on X. For a subgroup $S \subset G$ we let $[S]$ denote its conjugacy class. Note that an orbit type determines a corresponding member of $\mathfrak{C}_p$, but two different orbit types might determine the same member of $\mathfrak{C}_p$. There is a partial ordering of $\mathfrak{C}_p$ given by $[S] < [T]$ if S is conjugate to a proper subgroup of T.

We say that a subset $\mathfrak{F} \subset \mathfrak{C}_p$ is *full* if $[S] \in \mathfrak{F}$ and $[S] < [T] \Rightarrow [T] \in \mathfrak{F}$. Also, let

$$X^{\mathfrak{F}} = \bigcup_{[S]\in\mathfrak{F}} X^S.$$

Note that $X^{\mathfrak{F}}$ is closed when $\mathfrak{F}$ is full and that $\mathfrak{F} \subset \mathfrak{G} \Rightarrow X^{\mathfrak{F}} \subset X^{\mathfrak{G}}$. Also, $X^{\mathfrak{F}}$ is G-invariant.

14.5. Theorem. *Suppose that the compact Lie group G acts on the paracompact p-acyclic space X with $\dim_{\mathbb{Z}_p} X < \infty$. Let $\mathfrak{F}$ and $\mathfrak{G}$ be full subsets of $\mathfrak{C}_p$ with $\mathfrak{F} \subset \mathfrak{G}$ and with $\mathfrak{G} - \mathfrak{F}$ finite. Then $\tilde{H}^*(X^{\mathfrak{G}}/G, X^{\mathfrak{F}}/G; \mathbb{Z}_p) = 0$.*

Proof. The proof will be by induction on the cardinality of $\mathfrak{G} - \mathfrak{F}$ for a fixed $\mathfrak{F}$. Of course, the result is true when $\mathfrak{G} = \mathfrak{F}$. Thus assume that $\tilde{H}^*(X^{\mathfrak{G}'}/G, X^{\mathfrak{F}}/G; \mathbb{Z}_p) = 0$ if $\mathfrak{F} \subset \mathfrak{G}' \subsetneq \mathfrak{G}$. Suppose that $[S] \in \mathfrak{G} - \mathfrak{F}$ and is maximal with this property. Put $\mathfrak{G}' = \mathfrak{G} - \{[S]\} \subsetneq \mathfrak{G}$. Consider the map

$$\theta : X^S/N(S) \to X^{\mathfrak{G}}/G.$$

Let $x \in X^S$; then $G_x \supset S$. If $x \in X^{\mathfrak{G}} - X^{\mathfrak{G}'}$, then S is a maximal p-torus of G_x since $S \subset S'$ for some maximal p-torus S' of G_x, so that $[S'] \in \mathfrak{G}'$ (whence $x \in X^{\mathfrak{G}'}$) if $S \neq S'$. We claim that it follows that $G(x)^S/N(S)$ is a single point, whence θ is one-to-one on $(X^S - X^{\mathfrak{G}'})/N(S)$. Indeed, suppose that $g(x) \in G(x)^S$ for some $g \in G$. Then $Sg(x) = g(x)$, whence $g^{-1}Sg \subset G_x$. Since any two maximal p-tori of G_x are conjugate in G_x, there is an element $k \in G_x$ with $k^{-1}g^{-1}Sgk = S$, i.e., $gk \in N(S)$. But then $g(x) = gk(x) \in N(S)(x)$, whence $N(S)(g(x)) = N(S)(x)$ as claimed.

On the other hand, if $x \in X^{\mathfrak{G}'}$, then S is not a maximal p-torus in G_x. Then $\chi(G_x/S) \equiv 0 \pmod p$. Therefore $\chi((G_x/S)^S) \equiv 0 \pmod p$ by Smith theory. But $(G_x/S)^S = N_{G_x}(S)/S$, and it follows that $N_{G_x}(S)/S$ contains an element of order p.

These remarks show that $\theta : (X^S/N(S), P/N(S)) \to (X^{\mathfrak{G}}/G, X^{\mathfrak{G}'}/G)$ is a relative homeomorphism, where

$$P = \{x \in X^S \mid (N(S)/S)_x \text{ contains an element of order } p\}.$$

By 14.4, $\tilde{H}^*(X^S/N(S), P/N(S); \mathbb{Z}_p) = 0$. Since θ is a relative homeomorphism, $\tilde{H}^*(X^{\mathfrak{G}}/G, X^{\mathfrak{G}'}/G; \mathbb{Z}_p) = 0$ by II-12.5. This proves the inductive

step via the exact cohomology sequence of the triple $(X^{\mathfrak{G}}/G, X^{\mathfrak{G}'}/G, X^{\mathfrak{F}}/G)$. $\square$

The case $\mathfrak{F} = \varnothing$ and $\mathfrak{G} = \mathfrak{C}_p$ gives:

14.6. Corollary. *Suppose that the compact Lie group G acts on the paracompact space X with $\dim_{\mathbb{Z}_p} X < \infty$ and only finitely many orbit types. If X is p-acyclic, then X/G is also p-acyclic.* $\square$

If we define $\mathbb{Z}_0 = \mathbb{Q}$ and the 0-torus to be a torus (so that a maximal 0-torus is just a maximal torus) then all these arguments apply to the case $p = 0$. (Just replace the Smith theory arguments by ones using the Smith-Gysin sequence for rational coefficients.) Thus we have:

14.7. Corollary. *Suppose that the compact Lie group G acts on the paracompact space X with $\dim_{\mathbb{Q}} X < \infty$ and only finitely many orbit types. If X is $\mathbb{Q}$-acyclic, then X/G is also $\mathbb{Q}$-acyclic.* $\square$

For compact spaces, the universal coefficient theorem II-15.3 gives a corresponding result over the integers:

14.8. Corollary. *Suppose that the compact Lie group G acts on the compact space X with $\dim_{\mathbb{Z}} X < \infty$ and only finitely many orbit types. If X is acyclic, then X/G is also acyclic.*[33] $\square$

Finally, a technique of Conner [26] removes the assumption on finiteness of number of orbit types and that of finite cohomological dimension in the last corollary. First, note that the case of $G = \mathbb{S}^1$ and rational coefficients follows from the Smith-Gysin sequence. For $G = \mathbb{S}^1$ and $\mathbb{Z}_p$ coefficients, Smith theory gives that

$$H_c^*((X^{\mathbb{Z}_{p^n}} - X^{\mathbb{Z}_{p^{n+1}}})/\mathbb{Z}_{p^{n+1}}) \approx H^*(X^{\mathbb{Z}_{p^n}}/\mathbb{Z}_{p^{n+1}}, X^{\mathbb{Z}_{p^{n+1}}}) = 0,$$

since this is just ${}_{\sigma}H^*(X^{\mathbb{Z}_{p^n}})$ for the action of $\mathbb{Z}_p \approx \mathbb{Z}_{p^{n+1}}/\mathbb{Z}_{p^n}$ on $X^{\mathbb{Z}_{p^n}}$. (Also, we are using the fact that finite dimensionality is not needed for the Smith theory of actions on *compact* spaces.) Now, $(X^{\mathbb{Z}_{p^n}} - X^{\mathbb{Z}_{p^{n+1}}})/\mathbb{Z}_{p^{n+1}}$ is an orientable $\mathbb{Z}_p$-cohomology circle bundle over $(X^{\mathbb{Z}_{p^n}} - X^{\mathbb{Z}_{p^{n+1}}})/G$. Thus the Gysin sequence and II-Exercise 53 show that

$$H^*(X^{\mathbb{Z}_{p^n}}/G, X^{\mathbb{Z}_{p^{n+1}}}/G; \mathbb{Z}_p) \approx H_c^*((X^{\mathbb{Z}_{p^n}} - X^{\mathbb{Z}_{p^{n+1}}})/G; \mathbb{Z}_p) = 0.$$

Therefore, for $\mathbb{Z}_p$ coefficients,

$$\tilde{H}^*(X/G) = \tilde{H}^*(X^{\mathbb{Z}_{p^0}}/G) \approx \tilde{H}^*(X^{\mathbb{Z}_{p^1}}/G) \approx \cdots \approx \tilde{H}^*(X^G),$$

the last isomorphism following from continuity II-14.6. Similarly, Smith theory gives

$$0 = \tilde{H}^*(X) = \tilde{H}^*(X^{\mathbb{Z}_{p^0}}) \approx \tilde{H}^*(X^{\mathbb{Z}_{p^1}}) \approx \cdots \approx \tilde{H}^*(X^G).$$

[33] Also see 14.10.

Also, the result is true when G is finite by II-19.13.

Now, for any compact nonabelian group G, Oliver has constructed, in [66], an example of a smooth action of G on some disk D with no points left fixed by the whole group. Since such an action has only finitely many orbit types (see [6]) we know from 14.8 that D/H is acyclic for all $H \subset G$. Conner called such a G-space D an *acyclic model.*

Let $L = \mathbb{Z}$ or a prime field. For a general compact L-acyclic space X on which G acts, Conner's technique is to look at the twisted product $X \times_G D$. There are the maps

$$X/G \xleftarrow{\xi} X \times_G D \xrightarrow{\eta} D/G.$$

The "fibers" of ξ have the form D/G_x for $x \in X$. Consequently, ξ is a Vietoris map. The fibers of η are X/G_y for $y \in D$. Since $G_y \neq G$, we can assume, by induction on the dimension and number of components of G, that these fibers are all L-acyclic. Therefore, the Vietoris mapping theorem II-11.7 gives that $\tilde{H}^*(X/G; L) \approx \tilde{H}^*(X \times_G D; L) \approx \tilde{H}^*(D/G; L) = 0$. Thus, this argument gives the following main application of the Oliver transfer, the Conner conjecture, finally proved by Oliver:

14.9. Theorem. (R. Oliver and P. Conner.) *Let $L = \mathbb{Z}$ or a prime field. If the compact Lie group G acts on the compact L-acyclic space X, then X/G is also L-acyclic.* □

Further arguments due to Conner [26] show that any action of a compact Lie group on euclidean space or a disk has a contractible orbit space.

Finally, let us show that 14.8 holds with compactness replaced by paracompactness:

14.10. Corollary. *Suppose that the compact Lie group G acts on the paracompact space X with $\dim_{\mathbb{Z}} X < \infty$ and only finitely many orbit types. If X is acyclic, then X/G is also acyclic.*

Proof. In Conner's argument concerning the twisted product $X \times_G D$, each orbit of $H \subset G$ on D has an equivariant tubular neighborhood since G acts smoothly on D. It follows that each $r_z^* : \mathscr{H}^*(\eta; \mathbb{Z})_z \to H^*(z^\bullet; \mathbb{Z})$ is an isomorphism. Therefore, the Leray spectral sequence of η shows that η^* is an isomorphism provided we can show that each X/G_y is acyclic, $y \in D$. Thus the argument works for X paracompact with $\dim_{\mathbb{Z}} X < \infty$ and with only finitely many orbit types as soon as we show that the result holds for actions of $\mathbb{S}^1$. In the latter case, there is a finite subgroup $H \subset \mathbb{S}^1$ containing all finite isotropy groups.[34] By II-19.13, X/H is acyclic. Now, $\mathbb{S}^1/H$ acts semi-freely on X/H and so $X/\mathbb{S}^1 = (X/H)/(\mathbb{S}^1/H)$ is acyclic by 13.2.[35] □

[34] Here we are using that if an action of a compact Lie group G on a space X has only finitely many orbit types, then the same is true for the action of any subgroup $K \subset G$. This is true since K has only finitely many orbit types on each $G(x) \approx G/G_x$; see [6].

[35] See [65] for an alternative proof.

Exercises

1. Ⓢ Let $f : X \to Y$. Let Ψ and Φ be families of supports on X and Y respectively. Assume that Φ and $\Phi(\Psi)$ are both paracompactifying. Show that $f_\Psi \mathscr{A}$ is Φ-soft for any $\Phi(\Psi)$-soft sheaf $\mathscr{A}$ on X.
2. Ⓢ If $Y \subset X = \mathbb{S}^n$ separates X, then show that $\underline{\dim}_L^X Y \geq n-1$.
3. Ⓢ Let X be a compact metric space with $\dim_L X = n$, where L is a principal ideal domain. Assume that $X = A \cup B$, where B is totally disconnected. Then show that $\dim_L A \geq n-1$.
4. Let $f : X \to Y$, where X and Y are locally compact Hausdorff. Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and Y respectively, such that $\mathscr{A} \widehat{*} \mathscr{B} = 0$. Show that there is a natural exact sequence of sheaves on Y of the form
$$0 \to \mathscr{H}_c^p(f;\mathscr{A}) \otimes \mathscr{B} \to \mathscr{H}_c^p(f;\mathscr{A} \otimes f^*\mathscr{B}) \to \mathscr{H}_c^{p+1}(f;\mathscr{A}) * \mathscr{B} \to 0.$$
[*Hint*: $\mathscr{H}_c^p(f;\mathscr{A} \otimes f^*\mathscr{B}) = \mathscr{S}\!\mathit{heaf}\big(U \mapsto H_c^p(f^{-1}(\overline{U});\mathscr{A} \otimes (\mathscr{B}(U)))\big)$ by continuity II-14.4. Apply II-15.3.]
5. Ⓢ For differential sheaves $\mathscr{L}^*$ on a space Y define a natural homomorphism
$$\theta : H^n(\Gamma_\Phi(\mathscr{L}^*)) \to \Gamma_\Phi(\mathscr{H}^n(\mathscr{L}^*))$$
and show that the edge homomorphism
$$\xi : H^n_{\Phi(\Psi)}(X;\mathscr{A}) \twoheadrightarrow E_\infty^{0,n} \rightarrowtail E_2^{0,n} = \Gamma_\Phi(\mathscr{H}^n_\Psi(f;\mathscr{A}))$$
of the Leray spectral sequence of $f : X \to Y$ is just the composition
$$H^n_{\Phi(\Psi)}(X;\mathscr{A}) = H^n(\Gamma_{\Phi(\Psi)}\mathscr{C}^*(X;\mathscr{A})) = H^n(\Gamma_\Phi(f_\Psi\mathscr{C}^*(X;\mathscr{A}))) \xrightarrow{\theta} \Gamma_\Phi(\mathscr{H}^n(f_\Psi\mathscr{C}^*(X;\mathscr{A}))) = \Gamma_\Phi(\mathscr{H}^n_\Psi(f;\mathscr{A})).$$
Moreover, for $y \in Y$, let $i_y^* : \Gamma_\Phi(\mathscr{H}^n_\Psi(f;\mathscr{A})) \to H^n_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A})$ be the composition of the restriction $\Gamma_\Phi(\mathscr{H}^n_\Psi(f;\mathscr{A})) \to \mathscr{H}^n_\Psi(f;\mathscr{A})_y$ with the map $r_y^* : \mathscr{H}^n_\Psi(f;\mathscr{A})_y \to H^n_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}|y^\bullet)$. Then show that $i_y^* \circ \xi$ is just the restriction map $H^n_{\Phi(\Psi)}(X;\mathscr{A}) \to H^n_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}|y^\bullet)$. [Note that $\Psi \cap y^\bullet = \Phi(\Psi) \cap y^\bullet$ unless $\{y\} \notin \Phi$, in which case $\Phi(\Psi) \cap y^\bullet = 0$.]
6. Ⓢ In the spectral sequence 9.2 of Borel, show that the edge homomorphism
$$H^n_\Phi(X/G;\mathscr{A}) \twoheadrightarrow E_\infty^{0,n} \rightarrowtail E_2^{0,n} = \Gamma(\mathscr{H}^n_\Theta(X;f^*\mathscr{A})) \to \mathscr{H}^n_\Theta(X;f^*\mathscr{A})_y \approx H^n_\Theta(X;f^*\mathscr{A})$$
(where $y \in B_G$) is identical with f^*.
7. Let $f : X \to Y$ be a closed map that is an orientable k-sphere fibration, with $k \geq 1$, with closed singular set $F \subset Y$ as in Section 13. Let L be a principal ideal domain. Let Φ be a paracompactifying family of supports on Y with $\dim_{\Phi,L} Y < \infty$ and assume that
$$H^p_{f^{-1}\Phi}(X;L) \approx \begin{cases} L, & \text{for } p = n, \\ 0, & \text{for } p \neq n, \end{cases}$$
for some integer $n \geq 0$. Show that
$$H^p_{\Phi|F}(F;L) \approx \begin{cases} L, & \text{for } p = r, \\ 0, & \text{for } p \neq r, \end{cases}$$
for some integer r with $0 \leq r \leq n$ and also compute $H^*_\Phi(Y;L)$.

8. Let $f: X \to Y$ and let Ψ be a family of supports on X. If $\mathscr{R}$ is a sheaf of rings on Y, show that for any $\mathscr{R}$-module $\mathscr{B}$, $f_\Psi f^*\mathscr{B}$ is also an $\mathscr{R}$-module. Thus, if $\mathscr{B}$ is Φ-fine for any paracompactifying family Φ of supports on Y, $f_\Psi f^*\mathscr{B}$ is also Φ-fine. [*Hint*: Use 3.4.]

9. Ⓢ Let $f: X \to Y$ be a Ψ-closed map where Ψ is a family of supports on X. Assume that for each $y \in Y$, $f^{-1}(y)$ is Ψ-taut in X and that $\dim_\Psi f^{-1}(y) = 0$. Show that the functor f_Ψ from sheaves on X to sheaves on Y is *exact*. [In particular, for locally compact Hausdorff spaces, f_c is exact when each $f^{-1}(y)$ has dimension zero.]

10. Ⓢ With the notation of 7.9 show that $\tau \cup \pi^*(\omega) = \tau \cup \tau$. That is, the Thom isomorphism applied to the Euler class yields the square of the Thom class.

11. Ⓢ If $f: X \to Y$ is Ψ-closed for a paracompactifying family Ψ of supports on X and if $B \subset Y$ is locally closed, show that there is a natural isomorphism $f_\Psi(\mathscr{A}_{f^{-1}B}) \approx (f_\Psi\mathscr{A})_B$ for sheaves $\mathscr{A}$ on X.

12. Ⓢ Let Y be a k-dimensional component of the fixed point set of a differentiable involution on an m-manifold M. Suppose that $H^p(M;\mathbb{Z}_2) = 0$ for $0 < p \le m-k$. Show that the fixed-point set is connected and that the normal Stiefel-Whitney classes of Y in M vanish; Conner [25].

13. Ⓢ Let $\mathscr{A}^*$ be a differential sheaf with $\mathscr{H}^q(\mathscr{A}^*) = 0$ for $q < 0$. Assume either that $\mathscr{A}^*$ is bounded below or that $\dim_\Phi X < \infty$. Let h denote the differential $\mathscr{A}^{-1} \to \mathscr{A}^0$. If $\mathscr{A}^q$ is Φ-acyclic for all q, show that $\operatorname{Im} h$ and $\operatorname{Coker} h$ are Φ-acyclic. [*Hint*: Consider the spectral sequence of the differential sheaf $\mathscr{B}^*$ with $\mathscr{B}^q = \mathscr{A}^q$ for $q < 0$ and $\mathscr{B}^q = 0$ for $q \ge 0$.]

14. Let $0 \to \mathscr{L}^* \to \mathscr{M}^* \to \mathscr{N}^* \to 0$ be an exact sequence of differential sheaves. Assume either that $\dim_\Phi X < \infty$ or that $\mathscr{L}^*$, $\mathscr{M}^*$, and $\mathscr{N}^*$ are all bounded below. Also assume that $H^*(H^p_\Phi(X;\mathscr{L}^*)) = 0$ for $p > 0$ and similarly for $\mathscr{M}^*$ and $\mathscr{N}^*$. Show that the inclusion

$$\operatorname{Im}\{\Gamma_\Phi(\mathscr{M}^*) \to \Gamma_\Phi(\mathscr{N}^*)\} \hookrightarrow \Gamma_\Phi(\mathscr{N}^*)$$

induces an isomorphism in homology, so that there is an induced exact sequence

$$\cdots \to H^p(\Gamma_\Phi(\mathscr{L}^*)) \to H^p(\Gamma_\Phi(\mathscr{M}^*)) \to H^p(\Gamma_\Phi(\mathscr{N}^*)) \to H^{p+1}(\Gamma_\Phi(\mathscr{L}^*)) \to \cdots$$

Also show that the naturality relation (5) on page 199 is valid in this situation [see 2.1].

15. If $f: X \to Y$, let $s\mathscr{H}^q_\Psi(f;\mathscr{A}) = \mathscr{H}^q(f_\Psi(\mathscr{S}^*(X;\mathscr{A})))$. Investigate the properties of this sheaf. Define a spectral sequence with

$$E_2^{p,q} = H^p_\Phi(Y; s\mathscr{H}^q_\Psi(f;\mathscr{A})) \Longrightarrow sH^{p+q}_{\Phi(\Psi)}(X;\mathscr{A})$$

when Φ is paracompactifying.

16. Use Exercise 15 to prove that if $f: X \to Y$ is closed and surjective and if each fiber $f^{-1}(y)$ has a fundamental system of neighborhoods in X that are contractible, then there are natural isomorphisms

$$H^p_{f^{-1}\Phi}(X;L) \approx H^p_\Phi(Y;L) \approx sH^p_{f^{-1}\Phi}(X;L)$$

when Φ is paracompactifying.

17. With the hypotheses of Exercise 16 assume that Y is locally compact and hereditarily paracompact. Show that Y is clc_L^∞.

18. Let X and Y be spaces with support families Φ and Ψ respectively. Let $A \subset X$ and let $\mathscr{A}$ be a sheaf of L-modules on X, where L is a principal ideal domain. Assume that the Leray sheaf of the projection $\pi : X \times Y \to Y$ mod $A \times Y$ is the constant sheaf $H_\Phi^*(X, A; \mathscr{A})$ (e.g., Y locally compact, Hausdorff, and clc_L^∞). Also assume that $H_\Phi^p(X, A; \mathscr{A})$ is a *finitely generated* L-module for each p. Show that for Φ paracompactifying or for $\Phi = cld$, there is a split exact sequence

$$\bigoplus_{p+q=n} H_\Phi^p(X,A;\mathscr{A}) \otimes H_\Psi^q(Y;L) \rightarrowtail H_\Theta^n((X,A)\times Y;\mathscr{A}\times Y) \twoheadrightarrow \bigoplus_{p+q=n+1} H_\Phi^p(X,A;\mathscr{A}) * H_\Psi^q(Y;L)$$

where $\Theta = \Psi(\Phi \times Y)$. (Note that Θ does not coincide with $\Phi \times \Psi$ in general. It does when $\Phi = cld$ or when Ψ consists of compact sets.) [*Hint*: Consider the proof of 7.6 and note that $\Gamma_\Psi(\mathscr{B} \otimes M) = \Gamma_\Psi(\mathscr{B}) \otimes M$ when $\mathscr{B}$ is flabby and torsion-free and M is finitely generated; see II-15.4.]

19. Show that both 7.6 and Exercise 18 remain valid if the coefficient sheaf L on Y is replaced by a locally constant sheaf $\mathscr{L}$ with stalks L and $\mathscr{A} \times Y$ is replaced by $\mathscr{A} \widehat{\otimes} \mathscr{L}$.

20. Ⓢ Let G be a compact connected Lie group acting on a paracompact space X that is acyclic and of finite dimension over $\mathbb{Z}_p$ for some prime p. If $P \subset X$ is the union of the fixed-point sets of all subgroups of G of order p, then show that P is $\mathbb{Z}_p$-acyclic. (This is due to Oliver [65].)

21. Suppose that X and Y are locally compact Hausdorff spaces and that $f : X \to Y$ is an orientable k-sphere fibration with singular set F as in Section 13. If $H_c^p(X; f^*\mathscr{B}) = 0$ for $p > n$, show that $H_c^p(F; \mathscr{B}) = 0$ for $p > n$ and that $H_c^q(Y - F; \mathscr{B}) = 0$ for $q > n - k$.

22. Ⓢ If X is arcwise connected and M is an abelian group regarded as a constant sheaf on X, show that the edge homomorphism

$$\theta : H^0(X; M) \to H^0(X; {}_\Delta\mathscr{H}^0(X; M)) = E_2^{0,0} = E_\infty^{0,0} = {}_\Delta H^0(X; M)$$

of 2.6 is an isomorphism.

23. Ⓢ If X is locally arcwise connected, show that the map $\varphi : H_\Phi^1(X; M) \to H_\Phi^1(X; {}_\Delta\mathscr{H}^0(X; M))$ of 2.6 is an isomorphism.

24. Ⓢ Discuss the spectral sequence

$$E_2^{p,q} = H^p(X; {}_\Delta\mathscr{H}^q(X; M)) \Longrightarrow {}_\Delta H^{p+q}(X; M)$$

of 2.6 for X being the topologist's sine curve of II-10.9.

25. Ⓢ Discuss the spectral sequence

$$E_2^{p,q} = H^p(X; {}_\Delta\mathscr{H}^q(X; M)) \Longrightarrow {}_\Delta H^{p+q}(X; M)$$

of 2.6 for X being the union in the plane of circles of radius $1/n$, $n = 1, 2, 3, \ldots$, all tangent to the x-axis at the origin.

26. Let Y be one strand of Wilder's necklace and $\pi : Y \to T \approx \mathbb{S}^1$ the retraction to its thread; see II-17.14. Show that each stalk of the Leray sheaf $\mathscr{H}^1(\pi; L)$ is isomorphic to L, but that $H^p(T; \mathscr{H}^1(\pi; L)) = 0$ for all p. Describe the rest of the Leray spectral sequence of π. If $K \subset Y$ is the complement of a small open disk in the side of one of the beads, investigate the nature of the Leray spectral sequence of $\pi|K : K \to T$.

27. Ⓢ Consider double inverse systems $\{A_{i,j}\}$ as in II-Exercise 59. Show that there is a spectral sequence

$$E_2^{p,q} = \varprojlim_i^p(\varprojlim_j^q A_{i,j}) \Longrightarrow \varprojlim_{i,j}^{p+q} A_{i,j}.$$

Conclude that there is an exact sequence

$$0 \to \varprojlim_i^1(\varprojlim_j A_{i,j}) \to \varprojlim_{i,j}^1 A_{i,j} \to \varprojlim_i(\varprojlim_j^1 A_{i,j}) \to 0$$

and that $\varprojlim_i^1(\varprojlim_j^1 A_{i,j}) = 0$.

28. Let $\mathscr{A}^*$ be a flabby resolution of a sheaf $\mathscr{A}$ on X and let $\mathfrak{U}$ be an open covering of X. Discuss the spectral sequences of the double complex $C^{p,q} = \check{C}_\Phi^p(\mathfrak{U}; \mathscr{A}^q)$ for Φ paracompactifying.

29. Ⓢ Verify that for the identity map $1 : (X, cld) \to (X, \Psi)$, the edge homomorphism

$$1^\diamond : H_\Psi^n(X; \mathscr{A}) = E_2^{n,0} \twoheadrightarrow E_\infty^{n,0} \rightarrowtail H_\Psi^n(X; \mathscr{A})$$

in the Leray spectral sequence, is the identity.

30. Ⓢ Define a spectral sequence

$$E_2^{p,q}(\mathfrak{U}) = \check{H}^p(\mathfrak{U}; H^q(\bullet\,; \mathscr{A})) \Longrightarrow H^{p+q}(X; \mathscr{A}),$$

natural in the open coverings $\mathfrak{U}$, and rework III-Exercise 15 in this context.

31. Ⓢ Let $f : X \to Y$ be a surjective map between compact Hausdorff spaces. Let $F \subset B \subset Y$ be closed subspaces, and assume that F and Y are acyclic (with constant coefficient group $\mathbb{Z}$). Suppose that f is an orientable $\mathbb{S}^3$ bundle over $Y - B$, an orientable $\mathbb{S}^2$ bundle over $B - F$, and a homeomorphism over F. Then show that

$$\tilde{H}^p(X; \mathbb{Z}) \approx \tilde{H}^{p-2}(B; \mathbb{Z}) \oplus \tilde{H}^{p-4}(B; \mathbb{Z}).$$

32. Ⓢ Let $\mathscr{L}^*$ be a differential sheaf on X that is Φ-acyclic and bounded below. Suppose that $\mathscr{H}^q(\mathscr{L}^*) = 0$ for $0 \neq q < n$ and that each point $x \in X$ has a neighborhood U such that the restriction $H^n(\Gamma_\Phi(\mathscr{L}^*)) \to H^n(\Gamma_{\Phi \cap U}(\mathscr{L}^*|U))$ is zero. Then show that the edge homomorphism $\eta^k : H_\Phi^k(X; \mathscr{H}^0(\mathscr{L}^*)) = E_2^{k,0} \twoheadrightarrow E_\infty^{k,0} \rightarrowtail H^k(\Gamma_\Phi(\mathscr{L}^*))$ of the spectral sequence of IV-2.1 is an isomorphism for $k \leq n$. Also translate this into a statement about singular cohomology.

33. Ⓢ With the hypotheses of 2.2, suppose that $\Gamma_\Phi(\mathscr{H}^N(\mathscr{L}^*)) \to \Gamma_\Phi(\mathscr{H}^N(\mathscr{M}^*))$ is a monomorphism (respectively, an epimorphism). Then show that the map $H^N(\Gamma_\Phi(\mathscr{L}^*)) \to H^N(\Gamma_\Phi(\mathscr{M}^*))$ is a monomorphism (respectively, an epimorphism).

Chapter V

Borel-Moore Homology

Throughout this chapter all spaces dealt with are assumed to be locally compact Hausdorff spaces. The base ring L will be taken to be a principal ideal domain, and all sheaves are assumed to be sheaves of L-modules. Note that over a principal ideal domain (and, more generally, over a Dedekind domain) a module is injective if and only if it is divisible.

We shall develop a homology theory, the Borel-Moore homology theory, for locally compact pairs, with coefficients in a sheaf, and with supports in an arbitrary family. For *constant coefficients* and *compact supports* the theory satisfies the axioms of Eilenberg-Steenrod-Milnor on the full category of locally compact pairs and maps. Thus, it coincides with singular homology on locally finite CW-complexes. In Section 12 we show, more generally, that it coincides with singular homology on *HLC* spaces.

The usefulness of the Borel-Moore homology theory lies largely in the fact that this theory corrects some of the "defects" of classical homology theories. The Čech homology theory is not exact, even for compact pairs, and this is a major fault of that theory. Also, singular homology (and cohomology) does not behave well with respect to dimension. The Borel-Moore homology theory does not possess these defects. However, it achieves this by sacrificing other desirable properties (and we shall show that such a sacrifice is necessary). For example, the homology group in dimension zero is generally rather complicated. Also, the group in dimension -1 is not obviously zero, although it does turn out to be so in most cases of interest (no case is known where it fails to be zero). A somewhat troubling "defect" of Borel-Moore theory is that it fails to satisfy "change of rings," that is, the homology groups may depend on the choice of base ring. Although this fault can be circumvented if one is willing to have a theory that is not defined for arbitrary sheaves of coefficients, that sacrifice is, to the author's mind, too great. Also, we shall show that this sacrifice is necessary if one wishes to maintain certain other desirable qualities. All these faults are present only when dealing with spaces with "bad" local properties. On the category of clc_L^∞ spaces, we shall show that all these faults disappear; see Sections 12 through 15.

An important property of the Borel-Moore homology theory is that it is closely related to the sheaf-theoretic cohomology theory. In fact, it is a sort of co-cohomology based on sheaf cohomology. This relationship with sheaf cohomology is exemplified by the universal coefficient formulas (9) and 12.8, the mixed homology-cohomology Künneth formula (58), the

cap product (Section 10), the basic spectral sequences (Section 8), and the Poincaré duality theorem (Section 9). In Section 1, we consider the notion of a cosheaf, which is a type of dual to the notion of a sheaf, and in Section 2, we define the basic notion of the dual (differential sheaf) of a differential cosheaf. For most of the chapter the notion of a cosheaf plays a predominantly terminological role, while in Sections 12, 13, and 14 it takes on more significance. See Chapter VI for further development.

The homology theory itself is defined, in Section 3, from a canonical chain complex. In Section 4 it is shown that for suitable supports and coefficient sheaves, every map of spaces induces a natural map on the chain groups of the spaces in question. (This basic property is more complicated in Borel-Moore theory than in the classical theories.) In Section 5, relative homology is introduced, the main problem being to show that the chain groups of a subspace can be canonically embedded as a subcomplex of the chain groups of the ambient space. In the same section, the axiom of excision and the relationship of the homology of a space to that of the members of the support family are considered.

In Section 6 we prove a Vietoris mapping theorem for homology with an arbitrary coefficient sheaf, and this is used to verify the homotopy invariance property for this homology theory.

In Section 7 the homology sheaf of a map is defined. It is analogous to the Leray sheaf and generalizes the sheaf of local homology groups of a space defined in Section 3.

In Section 8 the (mixed homology-cohomology) spectral sequence of a map is defined. Unlike the Leray spectral sequence, the main case of interest is that of the identity map. In this case, the spectral sequence leads to the Poincaré duality theorem of Section 9, which is the central focus of this chapter.

In Section 10 we define the cap product and study its relationship to the cup product and to Poincaré duality. It is applied to intersection theory in Section 11.

In Sections 12 and 13 we prove, among other things, that the homology of an HLC space coincides with the classical singular homology of the space. A universal coefficient formula for clc_L^∞ spaces is also obtained.

A Künneth formula relating the homology of a product space to the homology of its factors is obtained in Section 14 for clc_L^∞ spaces.

The problem of change of rings is considered in Section 15. An example is given to show that the homology groups may depend on the base ring. It is then shown that in some cases (e.g., for clc_L^∞ spaces) the homology groups *are* independent of the base ring.

In Sections 16–18 we study homology (and cohomology) manifolds fairly extensively.

Finally, the transfer map and the Smith theory of periodic transformations are studied in Sections 19 and 20.

Throughout this chapter we will use $\dim_\Phi X$ as an abbreviation for $\dim_{\Phi,L} X$.

1 Cosheaves

In this section we introduce, and study to a small extent, some elementary notions dual to those of a sheaf and presheaf. As stated above, throughout this chapter L stands for a given base ring that is a principal ideal domain.

1.1. Definition. *A "precosheaf" $\mathfrak{A}$ on X is a covariant functor from the category of open subsets of X to that of L-modules. A precosheaf is a "cosheaf" if the sequence*

$$\bigoplus_{\langle\alpha,\beta\rangle} \mathfrak{A}(U_\alpha \cap U_\beta) \xrightarrow{g} \bigoplus_{\alpha} \mathfrak{A}(U_\alpha) \xrightarrow{f} \mathfrak{A}(U) \to 0$$

is exact for all collections $\{U_\alpha\}$ of open sets with $U = \bigcup_\alpha U_\alpha$, where $g = \sum_{\langle\alpha,\beta\rangle}(i_{U_\alpha,U_{\alpha,\beta}} - i_{U_\beta,U_{\alpha,\beta}})$ and $f = \sum_\alpha i_{U,U_\alpha}$ ($i_{U,V}$ being the canonical homomorphism $\mathfrak{A}(V) \to \mathfrak{A}(U)$ for $V \subset U$).

The *constant precosheaf* L is the precosheaf taking the value L on each U. There is no generally acceptable notion of a constant cosheaf on spaces that are not locally connected (a point to be commented upon later), but on locally connected spaces, a suitable notion of *constant cosheaf* is the precosheaf assigning to U the free L-module on the components of U. (To see this, use Exercise 3 and consider the simplicial Mayer-Vietoris sequence of the nerve of triples $(U, V, U \cap V)$.)

1.2. Definition. *A cosheaf $\mathfrak{A}$ is said to be "flabby" if $i_{X,U} : \mathfrak{A}(U) \to \mathfrak{A}(X)$ is a monomorphism for each open $U \subset X$.*

Note that the constant cosheaf (when it exists) is usually *not* flabby.

1.3. An important example is that of the *singular cosheaf*, which is defined as follows: Let $S_p(U)$ be the singular chain group[1] in degree p of U with coefficients in L. This gives a precosheaf but not a cosheaf. Take the direct limit under barycentric subdivision of $S_p(U)$. That is, if $A_n = S_p(U)$ for all $n = 1, 2, \ldots$ and $A_n \to A_{n+1}$ is the subdivision homomorphism, let $\mathfrak{S}_p(U) = \varinjlim A_n$. The reader may check that indeed, $\mathfrak{S}_p$ is a flabby cosheaf on X; see Exercises 3 and 4. Note that the canonical map $S_p(X) \to \mathfrak{S}_p(X)$ induces an isomorphism

$$H_p(S_*(X)) \xrightarrow{\approx} H_p(\mathfrak{S}_*(X))$$

since homology commutes with direct limits. When necessary, we shall denote the singular cosheaf $\mathfrak{S}_*$ by $\mathfrak{S}_*(X; L)$.

1.4. Definition. *Let $\mathfrak{A}$ be a flabby cosheaf and let $s \in \mathfrak{A}(U)$. Let $|s| \subset U$ (the "support" of s) be defined by: $x \notin |s|$ if there is an open set $V \subset U$ with $x \notin V$ and with $s \in \operatorname{Im}(i_{U,V} : \mathfrak{A}(V) \to \mathfrak{A}(U))$.*

[1] Also denoted by $\Delta_p^c(U)$ in I-Exercise 12.

1.5. Proposition. *Let $\mathfrak{A}$ be a flabby cosheaf and let $s \in \mathfrak{A}(U)$. Then $|s|$ is compact, and for $V \subset U$ open, $|s| \subset V \Leftrightarrow s \in \operatorname{Im} i_{U,V}$. Moreover, $|s| = \varnothing$ $\Leftrightarrow s = 0$.*

Proof. It is immediate that $s \in \operatorname{Im} i_{U,V} \Rightarrow |s| \subset V$. Let $\{U_\alpha\}$ be a covering of U by open sets that are relatively compact in U, and let $s \in \mathfrak{A}(U)$. Then, immediately from Definition 1.1, $s \in \operatorname{Im} i_{U,W}$ for some W that is the union of a finite number of the U_α. Thus $|s|$ is contained in a compact subset of U. If $x \notin |s|$, then $s \in \operatorname{Im} i_{U,U-\{x\}}$, whence $|s|$ is contained in some compact subset of $U - \{x\}$. It follows that x has a neighborhood disjoint from $|s|$, whence $|s|$ is compact.

Now suppose that $s = i_{U,P}(s_P)$ and $s = i_{U,Q}(s_Q)$ for some open sets $P, Q \subset U$. In the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{A}(P\cap Q) & \xrightarrow{g} & \mathfrak{A}(P)\oplus\mathfrak{A}(Q) & \xrightarrow{f} & \mathfrak{A}(P\cup Q) & \longrightarrow & 0 \\
\downarrow{\scriptstyle i_{U,P\cap Q}} & & \downarrow{\scriptstyle i_{U,P}\oplus i_{U,Q}} & & \downarrow{\scriptstyle i_{U,P\cup Q}} & & \\
\mathfrak{A}(U) & \longrightarrow & \mathfrak{A}(U)\oplus\mathfrak{A}(U) & \longrightarrow & \mathfrak{A}(U) & \longrightarrow & 0
\end{array}$$

the verticals are monomorphic since $\mathfrak{A}$ is flabby. The element $\langle s_P, -s_Q\rangle$ of $\mathfrak{A}(P)\oplus\mathfrak{A}(Q)$ maps to $f\langle s, -s\rangle = s - s = 0$ in $\mathfrak{A}(U)$, whence it maps to zero in $\mathfrak{A}(P\cup Q)$. Therefore there is an element $s_{P\cap Q} \in \mathfrak{A}(P\cap Q)$ with $\langle s_P, -s_Q\rangle = g(s_{P\cap Q}) = \langle i_{P,P\cap Q}(s_{P\cap Q}), -i_{Q,P\cap Q}(s_{P\cap Q})\rangle$, whence $s \in \operatorname{Im} i_{U,P\cap Q}$.

Suppose now that $|s| \subset V \subset U$. We wish to show that $s \in \operatorname{Im} i_{U,V}$. There is a relatively compact open set $W \subset U$ with $s \in \operatorname{Im} i_{U,W}$, and it is no loss of generality to assume that $V \subset W$. If $x \in \overline{W} - V$ then $x \notin |s|$, so that there is a compact neighborhood N_x of x with $s \in \operatorname{Im} i_{U,U-N_x}$. There is a finite set $\{x_1, \dots, x_n\}$ such that $\overline{W} - V \subset N_{x_1} \cup \cdots \cup N_{x_n}$. Let $V_i = U - N_{x_i}$. Since

$$s \in \operatorname{Im} i_{U,W} \cap \operatorname{Im} i_{U,V_1} \cap \cdots \cap \operatorname{Im} i_{U,V_n},$$

we have that $s \in \operatorname{Im} i_{U,Q}$, where $Q = W\cap V_1 \cap \cdots \cap V_n = W \cap \bigcap (U - N_{x_i}) = W - \bigcup N_{x_i} \subset V$, as required.

If $|s| = \varnothing$, then this shows that $s \in \operatorname{Im} i_{U,\varnothing} = 0$, so that $s = 0$. □

The following proposition, together with 1.8, characterizes the class of flabby cosheaves as the class of cosheaves of sections with compact support of c-soft sheaves.

1.6. Proposition. *If $\mathscr{L}$ is a c-soft sheaf then the precosheaf $\Gamma_c\mathscr{L}$, where*

$$\boxed{(\Gamma_c\mathscr{L})(U) = \Gamma_c(\mathscr{L}|U),}$$

is a flabby cosheaf. This cosheaf will also be denoted by[2]

$$\boxed{\Gamma_c\{\mathscr{L}\} = \Gamma_c\mathscr{L}.}$$

[2]Note that it is the absence of parentheses or the use of braces that distinguishes this from the *group* $\Gamma_c(\mathscr{L})$.

Proof. By Exercise 3 it suffices to show that

$$\Gamma_c(\mathscr{L}|U \cap V) \to \Gamma_c(\mathscr{L}|U) \oplus \Gamma_c(\mathscr{L}|V) \to \Gamma_c(\mathscr{L}|U \cup V) \to 0$$

is exact for U, V open, since part (b) of that exercise clearly holds for $\mathfrak{A} = \Gamma_c\mathscr{L}$. But this follows from the Mayer-Vietoris sequence (27) on page 94. □

Now we wish to prove the converse of 1.6. Let $\mathfrak{A}$ be a flabby cosheaf and let A be the presheaf defined by $A(U) = \mathfrak{A}(X)/\mathfrak{A}_{X-U}(X)$, where

$$\boxed{\mathfrak{A}_B(X) = \{s \in \mathfrak{A}(X) \mid |s| \subset B\}.}$$

Let $\mathscr{A} = \mathscr{S}\!\mathit{heaf}(A)$. Note that

$$\mathscr{A}_x = \mathfrak{A}(X)/\mathfrak{A}_{X-\{x\}}(X).$$

There is the canonical map

$$\boxed{\theta : \mathfrak{A}(X) \to \mathscr{A}(X).}$$

Clearly, $\theta(s)(x) = 0 \Leftrightarrow |s| \subset X - \{x\}$. Thus $|\theta(s)| = |s|$, and in particular, θ maps $\mathfrak{A}(X)$ into $\Gamma_c(\mathscr{A})$ monomorphically. We shall show that θ maps *onto* $\Gamma_c(\mathscr{A})$.

1.7. Lemma. *Let $U \subset X$ be open and let $t \in \mathscr{A}(U)$. Suppose that $s_1, s_2 \in \mathfrak{A}(X)$ are given such that $\theta(s_1)|U_i = t|U_i$ for some open sets $U_i \subset U$, $i = 1, 2$. If V_i is any open set with closure in U_i, $i = 1, 2$, then there exists an element $s \in \mathfrak{A}(X)$ such that $\theta(s)|(V_1 \cup V_2) = t|(V_1 \cup V_2)$.*

Proof. Since $|s_1 - s_2| = |\theta(s_1 - s_2)| = |\theta(s_1) - \theta(s_2)|$ is contained in $X - (U_1 \cap U_2)$, it is also contained in $(X - \overline{V}_1) \cup (X - \overline{V}_2)$. Since $\mathfrak{A}$ is a cosheaf, there exist elements $t_i \in \mathfrak{A}(X)$, $i = 1, 2$, with $|t_i| \subset X - \overline{V}_i$ and with $s_1 - s_2 = t_1 - t_2$. Let $s = s_1 - t_1 = s_2 - t_2$. Then

$$\theta(s)|V_i = \theta(s_i)|V_i = t|V_i,$$

for $i = 1, 2$, as claimed. □

1.8. Proposition. *Every flabby cosheaf $\mathfrak{A}$ has the form $\mathfrak{A} = \Gamma_c\mathscr{A}$ for a unique c-soft sheaf $\mathscr{A}$. Moreover, the sheaf $\mathscr{A}$ is torsion-free $\Leftrightarrow$ $|ms| = |s|$ for all $s \in \mathfrak{A}(X)$ and all $0 \neq m \in L$. (In this case, we say that $\mathfrak{A}$ is "torsion-free.")*

Proof. With the preceding notation, we claim that θ maps *onto* $\Gamma_c(\mathscr{A})$. Indeed, let $t \in \Gamma_c(\mathscr{A})$, and cover $|t|$ by open sets $U_1, \ldots, U_n$ such that there exist elements $s_i \in \mathfrak{A}(X)$ with $\theta(s_i)|U_i = t|U_i$. (Such a covering exists by the definition of $\mathscr{A}$.) Let $U_0 = X - |t|$ and $s_0 = 0$. There exists a covering of X by open sets $V_0, V_1, \ldots, V_n$ with $\overline{V}_i \subset U_i$, and an easy induction on Lemma 1.7 shows that there exists an element $s \in \mathfrak{A}(X)$ with $\theta(s) = t$ on all of X.

Since $\mathfrak{A}$ is flabby, $\mathfrak{A}(U) \approx \mathfrak{A}_U(X)$, and hence θ maps $\mathfrak{A}(U)$ isomorphically onto $\Gamma_{c|U}(\mathscr{A}) = \Gamma_c(\mathscr{A}|U)$.

To show that $\mathscr{A}$ is c-soft, let $K \subset X$ be compact and let $t \in \mathscr{A}(K)$. By II-9.5 there is an extension $t' \in \mathscr{A}(U)$ of t to some open neighborhood U of K. Let $\{U_i\}$, $i = 1, \ldots, n$, be an open covering of K in U such that there are elements $s_i \in \mathfrak{A}(X)$ with $\theta(s_i)|U_i = t'|U_i$. Again, an inductive use of Lemma 1.7 shows that there exists an element $s \in \mathfrak{A}(X)$ with $\theta(s)|V = t'|V$ for some neighborhood V of K. Thus $\theta(s)$ is an extension of t to $\Gamma_c(\mathscr{A})$.

To show that $\mathscr{A}$ is unique, let $\mathscr{B}$ be a c-soft sheaf such that $\mathfrak{A} \approx \Gamma_c\mathscr{B}$ naturally. That is, suppose that $\mathfrak{A}(U) \approx \Gamma_c(\mathscr{B}|U)$ naturally in U. Let $\mathscr{A}$ be, as in the construction, the sheaf generated by the presheaf

$$A : U \mapsto \frac{\mathfrak{A}(X)}{\mathfrak{A}_{X-U}(X)} \approx \frac{\Gamma_c\mathscr{B}}{\{s \in \Gamma_c(\mathscr{B}) \mid |s| \subset X - U\}}.$$

There is the canonical map $\lambda : \mathfrak{A}(X) \approx \Gamma_c(\mathscr{B}) \to \mathscr{B}(U)$, which fits in the exact sequence

$$0 \to \mathfrak{A}_{X-U}(X) \to \mathfrak{A}(X) \xrightarrow{\lambda} \mathscr{B}(U),$$

and so λ induces a monomorphism $A(U) \xrightarrow{\lambda'} \mathscr{B}(U)$ of presheaves and hence a monomorphism $\mathscr{A} \xrightarrow{\lambda''} \mathscr{B}$ of the generated sheaves. Now, the restriction of sections $\rho_x : \Gamma_c(\mathscr{B}) \to \mathscr{B}_x$ factors as $\Gamma_c(\mathscr{B}) \to A(U) \to \mathscr{B}_x$, for neighborhoods U of x. Hence, in the direct limit, ρ_x factors as $\Gamma_c(\mathscr{B}) \to \mathscr{A}_x \xrightarrow{\lambda''_x} \mathscr{B}_x$. Since $\mathscr{B}$ is c-soft, ρ_x is surjective, and so the monomorphism λ''_x is an isomorphism, whence $\lambda'' : \mathscr{A} \to \mathscr{B}$ is an isomorphism.

For the last statement, let $s \in \mathfrak{A}(X) = \Gamma_c(\mathscr{A})$. If $\mathscr{A}$ is torsion-free then it is clear that $|ms| = |s|$. Conversely, if $\mathscr{A}$ is not torsion-free, then some $\mathscr{A}_x$ has m-torsion for some integer $m > 1$; that is, there exists an element $0 \neq s_x \in \mathscr{A}_x$ with $ms_x = 0$. Since $\mathscr{A}$ is c-soft, there is a section $s \in \Gamma_c(\mathscr{A})$ having value s_x at x. Then $x \in |s| - |ms|$, so that $|s| \neq |ms|$. □

1.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be flabby cosheaves on X, where $\mathfrak{A} = \Gamma_c\mathscr{A}$ and $\mathfrak{B} = \Gamma_c\mathscr{B}$ with $\mathscr{A}$ and $\mathscr{B}$ c-soft. Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of cosheaves, i.e., a natural transformation of functors. It follows easily from the proof of 1.8 that since $|h(a)| \subset |a|$ for $a \in \mathfrak{A}(X)$, h induces a homomorphism of sheaves $\mathscr{A} \to \mathscr{B}$ and that the induced map $\Gamma_c\mathscr{A} \to \Gamma_c\mathscr{B}$ coincides with the original h. This also follows from the fact that, for example,

$$\mathscr{A} = \mathscr{S}\!\mathit{heaf}(U \mapsto \mathfrak{A}(X)/\mathfrak{A}_{X-\overline{U}}(X))$$

and that the monomorphism $\mathfrak{A}(X - \overline{U}) \rightarrowtail \mathfrak{A}(X)$ maps onto $\mathfrak{A}_{X-\overline{U}}(X)$.

1.10. Proposition. *Let $\mathfrak{A}$ be a cosheaf and M an L-module. Then the presheaf $U \mapsto \operatorname{Hom}(\mathfrak{A}(U), M)$ is a sheaf, denoted by $\mathcal{H}om(\mathfrak{A}, M)$. It is flabby if $\mathfrak{A}$ is flabby and M is an injective module. It is torsion free if each $\mathfrak{A}(U)$ is divisible.*

Proof. This is clear since $\operatorname{Hom}(\bullet, M)$ is left exact and is exact when M is injective. □

Recall from II-3.3 that $\Gamma_c(\mathscr{L}|U)$ is divisible if $\mathscr{L}$ is injective.

1.11. Lemma. *Let $\mathfrak{A}$ be a flabby cosheaf and M an L-module. Let $f \in \Gamma(\mathcal{H}om(\mathfrak{A}, M)) = \operatorname{Hom}(\mathfrak{A}(X), M)$. Then $|f|$ is the smallest closed set K such that $f(s) = 0$ for all $s \in \mathfrak{A}(X)$ with $|s| \cap K = \varnothing$.*

Proof. Let K be such that $f(s) = 0$ for $|s| \subset X - K$. Then $f|X - K = 0$, whence $|f| \subset K$. If $U = X - |f|$ and $s \in \mathfrak{A}(X)$ has $|s| \subset U$, then $s = i_{X,U}(s')$ for some $s' \in \mathfrak{A}(U)$ by 1.5. But $f(s) = (f|U)(s') = 0$ since $f|U = 0$. Thus $f(s) = 0$ whenever $|s| \cap |f| = \varnothing$, so that the set $|f|$ does satisfy the stated condition. □

1.12. Proposition. *Let $\mathfrak{A}$ be a torsion free flabby cosheaf and let M be an injective L-module. Then $\Gamma_\Phi(\mathcal{H}om(\mathfrak{A}, M))$ is divisible for any family Φ of supports.*

Proof. Let $f \in \Gamma(\mathcal{H}om(\mathfrak{A}, M)) = \operatorname{Hom}(\mathfrak{A}(X), M)$ and let

$$B = \operatorname{Coker}\{\mathfrak{A}(X - |f|) \to \mathfrak{A}(X)\}.$$

Then B is torsion-free by 1.8, and

$$\operatorname{Hom}(B, M) = \operatorname{Ker}\{\operatorname{Hom}(\mathfrak{A}(X), M) \to \operatorname{Hom}(\mathfrak{A}(X - |f|), M)\}.$$

If $0 \neq m \in L$, $0 \to B \xrightarrow{m} B$ is exact, so $\operatorname{Hom}(B, M) \xrightarrow{m} \operatorname{Hom}(B, M) \to 0$ is also exact, since M is injective; that is, $\operatorname{Hom}(B, M)$ is divisible. Thus there is an element $g \in \operatorname{Hom}(B, M) \subset \operatorname{Hom}(\mathfrak{A}(X), M)$ with $mg = f$. By definition, g is zero on $X - |f|$, whence $|g| \subset X - (X - |f|) = |f|$. □

1.13. For $\mathfrak{A}$ flabby and M injective, there is a canonical map $\mathfrak{A}(X) \to \operatorname{Hom}(\Gamma_c(\mathcal{H}om(\mathfrak{A}, M)), M)$ defined by $s \mapsto \hat{s}$, where $\hat{s}(f) = f(s)$ for $f \in \operatorname{Hom}(\mathfrak{A}(X), M)$ and $|f|$ compact. Now, $\hat{s}(f) = f(s) = 0$ whenever $|f| \subset X - |s|$, by 1.11, and it follows that $|\hat{s}| \subset |s|$. Hence $|\hat{s}|$ is compact. Thus we have the canonical map

$$\mathfrak{A}(X) \to \Gamma_c(\mathcal{H}om(\Gamma_c\mathcal{H}om(\mathfrak{A}, M), M)),$$

which is easily seen to extend to a homomorphism of cosheaves:

$$\boxed{\mathfrak{A} \to \Gamma_c\mathcal{H}om(\Gamma_c\mathcal{H}om(\mathfrak{A}, M), M).}$$

If $\mathfrak{A} = \Gamma_c\mathscr{A}$ for $\mathscr{A}$ c-soft, then this corresponds to a homomorphism

$$\boxed{\mathscr{A} \to \mathscr{H}om(\Gamma_c\mathscr{H}om(\Gamma_c\mathscr{A}, M), M)}$$

of sheaves by 1.9.

1.14. Let $\mathfrak{A}$ be a *torsion-free* flabby cosheaf, that is, $\mathfrak{A} = \Gamma_c\mathscr{A}$, where $\mathscr{A}$ is a c-soft torsion free sheaf. Note that by II-15.3, $\mathfrak{A}(U) \otimes M = \Gamma_c(\mathscr{A}|U) \otimes M \approx \Gamma_c((\mathscr{A} \otimes M)|U)$ for any L-module M. Moreover, for any sheaf $\mathscr{B}$ on X, $\mathscr{A} \otimes \mathscr{B}$ is c-soft by II-16.31. Consequently, we shall *define* $\mathfrak{A} \otimes \mathscr{B}$ to be the flabby cosheaf

$$\boxed{\mathfrak{A} \otimes \mathscr{B} = \Gamma_c\{\mathscr{A} \otimes \mathscr{B}\}.}$$

(This notation will not be essentially used until the end of Section 12.)

1.15. Definition. *If $f : X \to Y$ is a map (of locally compact spaces) and $\mathfrak{A}$ is a precosheaf on X, then we let $f\mathfrak{A}$ be the precosheaf on Y defined by*

$$\boxed{(f\mathfrak{A})(U) = \mathfrak{A}(f^{-1}(U)).}$$

If f is an inclusion map, then $f\mathfrak{A}$ will also be denoted by $\mathfrak{A}^Y$.

1.16. Proposition. *Let $\mathfrak{A}$ be a cosheaf. Then $f\mathfrak{A}$ is also a cosheaf. If $\mathfrak{A}$ is flabby, then $f\mathfrak{A}$ is flabby. If $\mathfrak{A} = \Gamma_c\mathscr{A}$, where $\mathscr{A}$ is c-soft, then $f\mathfrak{A} = \Gamma_c\{f_c\mathscr{A}\}$.*

Proof. The first two statements follow immediately from the definitions. For $\mathscr{A}$ c-soft, $f_c\mathscr{A}$ is also c-soft by IV-Exercise 1. Then

$$\begin{aligned}
\Gamma_c\{f_c\mathscr{A}\}(U) &= \Gamma_c((f_c\mathscr{A})|U) && \text{by definition}\\
&= \Gamma_c(f_c(\mathscr{A}|f^{-1}U)) && \text{since } (f_c\mathscr{A})|U = f_c(\mathscr{A}|f^{-1}U)\\
&= \Gamma_{c(c)}(\mathscr{A}|f^{-1}U) && \text{by IV-5.2}\\
&= \Gamma_c(\mathscr{A}|f^{-1}U) && \text{since } c(c) = c\\
&= (\Gamma_c\mathscr{A})(f^{-1}U) && \text{by definition}\\
&= \mathfrak{A}(f^{-1}U) && \text{since } \mathfrak{A} = \Gamma_c\mathscr{A}\\
&= (f\mathfrak{A})(U) && \text{by definition,}
\end{aligned}$$

proving the last statement. □

1.17. Corollary. *If $A \subset X$ is locally closed and $\mathfrak{A} = \Gamma_c\mathscr{A}$ for $\mathscr{A}$ c-soft, then $\mathfrak{A}^X = \Gamma_c\{\mathscr{A}^X\}$.*

Proof. If $i : A \hookrightarrow X$ is the inclusion, then $\mathfrak{A}^X = i\mathfrak{A} = \Gamma_c\{i_c\mathscr{A}\} = \Gamma_c\{\mathscr{A}^X\}$ by IV-3.8. □

If $\mathfrak{A}$ is a cosheaf, M is an L-module, and $U \subset Y$ is open, then we have that

$$\begin{aligned}
\mathscr{H}om(f\mathfrak{A}, M)(U) &= \operatorname{Hom}((f\mathfrak{A})(U), M) = \operatorname{Hom}(\mathfrak{A}(f^{-1}U), M)\\
&= \mathscr{H}om(\mathfrak{A}, M)(f^{-1}U) = (f\mathscr{H}om(\mathfrak{A}, M))(U),
\end{aligned}$$

so that we have the natural equality

$$\boxed{\mathscr{H}om(f\mathfrak{A}, M) = f\mathscr{H}om(\mathfrak{A}, M).} \tag{1}$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are precosheaves on X and Y respectively, we define an *f-homomorphism* $h : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ to be a collection of homomorphisms

$$h_U : \mathfrak{A}(f^{-1}(U)) \to \mathfrak{B}(U)$$

for U open in Y, commuting with inclusions. Clearly, f-homomorphisms $\mathfrak{A} \rightsquigarrow \mathfrak{B}$ correspond naturally and in a one-to-one manner with homomorphisms $f\mathfrak{A} \to \mathfrak{B}$ of precosheaves on Y.

1.18. We shall conclude this section with some further remarks on singular homology.

Let $A \subset X$ be a locally closed subspace. For $U \subset X$ open, the classical relative singular chain group $S_*(U, U \cap A)$ can be identified canonically with the free group generated by those singular simplices of U that do not lie entirely in A.[3] Thus $S_*(U, U \cap A) \to S_*(X, A)$ is a monomorphism, and therefore we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_*(U \cap A) & \to & S_*(U) & \to & S_*(U, U \cap A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_*(A) & \to & S_*(X) & \to & S_*(X, A) & \to & 0. \end{array}$$

This remains exact upon passage to the direct limit over subdivisions. It follows that the precosheaf $\mathfrak{S}_*(X, A; L) : U \mapsto \mathfrak{S}_*(U, U \cap A)$ [the limit over subdivisions of $S_*(U, U\cap A)$] is the cokernel of the canonical *monomorphism* $\mathfrak{S}_*(A; L)^X \rightarrowtail \mathfrak{S}_*(X; L)$ and it is a *flabby* and *torsion-free* cosheaf. [The reader should note that the cokernel of a homomorphism of cosheaves is itself a cosheaf. The fact that $\mathfrak{S}_*(X, A; L)$ is torsion-free in the sense of 1.10 is seen by noting that the limit over subdivisions of $S_*(X, A)/S_*(U, U \cap A)$ is torsion-free.]

Let $\mathscr{S}_*(X, A; L)$ denote the c-soft sheaf with

$$\mathfrak{S}_*(X, A; L) = \Gamma_c \mathscr{S}_*(X, A; L)$$

whose existence is guaranteed by 1.8. The exact sequence

$$0 \to \mathfrak{S}_*(A; L)^X \to \mathfrak{S}_*(X; L) \to \mathfrak{S}_*(X, A; L) \to 0$$

of precosheaves is clearly equivalent to an exact sequence

$$0 \to \mathscr{S}_*(A; L)^X \to \mathscr{S}_*(X; L) \to \mathscr{S}_*(X, A; L) \to 0$$

[3] Note, however, that this isomorphism is not preserved by subdivision.

of c-soft sheaves; see 1.17.

Let $\mathscr{A}$ be any sheaf on X, and define

$$\boxed{\mathscr{S}_*(X,A;\mathscr{A}) = \mathscr{S}_*(X,A;L)\otimes\mathscr{A}}$$

and

$$\boxed{\mathfrak{S}_*(X,A;\mathscr{A}) = \Gamma_c\mathscr{S}_*(X,A;\mathscr{A}) = \mathfrak{S}_*(X,A;L)\otimes\mathscr{A}}$$

(with the notation of 1.14). We have the exact sequence

$$0 \to \mathscr{S}_*(A;\mathscr{A}|A)^X \to \mathscr{S}_*(X;\mathscr{A}) \to \mathscr{S}_*(X,A;\mathscr{A}) \to 0.$$

Let Φ be a *paracompactifying* family of supports on X. Then, since a c-soft sheaf is Φ-soft by II-16.5, the sequence

$$0 \to \Gamma_{\Phi|A}(\mathscr{S}_*(A;\mathscr{A})) \to \Gamma_\Phi(\mathscr{S}_*(X;\mathscr{A})) \to \Gamma_\Phi(\mathscr{S}_*(X,A;\mathscr{A})) \to 0$$

is exact.

We *define* the singular homology group of X (mod A) with supports in Φ and coefficients in $\mathscr{A}$ to be

$$\boxed{{}_sH_p^\Phi(X,A;\mathscr{A}) = H_p(\Gamma_\Phi(\mathscr{S}_*(X,A;\mathscr{A}))).}$$

Then we have the exact sequence (for Φ paracompactifying)

$$\cdots \to {}_sH_p^{\Phi|A}(A;\mathscr{A}) \to {}_sH_p^\Phi(X;\mathscr{A}) \to {}_sH_p^\Phi(X,A;\mathscr{A}) \to {}_sH_{p-1}^{\Phi|A}(A;\mathscr{A}) \to \cdots$$

and for $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ exact, we also have the exact sequence

$$\begin{aligned}\cdots \to {}_sH_p^\Phi(X,A;\mathscr{A}') \to {}_sH_p^\Phi(X,A;\mathscr{A}) \to {}_sH_p^\Phi(X,A;\mathscr{A}'')\\ \to {}_sH_{p-1}^\Phi(X,A;\mathscr{A}') \to \cdots\end{aligned}$$

The derived sheaf $\mathscr{H}_p(\mathscr{S}_*(X,A;\mathscr{A}))$ will be denoted by ${}_s\mathscr{H}_p(X,A;\mathscr{A})$. See Exercise 16.

1.19. Consider the natural map of presheaves

$$\Delta_*^c(X,X-\overline{U};L) = S_*(X,X-\overline{U};L) \to \mathfrak{S}_*(X,X-\overline{U};L) \tag{2}$$

(which is the inclusion into the direct limit over subdivisions), where we use the notation of I-Exercise 12 for reasons that will become apparent. Passing to generated sheaves, this gives rise to a homomorphism

$$\varDelta_*(X;L) \to \mathscr{S}_*(X;L). \tag{3}$$

Since (2) induces an isomorphism in homology, so does (3). That is, the map

$$\mathscr{H}_*(\varDelta_*) \to \mathscr{H}_*(\mathscr{S}_*)$$

is an isomorphism.

Now, $\varDelta_*$ is homotopically fine, by II-Exercise 32, and $\mathscr{S}_*(X;L)$ is c-soft, and hence Φ-soft for Φ paracompactifying by II-16.5. Thus, by IV-2.2, (3) induces an isomorphism

$$\boxed{H_p(\Gamma_\Phi(\varDelta_*(X;L))) \xrightarrow{\approx} H_p(\Gamma_\Phi(\mathscr{S}_*(X;L))) = {}_SH_p^\Phi(X;L)}$$

when Φ is paracompactifying and $\dim_\Phi X < \infty$ (and similarly for arbitrary coefficient sheaves, since $\varDelta_*(X;L)$ and $\mathscr{S}_*(X;L)$ are both torsion-free). We also have this result, as already mentioned, for $\Phi = c$ without the condition on dimension.

Note that by I-Exercise 12, $\Gamma_\Phi(\varDelta_*(X;L))$ is the group of locally finite singular chains with support in Φ (in the obvious sense). Thus, when Φ is paracompactifying and $\dim_\Phi X < \infty$ (or $\Phi = c$), ${}_SH_p^\Phi(X;L)$ coincides with the classical singular homology group based on locally finite (finite if $\Phi = c$) singular chains with support in Φ. Similar remarks also apply to the relative case.

2 The dual of a differential cosheaf

Let M be an L-module. M has the canonical injective resolution $0 \to M \to M^0 \to M^1 \to 0$, where $M^0 = I(M)$ and $M^1 = I(M)/M$; see II-3. M^1 is injective since it is divisible and since L is a principal ideal domain.

A *differential cosheaf* $\mathfrak{A}_*$ is a graded cosheaf together with a differential $d : \mathfrak{A}_p \to \mathfrak{A}_{p-1}$ of degree -1 with $d^2 = 0$. For our purposes, $\mathfrak{A}_p$ will often vanish for $p > 0$.

2.1. The *dual* of the differential cosheaf $\mathfrak{A}_*$ with respect to the L-module M is defined to be the differential sheaf

$$\boxed{\mathscr{D}(\mathfrak{A}_*; M) = \mathscr{H}om(\mathfrak{A}_*, M^*),} \tag{4}$$

where as usual, the term in degree n is

$$\mathscr{D}^n(\mathfrak{A}_*; M) = \bigoplus_{p+q=n} \mathscr{H}om(\mathfrak{A}_p, M^q) = \mathscr{H}om(\mathfrak{A}_n, M^0) \oplus \mathscr{H}om(\mathfrak{A}_{n-1}, M^1)$$

and where the differential $\mathscr{D}^n \to \mathscr{D}^{n+1}$ is $d = d' - d''$, d' being the homomorphism induced by the differential $M^q \to M^{q+1}$ and $(-1)^n d''$ being that induced by $\mathfrak{A}_{p+1} \to \mathfrak{A}_p$. Explicitly, if

$$f \in \mathscr{D}^n(\mathfrak{A}_*, M)(U) = \bigoplus_{p+q=n} \operatorname{Hom}(\mathfrak{A}_p(U), M^q)$$

and if $a \in \mathfrak{A}_*(U)$, then

$$\boxed{(df)(a) = d(f(a)) - (-1)^n f(da).}$$

Since M^* is injective, $\mathscr{D}(\bullet; M)$ is an exact functor of differential cosheaves.

2.2. If $\mathscr{L}^*$ is a c-soft differential sheaf, then $\Gamma_c\mathscr{L}^*$ will be regarded as a differential cosheaf with the gradation

$$\boxed{(\Gamma_c\mathscr{L}^*)_p = \Gamma_c\mathscr{L}^{-p}.}$$

The differential sheaf $\mathscr{D}(\Gamma_c\mathscr{L}^*; M)$ will also be denoted by

$$\boxed{\mathscr{D}(\mathscr{L}^*; M) = \mathscr{D}(\Gamma_c\mathscr{L}^*; M).}$$

Note that

$$\boxed{\mathscr{D}(\mathscr{L}^*|U; M) = \mathscr{D}(\mathscr{L}^*; M)|U} \tag{5}$$

for open sets $U \subset X$. Also, as above, we let $\mathscr{D}_n$ stand for $\mathscr{D}^{-n}$. For our purposes, $\mathscr{D}_n$ will usually vanish for $n < -1$. Note that $\mathscr{D}(\bullet; M)$ is an exact functor of c-soft differential sheaves.

The differential cosheaf $\Gamma_c\mathscr{D}_*(\mathfrak{A}_*; M)$ will also be denoted by

$$\boxed{\mathfrak{D}(\mathfrak{A}_*; M) = \Gamma_c\mathscr{D}_*(\mathfrak{A}_*; M).}$$

When $M = L$ we shall often delete L from the notation.

For a flabby differential cosheaf $\mathfrak{A}_*$ the construction of 1.13 provides a natural homomorphism

$$\mathfrak{A}_* \to \mathfrak{D}(\mathfrak{D}(\mathfrak{A}_*; M); M) \tag{6}$$

of differential cosheaves. This will be used in Section 12.

Remark: The introduction of graded objects necessitates the use of a standard sign convention in the definition of (6) so that it will be a chain mapping. We shall indicate this for graded *modules*. The generalization to cosheaves follows immediately from the construction of 1.13. For graded modules A_* and M^*, the map $A_* \to \operatorname{Hom}(\operatorname{Hom}(A_*, M^*), M^*)$ is defined by $a \mapsto \hat{a}$, where

$$\hat{a}(f) = (-1)^{(\deg a)(\deg f)} f(a).$$

Note also that

$$\operatorname{Hom}(\operatorname{Hom}(A_*, M^*), M^*)_n = \bigoplus_s \operatorname{Hom}\left(\bigoplus_r \operatorname{Hom}(A_{n-r+s}, M^r), M^s\right)$$

and that more explicitly, we have

$$\hat{a}(f) = \begin{cases} (-1)^{n(n+s)} f(a), & \text{if } r = s, \\ 0, & \text{if } r \neq s \end{cases}$$

when $a \in A_n$ and $f \in \operatorname{Hom}(A_{n-r+s}, M^r)$.

2.3. For any differential cosheaf $\mathfrak{A}_*$ we have

$$\mathscr{D}(\mathfrak{A}_*; M)(U) = \operatorname{Hom}(\mathfrak{A}_*(U), M^*),$$

and standard homological algebra provides a natural exact sequence

$$\operatorname{Ext}(H_{p-1}(\mathfrak{A}_*(U)), M) \rightarrowtail H^p(\mathscr{D}^*(\mathfrak{A}_*; M)(U)) \twoheadrightarrow \operatorname{Hom}(H_p(\mathfrak{A}_*(U)), M),$$

which splits, nonnaturally; see [54] and [75].

2.4. If $\mathscr{L}^*$ is a c-soft differential sheaf, then with the conventional switching of indices, 2.3 becomes the exact sequence

$$\operatorname{Ext}(H^{p+1}(\Gamma_c(\mathscr{L}^*|U)), M) \rightarrowtail H_p(\mathscr{D}_*(\mathscr{L}^*; M)(U)) \twoheadrightarrow \operatorname{Hom}(H^p(\Gamma_c(\mathscr{L}^*|U)), M),$$

which is natural in the open set U and in M and splits.

2.5. Let $\mathscr{L}^*$ be a c-soft differential sheaf. By 1.10, $\mathscr{D}(\mathscr{L}^*; M)$ is flabby, and it is also torsion-free when $\mathscr{L}^*$ is injective.

Suppose that each $\mathscr{L}^p$ is a module over a given sheaf $\mathscr{R}$ of rings. Then since $\Gamma_c(\mathscr{L}^*|U)$ is an $\mathscr{R}(U)$-module, so is $\operatorname{Hom}(\Gamma_c(\mathscr{L}^*|U), M^*)$. It follows that $\mathscr{D}_p(\mathscr{L}^*; M)$ is an $\mathscr{R}$-module.

Taking $\mathscr{R} = \prod_p \mathscr{H}om(\mathscr{L}^p, \mathscr{L}^p)$ and recalling that a direct product of any family of Φ-soft sheaves is Φ-soft (II-Exercise 43), it follows that $\mathscr{D}_n(\mathscr{L}^*; M)$ is Φ-fine when each $\mathscr{L}^p$ is Φ-fine, where Φ is paracompactifying.

2.6. Note that we have the natural equalities

$$\boxed{\mathscr{D}(f\mathfrak{A}_*; M) = f\mathscr{D}(\mathfrak{A}_*; M)} \tag{7}$$

by (1), and

$$\boxed{\mathscr{D}(f_c\mathscr{L}^*; M) = f\mathscr{D}(\mathscr{L}^*; M),}$$

where $f : X \to Y$ is any map, since if $\mathfrak{L}^* = \Gamma_c\mathscr{L}^*$, then

$$\begin{aligned}
\mathscr{D}(f_c\mathscr{L}^*; M) &= \mathscr{D}(\Gamma_c\{f_c\mathscr{L}^*\}; M) && \text{by definition}\\
&= \mathscr{D}(f\mathfrak{L}^*; M) && \text{by 1.16}\\
&= f\mathscr{D}(\mathfrak{L}^*; M) && \text{by (7)}\\
&= f\mathscr{D}(\mathscr{L}^*; M) && \text{by definition.}
\end{aligned}$$

2.7. If M^* is replaced by another injective resolution of M, then $\mathscr{D}(\mathfrak{A}_*; M)$ changes by a chain equivalence (unique up to chain homotopy). Thus such a change does not affect homology and similarly does not affect any other matters that we shall deal with. Thus we may replace M^* by any injective resolution of M. For example, if $M = L$, it is sometimes convenient to let $L^0 = Q$, the field of quotients of L, and $L^1 = Q/L$.

3 Homology theory

Consider the canonical injective resolution $\mathscr{I}^*(X;L)$ of L (see II-3) and note that $\mathscr{I}^*(X;L)|U = \mathscr{I}^*(U;L)$ for $U \subset X$ open. For a sheaf $\mathscr{A}$ on X, we define

$$\boxed{\mathscr{C}_*(X;\mathscr{A}) = \mathscr{D}_*(\mathscr{I}^*(X;L)) \otimes \mathscr{A},}$$

$$\boxed{C_*^\Phi(X;\mathscr{A}) = \Gamma_\Phi(\mathscr{C}_*(X;\mathscr{A})),}$$

and

$$\boxed{H_p^\Phi(X;\mathscr{A}) = H_p(C_*^\Phi(X;\mathscr{A})).}$$

[Caution: $H_p(X;\mathscr{A})$ corresponds to homology based on infinite, locally finite chains, while $H_p^c(X;\mathscr{A})$ is analogous to classical homology.] Recall that by our notational conventions,

$$C_p(U;L) = \operatorname{Hom}(\Gamma_c(\mathscr{I}^p(U;L)), L^0) \oplus \operatorname{Hom}(\Gamma_c(\mathscr{I}^{p+1}(U;L)), L^1).$$

Thus $C_p^\Phi(X;\mathscr{A}) = 0$ for $p < -1$. Since $\mathscr{I}^*(X;L)$ is a module over the flabby sheaf $\mathscr{C}^0(X;L)$ of rings, 2.5 gives the following facts:

3.1. Proposition. *$\mathscr{C}_*(X;\mathscr{A})$ is a $\mathscr{C}^0(X;L)$-module and hence is Φ-fine for any paracompactifying family of supports Φ. Also, $\mathscr{C}_*(X;L)$ is flabby and torsion-free. Consequently, $\mathscr{C}_*(X;\mathscr{A})$ is an exact functor of $\mathscr{A}$, as is $C_*^\Phi(X;\mathscr{A})$ when Φ is paracompactifying.* □

Thus, when Φ is *paracompactifying* and $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is exact, we have the induced exact homology sequence

$$\boxed{\cdots \to H_p^\Phi(X;\mathscr{A}') \to H_p^\Phi(X;\mathscr{A}) \to H_p^\Phi(X;\mathscr{A}'') \to H_{p-1}^\Phi(X;\mathscr{A}') \to \cdots} \qquad (8)$$

Danger: It is important to realize that this sequence is not generally valid without the assumption that Φ is paracompactifying; see 3.12.

For $U \subset X$ open, (5) implies that $\mathscr{C}_*(X;\mathscr{A})|U = \mathscr{C}_*(U;\mathscr{A})$. Hence, for any family Φ of supports on X, we have the natural restriction homomorphism

$$C_*^\Phi(X;\mathscr{A}) \to \Gamma_{\Phi\cap U}(\mathscr{C}_*(X;\mathscr{A})|U) = C_*^{\Phi\cap U}(U;\mathscr{A})$$

and hence

$$\boxed{i_*^{U,X} : H_*^\Phi(X;\mathscr{A}) \to H_*^{\Phi\cap U}(U;\mathscr{A}).}$$

We remark that one may think *intuitively* of $H_*(U;\mathscr{A})$ as the homology of the pair $(X^+, X^+ - U)$, where X^+ is the one-point compactification of X; see 5.10.

By 2.4 with $\mathscr{L}^* = \mathscr{I}^*$, we have the fundamental exact sequence[4]

$$\boxed{0 \to \operatorname{Ext}(H_c^{p+1}(U;L), L) \to H_p(U;L) \to \operatorname{Hom}(H_c^p(U;L), L) \to 0,} \qquad (9)$$

[4]This sequence is the main resource for explicit computations.

which is natural with respect to inclusions $U \hookrightarrow V$ of open subsets of X [that is, with respect to the induced maps $H_p(V;L) \to H_p(U;L)$ and $H^p_c(U;L) \to H^p_c(V;L)$] and which is known to split by standard homological algebra. If $\mathscr{L}^*$ is any c-soft resolution of L, we have a similar sequence 2.4 (for $M = L$). There is a map $\mathscr{L}^* \to \mathscr{I}^*(X;L)$ of resolutions, unique up to chain homotopy, inducing a canonical map of (9) into the sequence of 2.4 that is an isomorphism on the ends. Thus the 5-lemma implies that

$$\boxed{H_p(X;L) \approx H_p(\Gamma(\mathscr{D}_*(\mathscr{L}^*;L)))} \tag{10}$$

when $\mathscr{L}^*$ is any c-soft resolution of L. (We shall generalize this later.)

By (9),

$$\boxed{H_{-1}(X;L) = \operatorname{Ext}(H^0_c(X;L),L) = 0}$$

because of Nöbeling's result implying that $H^0_c(X;L)$ is free; see the remarks in II-Exercise 34. Also see 5.13 for the case of arbitrary support families and coefficient sheaves.

The derived sheaf of the differential sheaf $\mathscr{C}_*(X;\mathscr{A})$ is called the *sheaf of local homology groups*, or simply the *homology sheaf*, and is denoted by

$$\boxed{\mathscr{H}_*(X;\mathscr{A}) = H_*(\mathscr{C}_*(X;\mathscr{A})).}$$

Since $\mathscr{C}_*(X;\mathscr{A})|U = \mathscr{C}_*(U;\mathscr{A})$, we have that

$$\boxed{\mathscr{H}_*(X;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}\,(U \mapsto H_*(U;\mathscr{A}))\,.}$$

The stalk

$$\boxed{\mathscr{H}_*(X;\mathscr{A})_x = \varinjlim H_*(U;\mathscr{A})}$$

(U ranging over the open neighborhoods of x) is called the *local homology group* at x of X.

Note that for coefficients in L, the local homology group at x is given by the sequence (9) upon passage to the limit over the neighborhoods U of x. That is, we have the exact sequence

$$\boxed{\varinjlim \operatorname{Ext}(H^{p+1}_c(U;L),L) \rightarrowtail \mathscr{H}_p(X;L)_x \twoheadrightarrow \varinjlim \operatorname{Hom}(H^p_c(U;L),L).} \tag{11}$$

It will be convenient for us to consider resolutions of L consisting of sheaves with the following properties:

3.2. Definition. *A sheaf $\mathscr{L}$ of L-modules on the locally compact space X is said to be "replete" if it is c-soft and if $\Gamma_c(\mathscr{L}|U)$ is divisible for all open sets $U \subset X$.*

3.3. Proposition. *If $\mathscr{L}$ is a replete sheaf on X and if $A \subset X$ is locally closed, then $\mathscr{L}|A$ and $\mathscr{L}_A$ are replete. An injective sheaf is replete.*

Proof. By I-6.6, $\Gamma_c(\mathscr{L}_A|U) = \Gamma_c(\mathscr{L}|A \cap U)$, which is divisible when A is open, by assumption. By II-9.3(iii), it is a quotient of $\Gamma_c(\mathscr{L}|U)$ (and hence it is divisible) when A is closed. The result follows from the fact that every locally closed set is the intersection of an open set with a closed set. The last statement follows from II-3.3. □

By 1.10 and 1.12 we have the following simple but basic fact:

3.4. Proposition. *If $\mathscr{L}^*$ is a replete differential sheaf, then $\mathscr{D}(\mathscr{L}^*)$ is torsion-free and flabby. If $\mathfrak{L}_*$ is a torsion-free flabby differential cosheaf, then $\mathscr{D}(\mathfrak{L}_*)$ is replete.* □

Let $\mathscr{L}^*$ be a replete resolution of L. Then $\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A} = \mathscr{S}\!heaf(U \mapsto \mathscr{D}(\mathscr{L}^*)(U) \otimes \mathscr{A}(U))$. Thus the derived sheaf is

$$\mathscr{H}_*(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) = \mathscr{S}\!heaf(U \mapsto H_*(\mathscr{D}(\mathscr{L}^*)(U) \otimes \mathscr{A}(U))).$$

Since $\mathscr{D}(\mathscr{L}^*)$ is torsion-free, we have the natural exact sequence

$$H_p(\mathscr{D}(\mathscr{L}^*)(U)) \otimes \mathscr{A}(U) \rightarrowtail H_p(\mathscr{D}(\mathscr{L}^*)(U) \otimes \mathscr{A}(U)) \twoheadrightarrow H_{p-1}(\mathscr{D}(\mathscr{L}^*)(U)) * \mathscr{A}(U).$$

Passing to the associated sheaves, we obtain an exact sequence of sheaves:

$$0 \to \mathscr{H}_p(\mathscr{D}(\mathscr{L}^*)) \otimes \mathscr{A} \to \mathscr{H}_p(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) \to \mathscr{H}_{p-1}(\mathscr{D}(\mathscr{L}^*)) * \mathscr{A} \to 0. \quad (12)$$

If we take $\mathscr{L}^* = \mathscr{I}^*(X;L)$, then (12) becomes

$$\boxed{0 \to \mathscr{H}_p(X;L) \otimes \mathscr{A} \to \mathscr{H}_p(X;\mathscr{A}) \to \mathscr{H}_{p-1}(X;L) * \mathscr{A} \to 0.} \quad (13)$$

On the stalk at x, (13) is the universal coefficient sequence of the chain complex $\mathscr{C}_*(X;L)_x \otimes \mathscr{A}_x$ and hence (13) is at least *pointwise* split. If $\dim_L X = n$, then $\mathscr{H}_n(X;L)$ is torsion-free since it is generated by $U \mapsto H_n(U;L) = \operatorname{Hom}(H_c^n(U;L), L)$. Therefore

$$\boxed{\mathscr{H}_k(X;\mathscr{A}) = 0 \quad \text{for} \quad k > \dim_L X} \quad (14)$$

by (13).

We already know from (10) that any homomorphism $\mathscr{L}^* \to \mathscr{I}^*(X;L)$ of resolutions, with $\mathscr{L}^*$ c-soft, induces a natural isomorphism

$$H_*(U;L) \xrightarrow{\approx} H_*(\mathscr{D}(\mathscr{L}^*)(U)). \quad (15)$$

Thus, for $\mathscr{L}^*$ replete we obtain a map of the sequence (13) into (12) that is an isomorphism on the ends. By the 5-lemma, the homomorphism in the middle is also an isomorphism:

$$\boxed{\mathscr{H}_p(X;\mathscr{A}) \xrightarrow{\approx} \mathscr{H}_p(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) \text{ for } \mathscr{L}^* \text{ replete.}} \quad (16)$$

3.5. Theorem. *Let $\mathscr{L}^*$ be a replete resolution of L and Φ a paracompactifying family of supports on X. If* $\dim_\Phi X < \infty$, *then the map*

$$H_p^\Phi(X;\mathscr{A}) \to H_p(\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}))$$

is an isomorphism for any sheaf $\mathscr{A}$ on X. (Also see 5.6.) *If $\mathscr{L}^*$ is injective, then this holds without the condition on dimension.*[5]

Proof. Since Φ is paracompactifying, $\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{A}$ is Φ-soft by II-16.31, and the result follows from (16) and IV-2.2. For the last statement note that for $\mathscr{I}^* = \mathscr{I}^*(X;L)$ there are homomorphisms $\varphi : \mathscr{I}^* \to \mathscr{L}^*$ and $\psi : \mathscr{L}^* \to \mathscr{I}^*$ of resolutions, and the compositions $\varphi\psi$ and $\psi\varphi$ are chain homotopic to the identity. This persists on passing to duals, and so $\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A})$ is chain equivalent to $\Gamma_\Phi(\mathscr{D}(\mathscr{I}^*) \otimes \mathscr{A}) = C_*^\Phi(X;\mathscr{A})$. □

It will be important to have this isomorphism for families Φ that are not paracompactifying (e.g., the family $\Phi|F$ on X where $F \subset X$ is closed). We also wish to have this result in certain cases without the condition on dimension. For this, the coefficient sheaf must be drastically restricted. In order to retain some degree of generality we make the following definition:

3.6. Definition. *A sheaf $\mathscr{M}$ on X will be called "elementary" if it is locally constant with finitely generated stalks (over L). For any sheaf $\mathscr{M}$ on X let $\Omega_\mathscr{M}$ be the smallest collection of open subsets $U \subset X$ satisfying the following three properties:*

(a) *$\mathscr{M}$ constant on $U \Rightarrow U \in \Omega_\mathscr{M}$,*

(b) *$U, V, U \cap V \in \Omega_\mathscr{M} \Rightarrow U \cup V \in \Omega_\mathscr{M}$,*

(c) *$U = \coprod_\alpha U_\alpha$, $U_\alpha \in \Omega_\mathscr{M}$ for all $\alpha \Rightarrow U \in \Omega_\mathscr{M}$.*

[*Clearly $\Omega_\mathscr{M}$ is the collection of all open sets that can be reached from those of type* (a) *by a finite number of operations of types* (b) *and* (c).]

We say that "the pair $(\mathscr{M}, \Phi)$ is elementary" if $\mathscr{M}$ is elementary and each $K \in \Phi$ is contained in a member of $\Omega_\mathscr{M}$.

Note that $(\mathscr{M}, \Phi)$ elementary $\Rightarrow$ $(\mathscr{M}|A, \Phi|A)$ elementary for $A \subset X$ locally closed.

Also note that if $\mathscr{M}$ is elementary then $(\mathscr{M}, \Phi)$ is elementary in any of the following three cases:

(i) $\Phi = c$,

(ii) $\mathscr{M}$ is constant,

(iii) each member of Φ is paracompact.

[5] By (15) we also have this for $\mathscr{A} = L$ and $\Phi = cld$ without the condition on dimension.

Case (i) is obvious and uses only properties (a) and (b). Case (ii) is because $X \in \Omega_{\mathscr{M}}$ in that case. Case (iii) follows from the next lemma:

3.7. Lemma. *Let Ω be a collection of open sets in a paracompact, locally compact space X. Suppose that Ω satisfies the following three properties:*

(a) *Every point of X has a neighborhood N such that $U \subset N \Rightarrow U \in \Omega$.*

(b) $U, V, U \cap V \in \Omega \Rightarrow U \cup V \in \Omega$.

(c) $U = \mathop{+}\limits_{\alpha} U_\alpha$, $U_\alpha \in \Omega$ *for all* $\alpha \Rightarrow U \in \Omega$.

Then $X \in \Omega$.

Proof. It is well known (see [19, I-12.11]) that a locally compact space is paracompact if and only if it is the topological sum of σ-compact subspaces. Thus, by (c) it suffices to assume that X is σ-compact. We can express $X = \bigcup_{i=1}^{\infty} F_i$, where the F_i are compact and $F_i \subset \operatorname{int} F_{i+1}$, using σ-compactness. By use of (a) and (b) we can find open sets $V_i \in \Omega$ that are finite unions of sets of the form in (a) such that V_i contains the compact set $\overline{F_{i+1} - F_i}$ and such that the V_{2j} are all disjoint and the V_{2j-1} are also all disjoint. By (c) $\bigcup V_{2j} \in \Omega$ and $\bigcup V_{2j-1} \in \Omega$. Each $V_i \cap V_{i+1}$ is a finite union of sets of the form in (a), and so by (b), $V_i \cap V_{i+1} \in \Omega$. Now, $(\bigcup V_{2j}) \cap (\bigcup V_{2j-1})$ is the *disjoint* union of the $V_i \cap V_{i+1}$, and so $X = (\bigcup V_{2j}) \cup (\bigcup V_{2j-1}) \in \Omega$ by (c) and (b). □

By II-5.13 we have the following result:

3.8. Lemma. *If $\mathscr{M}$ is elementary and $\mathscr{A}$ is flabby and torsion-free, then $\mathscr{A} \otimes \mathscr{M}$ is flabby.* □

We shall need the following result particularly for inclusion maps:

3.9. Proposition. *If $f : X \to Y$, $\mathscr{A}$ is a flabby, torsion-free sheaf on X, and $\mathscr{M}$ is an elementary sheaf on Y, then there is a natural isomorphism*

$$\boxed{(f\mathscr{A}) \otimes \mathscr{M} \approx f(\mathscr{A} \otimes f^*\mathscr{M}).}$$

(*Also see* 4.3.)

Proof. For $U \subset Y$ open, we have the natural map

$$\begin{aligned}(f\mathscr{A})(U) \otimes \mathscr{M}(U) &\to \mathscr{A}(f^{-1}U) \otimes (f^*\mathscr{M})(f^{-1}U) \\ &\to (\mathscr{A} \otimes f^*\mathscr{M})(f^{-1}U) \\ &= f(\mathscr{A} \otimes f^*\mathscr{M})(U),\end{aligned}$$

and hence we have the following map of the associated sheaves on Y:

$$f\mathscr{A} \otimes \mathscr{M} \to f(\mathscr{A} \otimes f^*\mathscr{M}). \tag{17}$$

Both sides of (17) are exact functors of *elementary* sheaves $\mathscr{M}$ [the right-hand side is exact since $\mathscr{A} \otimes f^*\mathscr{M}$ is flabby for $\mathscr{M}$, and hence $f^*\mathscr{M}$, elementary and since $f(\mathscr{A} \otimes f^*\mathscr{M})(U) = (\mathscr{A} \otimes f^*\mathscr{M})(f^{-1}U)$]. The assertion is clearly a local matter, in Y, so that we may assume $\mathscr{M}$ to be constant. Since the assertion is clear for $\mathscr{M}$ free and finitely generated, it follows for general $\mathscr{M}$ by passing to a quotient and using the exactness of both sides of (17). □

3.10. If $0 \to \mathscr{L}^* \to \mathscr{M}^* \to \mathscr{N}^* \to 0$ is an exact sequence of replete differential sheaves on X and $\mathscr{A}$ is any sheaf on X, then

$$\boxed{0 \to \mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A} \to \mathscr{D}(\mathscr{M}^*) \otimes \mathscr{A} \to \mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A} \to 0}$$

is exact because of 3.4. Also,

$$\boxed{0 \to \Gamma_\Phi(\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{M}^*) \otimes \mathscr{A}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) \to 0}$$

is exact for Φ paracompactifying since $\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A}$ is Φ-soft by II-9.18. It is also exact for arbitrary Φ when $\mathscr{A}$ is elementary, by 3.8.

3.11. Let $\mathscr{L}^*$ be a replete resolution of L, and $\mathscr{M}$ an elementary sheaf on X. Then $\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}$ is flabby, and it follows, as in 3.5, that the natural map

$$H_p^\Phi(X; \mathscr{M}) \to H_p(\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}))$$

is an isomorphism for *any* family Φ of supports on X for which $\dim_\Phi X < \infty$. We shall show that the condition on dimension can be replaced by the condition that $(\mathscr{M}, \Phi)$ be elementary, but the proof of that must be deferred until Section 5.

3.12. We remark that if $0 \to \mathscr{M} \to \mathscr{A} \to \mathscr{B} \to 0$ is exact, with $\mathscr{M}$ elementary, and if $\mathscr{L}^*$ is a replete resolution of L, then

$$0 \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{B}) \to 0$$

is exact for any family Φ, since $\mathscr{D}(\mathscr{L}^*)$ is torsion-free and $\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}$ is flabby. In particular, when $\mathscr{L}^* = \mathscr{I}^*(X; L)$, we obtain the exact sequence

$$\boxed{\cdots \to H_p^\Phi(X; \mathscr{M}) \to H_p^\Phi(X; \mathscr{A}) \to H_p^\Phi(X; \mathscr{B}) \xrightarrow{\partial} H_{p-1}^\Phi(X; \mathscr{M}) \to \cdots}$$

for *any* family Φ of supports on X.

Danger: It is important to realize that this exact sequence is not valid without the assumption that $\mathscr{M}$ be elementary (or that Φ be paracompactifying). For example, let $F \subset X$ be closed, $\Phi = c|F$, and consider the exact sequence $0 \to L_{X-F} \to L \to L_F \to 0$. Then (using some items later in the chapter)

$$\begin{aligned} H_p^{c|F}(X; L_{X-F}) &\approx H_p^{(c|F)|X-F}(X - F; L) && \text{by (34) on page 306} \\ &= 0 && \text{since supports are empty.} \end{aligned}$$

Also,

$$H_p^{c|F}(X;L) \approx H_p^c(F;L)$$

by 5.7, and

$$\begin{aligned} H_p^{c|F}(X;L_F) &= H_p(\Gamma_{c|F}(\mathscr{C}_*(X;L)\otimes L_F)) && \text{by definition} \\ &= H_p(\Gamma_c(\mathscr{C}_*(X;L)\otimes L_F)) && \text{clearly} \\ &= H_p^c(X;L_F) && \text{by definition} \\ &\approx H_p^c(X, X-F;L) && \text{by (35) on page 306.} \end{aligned}$$

Therefore, if the homology sequence for this example is exact, then we would have that $H_p^c(F;L) \approx H_p^c(X, X-F;L)$, which, of course, is almost never true.

3.13. If M is any L-module, then

$$\Gamma_c(\mathscr{D}(\mathscr{I}^*(X;L)))\otimes M \approx \Gamma_c(\mathscr{D}(\mathscr{I}^*(X;L))\otimes M)$$

by II-15.3, since $\mathscr{D}(\mathscr{I}^*(X;L))$ is flabby and torsion-free. Thus

$$\boxed{C_*^c(X;M) \approx C_*^c(X;L)\otimes M,}$$

and, by the algebraic universal coefficient theorem we obtain the split exact sequence[6]

$$\boxed{0 \to H_p^c(X;L)\otimes M \to H_p^c(X;M) \to H_{p-1}^c(X;L) * M \to 0.}$$

3.14. Let K be a principal ideal domain that is also an L-module. Then homology with coefficients in K (or in any sheaf of K-modules) has two interpretations depending on whether we regard K or L as the base ring. It is not always the case that these interpretations give isomorphic homology groups (as K-modules). In certain cases, however, they do, as we shall show in Section 15.

3.15. Suppose that L is a *field* and that X is compact. If $X = \varprojlim X_\alpha$ for an inverse system of finite polyhedra X_α, then

$$\begin{aligned} H_p(X;L) &\approx \operatorname{Hom}(H^p(X;L), L) \approx \operatorname{Hom}(\varinjlim H^p(X_\alpha;L), L) \\ &\approx \varprojlim \operatorname{Hom}(H^p(X_\alpha;L), L) \approx \varprojlim H_p(X_\alpha;L) \approx \check{H}_p(X;L), \end{aligned}$$

the classical Čech homology group of X. Since any compact space is the inverse limit of polyhedra, we conclude that

$$\boxed{H_p(X;L) \approx \check{H}_p(X;L) \text{ for } X \text{ compact and } L \text{ a field.}}$$

This is not generally true for $L = \mathbb{Z}$ since Borel-Moore homology is exact and Čech homology, over $\mathbb{Z}$, is not. In 5.18 and 5.19 we give conditions under which it does hold for $L = \mathbb{Z}$.

[6] Also see 15.5.

4 Maps of spaces

Let $f : X \to Y$ be a map between locally compact spaces. If $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X and Y respectively and if $k : \mathscr{B} \rightsquigarrow \mathscr{A}$ is an f-cohomomorphism, then the induced maps $I(\mathscr{B}_{f(x)}) \to I(\mathscr{A}_x)$ give rise to an f-cohomomorphism $k^0 : \mathscr{I}^0(Y;\mathscr{B}) \rightsquigarrow \mathscr{I}^0(X;\mathscr{A})$. This, in turn, induces a compatible f-cohomomorphism of the quotient sheaves: $\mathscr{J}^1(Y;\mathscr{B}) \rightsquigarrow \mathscr{J}^1(X;\mathscr{A})$. Continuing inductively, we obtain an f-cohomomorphism $k^* : \mathscr{I}^*(Y;\mathscr{B}) \rightsquigarrow \mathscr{I}^*(X;\mathscr{A})$ of resolutions.

We now restrict attention to the case in which $\mathscr{A}$ and $\mathscr{B}$ are the constant sheaves with stalks L. For $U \subset Y$ open, we obtain the canonical chain map

$$k_U^* : \Gamma_c(\mathscr{I}^*(U;L)) \to \Gamma_{f^{-1}c}(\mathscr{I}^*(f^{-1}(U);L))$$

and consequently the chain map

$$\kappa_U : \operatorname{Hom}(\Gamma_{f^{-1}c}(\mathscr{I}^*(f^{-1}U;L)),L^*) \to \operatorname{Hom}(\Gamma_c(\mathscr{I}^*(U;L)),L^*). \qquad (18)$$

If f were *proper*, we would have that $f^{-1}c = c$, so that (18), for $U = Y$, would provide the chain map $C_*(X;L) \to C_*(Y;L)$ and consequently a canonical homomorphism $f_* : H_*(X;L) \to H_*(Y;L)$. It is worthwhile to record the following immediate consequence of this definition.

4.1. Proposition. *If $f : X \to Y$ is proper, then there is the natural commutative diagram*

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Ext}(H_c^{p+1}(U^\bullet;L),L) & \to & H_p(U^\bullet;L) & \to & \operatorname{Hom}(H_c^p(U^\bullet;L),L) & \to 0 \\
& \downarrow{\scriptstyle \operatorname{Ext}(f^*,L)} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \operatorname{Hom}(f^*,L)} & \\
0 \to & \operatorname{Ext}(H_c^{p+1}(U;L),L) & \to & H_p(U;L) & \to & \operatorname{Hom}(H_c^p(U;L),L) & \to 0
\end{array}$$

with exact rows, for open sets $U \subset Y$, and where $U^\bullet = f^{-1}(U)$. □

To define f_* in the *general case* we must digress for a moment.

4.2. Definition. *If Φ is a family of supports on X, let $\Phi^\#$ (the "dual" of Φ) denote that family of supports on X consisting of all closed sets $K \subset X$ such that $K \cap K'$ is compact for every $K' \in \Phi$.*

Obviously, $c^\# = cld$ and $cld^\# = c$. By Exercise 17,

$$\boxed{(f^{-1}c)^\# = cld(c) = c(Y)}$$

is the family of fiberwise compact supports on X.

Now let $\mathscr{L}$ be a c-soft sheaf on X, let M be an L-module, and let $\Phi \supset c$ be a family of supports on X. The inclusion $\Gamma_c(\mathscr{L}) \hookrightarrow \Gamma_\Phi(\mathscr{L})$ induces the map

$$\tau : \operatorname{Hom}(\Gamma_\Phi(\mathscr{L}),M) \to \operatorname{Hom}(\Gamma_c(\mathscr{L}),M) = \Gamma(\mathscr{H}\!om(\Gamma_c\mathscr{L},M)).$$

We shall now define a natural homomorphism

$$\boxed{\eta : \Gamma_{\Phi^{\#}}(\mathscr{H}\!om(\Gamma_c\mathscr{L}, M)) \to \operatorname{Hom}(\Gamma_\Phi(\mathscr{L}), M)} \tag{19}$$

such that

$$\tau\eta = \varepsilon : \Gamma_{\Phi^{\#}}(\bullet) \hookrightarrow \Gamma(\bullet)$$

is the inclusion. Let $f \in \Gamma_{\Phi^{\#}}(\mathscr{H}\!om(\Gamma_c\mathscr{L}, M))$ with $|f| = K \in \Phi^{\#}$. If $s \in \Gamma_\Phi(\mathscr{L})$ with $|s| = K' \in \Phi$, then $s|K \in \Gamma_c(\mathscr{L}|K)$, because $K \cap K' \in c$. Since $\mathscr{L}$ is c-soft, it follows that there exists an element $s' \in \Gamma_c(\mathscr{L})$ such that $s'|K = s|K$. If $s'' \in \Gamma_c(\mathscr{L})$ also satisfies the equation $s''|K = s|K$, then $(s' - s'')|K = 0$, so that $|s' - s''| \cap |f| = \varnothing$, and by 1.11 it follows that $f(s') = f(s'')$. Thus it makes sense to define, for any such s',

$$\boxed{\eta(f)(s) = f(s').}$$

If $\mathscr{L}$ and $\mathscr{N}$ are c-soft sheaves on X and Y respectively and if $k : \mathscr{N} \rightsquigarrow \mathscr{L}$ is an f-cohomomorphism, then we have the induced homomorphism

$$\kappa : \operatorname{Hom}(\Gamma_{f^{-1}c}(\mathscr{L}), M) \to \operatorname{Hom}(\Gamma_c(\mathscr{N}), M)$$

and thus

$$\kappa\eta : \Gamma_{c(Y)}(\mathscr{H}\!om(\Gamma_c\mathscr{L}, M)) \to \operatorname{Hom}(\Gamma_c(\mathscr{N}), M).$$

Replacing Y by its open subsets U, this yields a natural homomorphism of sheaves:

$$\kappa\eta : f_c(\mathscr{H}\!om(\Gamma_c\mathscr{L}, M)) \to \mathscr{H}\!om(\Gamma_c\mathscr{N}, M).$$

If $\mathscr{L}^*$ and $\mathscr{N}^*$ are c-soft differential sheaves and $\kappa : \mathscr{N}^* \rightsquigarrow \mathscr{L}^*$ is an f-cohomomorphism, then we obtain the homomorphism

$$f_c\mathscr{D}(\mathscr{L}^*) \to \mathscr{D}(\mathscr{N}^*) \tag{20}$$

of differential sheaves, which specializes to

$$f_c\mathscr{C}_*(X; L) \to \mathscr{C}_*(Y; L)$$

when $\mathscr{L}^* = \mathscr{I}^*(X; L)$ and $\mathscr{N}^* = \mathscr{I}^*(Y; L)$. We shall now generalize this map to the case of general coefficient sheaves.

4.3. Lemma. *Let $\mathscr{L}$ be a sheaf on X and $\mathscr{A}$ a sheaf on Y. Then the natural homomorphism $f_c(\mathscr{L}) \otimes \mathscr{A} \to f_c(\mathscr{L} \otimes f^*\mathscr{A})$ of* IV-(21) *on page* 215 *is an isomorphism when $\mathscr{L}$ is c-soft and torsion-free or when $\mathscr{A}$ is torsion-free.*[7]

[7] Note that this is a special case of IV-Exercise 4.

Proof. Recall that this homomorphism is induced by the cup product

$$\Gamma_{c(U)}(\mathscr{L}|U^\bullet)\otimes\mathscr{A}(U)\to\Gamma_{c(U)}(\mathscr{L}|U^\bullet)\otimes(f^*\mathscr{A})(U^\bullet)\to\Gamma_{c(U)}(\mathscr{L}\otimes f^*\mathscr{A}|U^\bullet),$$

where $U^\bullet = f^{-1}(U)$, and that on the stalks at $y\in Y$ it becomes the cup product

$$\Gamma_c(\mathscr{L}|y^\bullet)\otimes M\to\Gamma_c(\mathscr{L}\otimes M|y^\bullet),$$

where M is the L-module $\mathscr{A}_y$. The latter map is an isomorphism by the universal coefficient theorem II-15.3 under our hypotheses. □

If $\mathscr{L}^*$ is a *replete* differential sheaf on X, then $\mathscr{D}(\mathscr{L}^*)$ is flabby and torsion-free, so that for any sheaf $\mathscr{A}$ on Y,

$$f_c(\mathscr{D}(\mathscr{L}^*))\otimes\mathscr{A}\approx f_c(\mathscr{D}(\mathscr{L}^*)\otimes f^*\mathscr{A}) \tag{21}$$

by 4.3. Combining this with (20), we obtain the natural homomorphism

$$k': f_c(\mathscr{D}(\mathscr{L}^*)\otimes f^*\mathscr{A})\to\mathscr{D}(\mathscr{N}^*)\otimes\mathscr{A}, \tag{22}$$

which specializes to

$$f_c\mathscr{C}_*(X;f^*\mathscr{A})\to\mathscr{C}_*(Y;\mathscr{A}). \tag{23}$$

The reader may verify that (23) is a $\mathscr{C}^0(Y;L)$-module homomorphism; see IV-3.4.

For any family Ψ of supports on Y, we apply the functor Γ_Ψ to (22) and use the fact that $\Gamma_\Psi f_c = \Gamma_{\Psi(c)}$, from IV-5.2, to obtain the chain map

$$\Gamma_{\Psi(c)}(\mathscr{D}(\mathscr{L}^*)\otimes f^*\mathscr{A})\to\Gamma_\Psi(\mathscr{D}(\mathscr{N}^*)\otimes\mathscr{A}). \tag{24}$$

This specializes to a canonical chain map

$$\boxed{C_*^{\Psi(c)}(X;f^*\mathscr{A})\to C_*^{\Psi}(Y;\mathscr{A}).} \tag{25}$$

In turn, (25) induces the homomorphism

$$\boxed{f_*: H_p^{\Psi(c)}(X;f^*\mathscr{A})\to H_p^{\Psi}(Y;\mathscr{A}).} \tag{26}$$

(Also see Exercise 7.)

4.4. Definition. *Let $f: X\to Y$ and let Φ and Ψ be families of supports on X and Y respectively. Then we say that f is c-proper, with respect to Φ and Ψ, if $\Phi\subset\Psi(c)$. A homotopy $F: X\times\mathbb{I}\to Y$ is said to be c-proper, with respect to Φ and Ψ, if it is a c-proper map with respect to the families $\Phi\times\mathbb{I}$ and Ψ.*

Recall from IV-5.3 that a family Φ of supports on X is contained in $\Psi(c)$ $\Leftrightarrow$ ($f(\Phi)\subset\Psi$ and $f|K: K\to Y$ is proper for each $K\in\Phi$). Therefore we have:

4.5. Proposition. *A map $f : X \to Y$ between locally compact Hausdorff spaces is c-proper, with respect to Φ and Ψ, $\Leftrightarrow$ for each $K \in \Phi$, $f|K : K \to Y$ is proper and $f(K) \in \Psi$.* □

In this situation we have the natural induced homomorphisms

$$\boxed{f_* : H_p^\Phi(X; f^*\mathscr{A}) \to H_p^\Psi(Y; \mathscr{A}) \quad \text{for } c\text{-proper maps } f : X \to Y.}$$

In particular, f_* is always defined for *compact* supports (both Φ and Ψ), as should be expected. For closed supports, f_* is defined when f is a proper map.

Let $\mathscr{L}^*$ and $\mathscr{N}^*$ be replete resolutions of L on X and Y respectively, and let $\mathscr{L}^* \to \mathscr{I}^*(X; L)$ and $\mathscr{N}^* \to \mathscr{I}^*(Y; L)$ be homomorphisms of resolutions. For any f-cohomomorphism $k : \mathscr{N}^* \rightsquigarrow \mathscr{L}^*$ of resolutions, the diagram

$$\begin{array}{ccc} \mathscr{N}^* & \rightsquigarrow & \mathscr{L}^* \\ \downarrow & & \downarrow \\ \mathscr{I}^*(Y; L) & \rightsquigarrow & \mathscr{I}^*(X; L) \end{array}$$

of cohomomorphisms commutes up to chain homotopy, and it follows that the induced diagram

$$\begin{array}{ccc} C_*^{\Psi(c)}(X; f^*\mathscr{A}) & \longrightarrow & C_*^\Psi(Y; \mathscr{A}) \\ \downarrow & & \downarrow \\ \Gamma_{\Psi(c)}(\mathscr{D}(\mathscr{L}^*) \otimes f^*\mathscr{A}) & \to & \Gamma_\Psi(\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A}) \end{array}$$

also commutes up to chain homotopy. Thus the following diagram commutes:

$$\begin{array}{ccc} H_*^{\Psi(c)}(X; f^*\mathscr{A}) & \xrightarrow{f_*} & H_*^\Psi(Y; \mathscr{A}) \\ \Big\downarrow & & \Big\downarrow \\ H_*(\Gamma_{\Psi(c)}(\mathscr{D}(\mathscr{L}^*) \otimes f^*\mathscr{A})) & \xrightarrow{k'_*} & H_*(\Gamma_\Psi(\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A})). \end{array} \tag{27}$$

Note that the vertical maps in (27) are often isomorphisms; see 3.5 and 5.6.

The reader may make the straightforward verification of the fact that for

$$W \xrightarrow{g} X \xrightarrow{f} Y$$

we have

$$\boxed{(fg)_* = f_* g_*.}$$

Mapping X to a point induces, for any L-module M, the canonical "augmentation"

$$\boxed{\varepsilon : H_0^c(X; M) \to M,}$$

which is surjective because of the composition $\star \to X \to \star$. The kernel of ε is, of course, called the *reduced* homology group of X in degree zero and is denoted by $\tilde{H}_0^c(X; M)$.

5 Subspaces and relative homology

Let $A \subset X$ be locally closed and hence locally compact. If Φ is a family of supports on X, note that for the inclusion $i = i^{X,A} : A \hookrightarrow X$ we have that $\Phi(c) = \Phi|A$; see IV-5.3(b). Therefore, Section 4 provides the chain map

$$\boxed{C_*^{\Phi|A}(A; \mathscr{A}|A) \to C_*^{\Phi}(X; \mathscr{A})} \tag{28}$$

and the induced homomorphism

$$\boxed{i_* : H_p^{\Phi|A}(A; \mathscr{A}|A) \to H_p^{\Phi}(X; \mathscr{A})}$$

for any sheaf $\mathscr{A}$ on X. In order to define relative homology, we shall show that (28) is a monomorphism. To do this we must consider the construction in Section 4 in more detail for this special case.

Suppose that $\mathscr{A}$ is a sheaf on X and that $\mathscr{B}$ is a sheaf on A for which there is a surjective i-cohomomorphism $k : \mathscr{A} \rightsquigarrow \mathscr{B}$ (i.e., an epimorphism $\mathscr{A}|A \to \mathscr{B}$). Suppose, moreover, that for each $x \in A$ we are given a splitting homomorphism $j_x : \mathscr{B}_x \to \mathscr{A}_x$, i.e., such that $k_x j_x$ is the identity. We then have the induced maps

$$I(\mathscr{A}_x) \underset{I(j_x)}{\overset{I(k_x)}{\rightleftarrows}} I(\mathscr{B}_x).$$

Taking the product of these maps, we obtain the maps

$$\mathscr{I}^0(X; \mathscr{A})(U) = \prod_{x \in U} I(\mathscr{A}_x) \underset{j_U^0}{\overset{k_U^0}{\rightleftarrows}} \prod_{x \in U \cap A} I(\mathscr{B}_x) = \mathscr{I}^0(A; \mathscr{B})(U \cap A)$$

with $k_U^0 j_U^0 = 1$, for $U \subset X$ open. These induce

$$\mathscr{I}^0(X; \mathscr{A})_x \underset{j_x^0}{\overset{k_x^0}{\rightleftarrows}} \mathscr{I}^0(A; \mathscr{B})_x$$

with $k_x^0 j_x^0 = 1$ and hence

$$\Gamma_\Phi(\mathscr{I}^0(X; \mathscr{A})) \underset{j^0}{\overset{k^0}{\rightleftarrows}} \Gamma_{\Phi \cap A}(\mathscr{I}^0(A; \mathscr{B}))$$

with $k^0 j^0 = 1$. (Note that j^0 does not commute with the augmentations, so that we cannot pass directly to quotient sheaves here.) Now,

$$\operatorname{Ker} k_U^0 = \prod_{x \in U \cap A} \operatorname{Ker} I(k_x) \times \prod_{x \in U - A} I(\mathscr{A}_x),$$

and $\operatorname{Ker} I(k_x) \approx I(\mathscr{A}_x)/\operatorname{Im} I(j_x)$ is an injective L-module, since it is divisible. Also, $\operatorname{Ker} k_U^0$ is the value on U of the kernel $\mathscr{K}$ of the homomorphism

$\mathscr{I}^0(X;\mathscr{A}) \to i\mathscr{I}^0(A;\mathscr{B})$ canonically associated with the i-cohomomorphism k^0. From this and II-(6) on page 41, we deduce that $\mathscr{K}$ is an injective sheaf. In particular, each stalk $\mathscr{K}_x$ is divisible by II-3.3 and hence is injective. Let $x \in A$, and consider the diagram

$$\begin{array}{ccccccccc} & & \mathscr{A}_x & \xrightarrow{k_x} & \mathscr{B}_x & \to 0 \\ & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{K}_x & \to \mathscr{I}^0(X;\mathscr{A})_x & \xrightarrow{k_x^0} & \mathscr{I}^0(A;\mathscr{B})_x & \to 0 \\ & \downarrow g & \downarrow & & \downarrow & \\ 0 \to & K & \to \mathscr{J}^1(X;\mathscr{A})_x & \xrightarrow{'k_x} & \mathscr{J}^1(A;\mathscr{B})_x & \to 0 \\ & & \downarrow & & \downarrow & \\ & & 0 & & 0, & \end{array}$$

which is commutative (defining $'k_x$) and which has exact rows and columns (defining K). It follows by a diagram chase that g is surjective. Thus K is divisible and hence injective. Since K is injective, the bottom row of this diagram splits, that is, for each $x \in A$ there is a homomorphism $'j_x : \mathscr{J}^1(A;\mathscr{B})_x \to \mathscr{J}^1(X;\mathscr{A})_x$ with $'k_x{'j_x} = 1$. But this is just our initial situation with $\mathscr{J}^1(X;\mathscr{A})$ replacing $\mathscr{A}$ and $\mathscr{J}^1(A;\mathscr{B})$ replacing $\mathscr{B}$, etc. It follows that by induction and then taking $\mathscr{A} = L$ and $\mathscr{B} = L$, we have the maps

$$\Gamma_\Phi(\mathscr{I}^*(X;L)) \underset{j^*}{\overset{k^*}{\rightleftarrows}} \Gamma_{\Phi\cap A}(\mathscr{I}^*(A;L))$$

with $k^*j^* = 1$. (Note, however, that j^* is *not* a chain map.)

Translating (18) and (19), we have the diagram

$$\begin{array}{ccc} \Gamma_{cld|A}(\mathscr{H}om(\Gamma_c\mathscr{I}^*(A;L),L^*)) & \xrightarrow{\varepsilon} & \operatorname{Hom}(\Gamma_c(\mathscr{I}^*(A;L)),L^*) \\ \downarrow \eta & \nearrow \tau & \\ \operatorname{Hom}(\Gamma_{c\cap A}(\mathscr{I}^*(A;L)),L^*) & \underset{\gamma}{\overset{\kappa}{\rightleftarrows}} & \operatorname{Hom}(\Gamma_c(\mathscr{I}^*(X;L)),L^*) \end{array}$$

in which ε is the inclusion, $\varepsilon = \tau\eta$, and $\gamma\kappa = 1$ (where κ and γ are induced by k^* and j^* respectively). Note that $cld|A = (c \cap A)^{\#}$ by Exercise 17(c).

Let $f \in \operatorname{Hom}(\Gamma_c(\mathscr{I}^*(A;L)),L^*) = \Gamma(\mathscr{H}om(\Gamma_c\mathscr{I}^*(A;L),L^*))$. Since this is a torsion-free sheaf on A, we have that $|mf| = |f|$ for all $0 \neq m \in L$. Assume that $f = \varepsilon(g)$ and that $\kappa\eta(g) = mh$ for some $m \in L$ and

$$h \in \operatorname{Hom}(\Gamma_c(\mathscr{I}^*(X;L)),L^*).$$

Let $f' = \tau\gamma(h)$. Then $|f'| = |mf'| = |\tau\gamma\kappa\eta(g)| = |\tau\eta(g)| = |\varepsilon(g)| = |f|$, and it follows that $f' = \varepsilon(g')$ for some g'. We have $\varepsilon(mg') = mf' = \tau\gamma(mh) = \tau\gamma\kappa\eta(g) = \varepsilon(g)$ and hence $g = mg'$. Therefore $m(\kappa\eta(g')) = \kappa\eta(g) = mh$, and it follows that $h = \kappa\eta(g')$ by torsion-freeness. We have shown that if $mh \in \operatorname{Im}(\kappa\eta)$, then $h \in \operatorname{Im}(\kappa\eta)$. It follows from this fact that the quotient of $\operatorname{Hom}(\Gamma_c(\mathscr{I}^*(X;L)),L^*)$ by the image of the monomorphism

$\kappa\eta$ is torsion-free. This also holds if we replace X by an open subset $U \subset X$ and A by $A \cap U$. Thus, the map

$$i_c \mathscr{C}_*(A; L) \to \mathscr{C}_*(X; L)$$

is a *monomorphism*, and the quotient presheaf, and hence the quotient sheaf, of this is *torsion-free*. Recall that $i_c \mathscr{B} = \mathscr{B}^X$ for any sheaf $\mathscr{B}$ on A by IV-3.8. Thus we have the canonical *monomorphism*

$$\mathscr{C}_*(A; L)^X \otimes \mathscr{A} \rightarrowtail \mathscr{C}_*(X; L) \otimes \mathscr{A}$$

for any sheaf $\mathscr{A}$ on X. By (21) this map can be identified with a map

$$\mathscr{C}_*(A; \mathscr{A}|A)^X \rightarrowtail \mathscr{C}_*(X; \mathscr{A}).$$

We shall regard $\mathscr{C}_*(A; \mathscr{A}|A)^X$ as a *subsheaf* of $C_*(X; \mathscr{A})$.[8]

For any family Φ of supports on X, the map (28) arises from this by applying Γ_Φ and noting that $\Gamma_\Phi(\mathscr{B}^X) = \Gamma_{\Phi|A}(\mathscr{B})$ for sheaves $\mathscr{B}$ on A by I-6.6. It follows that the canonical map (28) is a *monomorphism* for each $\mathscr{A}$. Thus we shall define the relative chain groups by the exactness of the sequence

$$\boxed{0 \to C_*^{\Phi|A}(A; \mathscr{A}|A) \to C_*^{\Phi}(X; \mathscr{A}) \to C_*^{\Phi}(X, A; \mathscr{A}) \to 0,} \tag{29}$$

and we also define the relative homology as

$$\boxed{H_p^{\Phi}(X, A; \mathscr{A}) = H_p(C_*^{\Phi}(X, A; \mathscr{A})).}$$

As usual, we obtain from (29) the exact homology sequence

$$\boxed{\cdots \to H_p^{\Phi|A}(A; \mathscr{A}) \to H_p^{\Phi}(X; \mathscr{A}) \to H_p^{\Phi}(X, A; \mathscr{A}) \to H_{p-1}^{\Phi|A}(A; \mathscr{A}) \to \cdots}$$

of the pair (X, A).

We also define

$$\boxed{\mathscr{C}_*(X, A; \mathscr{A}) = \mathscr{C}_*(X; \mathscr{A}) / \mathscr{C}_*(A; \mathscr{A})^X \approx \mathscr{C}_*(X, A; L) \otimes \mathscr{A}.}$$

Note that this sheaf vanishes on $\operatorname{int}(A)$ and that it is a $\mathscr{C}^0(X; L)$-module. In general, $\mathscr{C}_*(A; \mathscr{A})^X$ is not Φ-acyclic, so that we *cannot* generally assert that

$$C_*^{\Phi}(X, A; \mathscr{A}) \approx \Gamma_\Phi(\mathscr{C}_*(X, A; \mathscr{A})). \tag{30}$$

However, (30) does hold when Φ is paracompactifying, since $\mathscr{C}_*(A; \mathscr{A})^X$ is then Φ-soft, and it also holds when A is closed and $\mathscr{A}$ is elementary, by II-5.4, since $\mathscr{C}_*(A; \mathscr{A})^X$ is then flabby by 3.8 and II-5.8. We record these remarks for future reference:

[8] Note that these sheaves coincide on $\operatorname{int}(A)$.

5.1. Proposition. *The sheaf $\mathscr{C}_*(X,A;L)$ is torsion-free. Also, the sheaf $\mathscr{C}_*(X,A;\mathscr{A})$ vanishes on* $\operatorname{int}(A)$ *and it is a $\mathscr{C}^0(X;L)$-module (whence it is Φ-fine when Φ is paracompactifying). For A closed and $\mathscr{M}$ elementary, $\mathscr{C}_*(X,A;\mathscr{M})$ is flabby.* □

We also let

$$\boxed{\mathscr{H}_p(X,A;\mathscr{A}) = \mathscr{H}_p(\mathscr{C}_*(X,A;\mathscr{A})).}$$

Now, $\mathscr{C}_*(X,A;\mathscr{A})$ can also be described as the sheaf generated by the presheaf

$$\begin{aligned} U \mapsto \mathscr{C}_*(X;\mathscr{A})(U)/\mathscr{C}_*(A;\mathscr{A})^X(U) &= C_*(U;\mathscr{A})/C_*^{cld_U|A}(U\cap A;\mathscr{A}) \\ &= C_*(U,U\cap A;\mathscr{A}). \end{aligned}$$

It follows that

$$\mathscr{H}_p(X,A;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}(U \mapsto H_p(U,U\cap A;\mathscr{A})).$$

Since A is locally closed, we have

$$\mathscr{C}_*(X;\mathscr{A}_A) = \mathscr{C}_*(X;\mathscr{A})_A \tag{31}$$

since

$$\mathscr{C}_*(X;L)\otimes\mathscr{A}_A = \mathscr{C}_*(X;L)\otimes\mathscr{A}\otimes L_A = \mathscr{C}_*(X;\mathscr{A})\otimes L_A = \mathscr{C}_*(X;\mathscr{A})_A.$$

For U open in X we have

$$\mathscr{C}_*(X;\mathscr{A})_U = (\mathscr{C}_*(X;\mathscr{A})|U)^X = \mathscr{C}_*(U;\mathscr{A})^X,$$

so that

$$\mathscr{C}_*(X;\mathscr{A}_U) = \mathscr{C}_*(U;\mathscr{A})^X \tag{32}$$

by (31). It follows that for F closed in X and with $U = X - F$ we have

$$\mathscr{C}_*(X;\mathscr{A}_F) = \mathscr{C}_*(X,U;\mathscr{A}). \tag{33}$$

Applying Γ_Φ to (32) and using I-6.6, we obtain

$$\boxed{H_*^\Phi(X;\mathscr{A}_U) \approx H_*^{\Phi|U}(U;\mathscr{A}|U)} \tag{34}$$

for arbitrary Φ. Similarly, from (33) we obtain

$$\boxed{H_*^\Phi(X;\mathscr{A}_F) \approx H_*^\Phi(X,U;\mathscr{A})} \tag{35}$$

when Φ is paracompactifying.

Moreover, when Φ is paracompactifying, the homology sequence of the pair (X, U) can be identified, by (34) and (35), with the sequence (8) associated with the coefficient sequence

$$0 \to \mathscr{A}_U \to \mathscr{A} \to \mathscr{A}_F \to 0.$$

If $f : (X, A) \to (Y, B)$ is a map of pairs and if Φ and Ψ are families of supports on X and Y respectively with $\Phi \subset \Psi(c)$, then $\Phi|A \subset (\Psi|B)(c)$, and for any sheaf $\mathscr{A}$ on Y we obtain a canonical chain map

$$\boxed{C_*^\Phi(X, A; f^*\mathscr{A}) \to C_*^\Psi(Y, B; \mathscr{A})}$$

and a map of the homology sequence of (X, A) into that of (Y, B).

The following is the basic excision result for Borel-Moore homology:

5.2. Theorem. *Let $A \subset X$ and suppose that Φ is paracompactifying. Let V be a subset of X with $X - V$ locally closed and $\overline{V} \subset \operatorname{int}(A)$. Then the map*

$$H_p^{\Phi|X-V}(X - V, A - V; \mathscr{A}) \to H_p^\Phi(X, A; \mathscr{A}),$$

induced by inclusion, is an isomorphism for any sheaf $\mathscr{A}$ on X. This also holds for arbitrary Φ when A is closed and $\mathscr{A}$ is elementary.

Proof. We shall use the relatively obvious fact that for an open subset $U \subset X$, the canonical map $\mathscr{C}_*(U; \mathscr{A}|U)^X \to \mathscr{C}_*(X; \mathscr{A})$ of Section 4 is a monomorphism onto the subsheaf $\mathscr{C}_*(X; \mathscr{A})_U$. Consider the diagram (coefficients in $\mathscr{A}$)

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{C}_*(A - \overline{V})^X & \to & \mathscr{C}_*(X - \overline{V})^X & \to & \mathscr{C}_*(X - \overline{V}, A - \overline{V})^X & \to & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & \mathscr{C}_*(A)^X & \xrightarrow{i} & \mathscr{C}_*(X) & \longrightarrow & \mathscr{C}_*(X, A) & \longrightarrow & 0
\end{array}$$

in which the rows are exact.

For $x \in X - \overline{V}$, the maps f_x and g_x are isomorphisms (by the preceding remark), and hence h_x is also an isomorphism. For $x \in \overline{V} \subset \operatorname{int}(A)$, i_x is an isomorphism, so that $\mathscr{C}_*(X, A)_x = 0$. Now, $\mathscr{C}_*(X - \overline{V}, A - \overline{V})_x^X = 0$ for $x \in \overline{V}$, so that h_x is trivially an isomorphism for $x \in \overline{V}$.

Thus h is an isomorphism. Similarly, we see that the map

$$\mathscr{C}_*(X - \overline{V}, A - \overline{V})^{X-V} \to \mathscr{C}_*(X - V, A - V)$$

is an isomorphism. It follows that the canonical map

$$\mathscr{C}_*(X - V, A - V; \mathscr{A})^X \to \mathscr{C}_*(X, A; \mathscr{A})$$

is an isomorphism, and hence that the induced map

$$C_*^{\Phi|X-V}(X - V, A - V; \mathscr{A}) \to C_*^\Phi(X, A; \mathscr{A})$$

(obtained by applying Γ_Φ) is also an isomorphism, which is an even stronger result than that claimed by the theorem. □

5.3. Theorem. *If Φ is a paracompactifying family of supports on X and $\mathscr{A}$ is any sheaf on X, then the canonical map*

$$\varinjlim_{K\in\Phi} H_*(K;\mathscr{A}|K) \to H_*^{\Phi}(X;\mathscr{A})$$

is an isomorphism.[9]

Proof. It is sufficient to show that $\varinjlim C_*^{\Phi}(X,K;\mathscr{A}) = 0$, whence $\varinjlim H_*^{\Phi}(X,K;\mathscr{A}) = 0$. But $C_*^{\Phi}(X,K;\mathscr{A}) = \Gamma_{\Phi}(\mathscr{C}_*(X,K;\mathscr{A}))$ and the sheaf $\mathscr{C}_*(X,K;\mathscr{A})$ vanishes on $\operatorname{int}(K)$. The assertion follows from the fact that each member of Φ has a neighborhood in Φ. [If $s \in \Gamma_{\Phi}(\mathscr{C}_*(X,K;\mathscr{A}))$ has $|s| \subset \operatorname{int}(K')$, $K' \in \Phi$, then the image of s in $\Gamma_{\Phi}(\mathscr{C}_*(X,K';\mathscr{A}))$ will be zero.] □

We wish to obtain an analogue of 5.3 for nonparacompactifying families of supports. For this, the coefficient sheaf will have to be taken to be elementary. We shall now work towards this goal.

5.4. Let $\mathscr{L}^*$ be a replete differential sheaf on X and let $A \subset X$ be locally closed with $i : A \hookrightarrow X$ the inclusion. Since $i_c(\mathscr{L}^*|A) = \mathscr{L}^*_A$ by IV-3.8, 2.6 and 3.9 yield the isomorphism

$$\mathscr{D}(\mathscr{L}^*_A) \otimes \mathscr{M} \approx i(\mathscr{D}(\mathscr{L}^*|A) \otimes \mathscr{M}|A)$$

when $\mathscr{M}$ is elementary. If $A = F$ is closed, then $i\mathscr{B} = i_c\mathscr{B} = \mathscr{B}^X$ for any sheaf $\mathscr{B}$ on F, so that 2.6 and 4.3 yield the isomorphism

$$\mathscr{D}(\mathscr{L}^*_F) \otimes \mathscr{A} \approx (\mathscr{D}(\mathscr{L}^*|F) \otimes \mathscr{A}|F)^X$$

for any sheaf $\mathscr{A}$ on X.

5.5. Proposition. *If $\mathscr{L}^*$ is replete, $F\subset X$ is closed, and $\mathscr{M}$ is elementary, then there are natural isomorphisms*

$$\Gamma_{\Phi|F}(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}) \approx \Gamma_{\Phi}(\mathscr{D}(\mathscr{L}^*_F) \otimes \mathscr{M}) \approx \Gamma_{\Phi|F}(\mathscr{D}(\mathscr{L}^*|F) \otimes \mathscr{M}|F)$$

for any family Φ of supports on X. This also holds when $\mathscr{L}^*$ is only c-soft and $\mathscr{M} = L$.

Proof. The second isomorphism follows from 5.4 and I-6.6. Let $U = X - F$. The exact sequence $0 \to \mathscr{L}^*_U \to \mathscr{L}^* \to \mathscr{L}^*_F \to 0$ induces an exact sequence $0 \to \mathscr{D}(\mathscr{L}^*_F) \otimes \mathscr{M} \to \mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M} \to \mathscr{D}(\mathscr{L}^*_U) \otimes \mathscr{M} \to 0$ since $\mathscr{L}^*_U$ is replete [whence $\mathscr{D}(\mathscr{L}^*_U)$ is torsion-free]. These sheaves are all flabby, since $\mathscr{M}$ is elementary, by 3.8. Consider the commutative diagram

$$\begin{array}{ccc} (\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M})(X) & \to & (\mathscr{D}(\mathscr{L}^*_U) \otimes \mathscr{M})(X) \\ \downarrow & & \downarrow \\ (\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M})(U) & \to & (\mathscr{D}(\mathscr{L}^*_U) \otimes \mathscr{M})(U). \end{array}$$

[9] Also see 5.6.

The bottom map is an isomorphism since $(\mathscr{D}(\mathscr{L}_F^*)\otimes\mathscr{M})(U)=0$. The vertical map on the right is also an isomorphism by 5.4. Consideration of supports shows that we have the commutative diagram

$$\begin{array}{ccc}
\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}) & \xrightarrow{h} & \Gamma_\Phi(\mathscr{D}(\mathscr{L}_U^*)\otimes\mathscr{M}) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle\approx} \\
\Gamma_{\Phi\cap U}((\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M})|U) & \xrightarrow{\approx} & \Gamma_{\Phi\cap U}((\mathscr{D}(\mathscr{L}_U^*)\otimes\mathscr{M})|U).
\end{array}$$

Thus $\Gamma_{\Phi|F}(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M})=\operatorname{Ker} g=\operatorname{Ker} h\approx\Gamma_\Phi(\mathscr{D}(\mathscr{L}_F^*)\otimes\mathscr{M})$, as claimed. The last statement follows easily from the foregoing proof with minor changes. □

The following is our major result on elementary coefficient sheaves.

5.6. Theorem. *Let $(\mathscr{M},\Phi)$ be elementary.*[10] *Then:*

(i) *The natural map* $\varinjlim_{K\in\Phi} H_*(K;\mathscr{M})\to H_*^\Phi(X;\mathscr{M})$ *is an isomorphism.*

(ii) *The canonical map* $H_*^\Phi(X;\mathscr{M})\to H_*(\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}))$ *is an isomorphism for any replete resolution* $\mathscr{L}^*$ *of* L.

If $\Phi=c$, *then these statements hold for arbitrary sheaves* $\mathscr{M}$.[11]

Proof. Note that we already have part (ii) for $\mathscr{M}=L$ and $\Phi=cld$ by (10) on page 293.

Let $\mathscr{L}^*$ be a replete resolution of L on X and let $\mathscr{L}^*\to\mathscr{I}^*(X;L)$ be a homomorphism of resolutions. This induces a homomorphism $\mathscr{L}^*|K\to\mathscr{I}^*(X;L)|K\to\mathscr{I}^*(K;L)$ for $K\in\Phi$, which in turn induces

$$\begin{aligned}\mathscr{C}_*(K;\mathscr{M})=\mathscr{D}(\mathscr{I}^*(K;L))\otimes\mathscr{M}|K&\to\mathscr{D}(\mathscr{I}^*(X;L)|K)\otimes\mathscr{M}|K\\&\to\mathscr{D}(\mathscr{L}^*|K)\otimes\mathscr{M}|K.\end{aligned}$$

Using 5.4, this gives rise to the commutative diagram

$$\begin{array}{ccccc}
\mathscr{C}_*(K;\mathscr{M})^X & \to & (\mathscr{D}(\mathscr{I}^*(X;L)|K)\otimes\mathscr{M}|K)^X & \to & (\mathscr{D}(\mathscr{L}^*|K)\otimes\mathscr{M}|K)^X \\
\Big| & & \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\approx} \\
\Big| & & \mathscr{D}(\mathscr{I}^*(X;L)_K)\otimes\mathscr{M} & \longrightarrow & \mathscr{D}(\mathscr{L}_K^*)\otimes\mathscr{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{C}_*(X;\mathscr{M}) & \xrightarrow{=} & \mathscr{D}(\mathscr{I}^*(X;L))\otimes\mathscr{M} & \longrightarrow & \mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}.
\end{array}$$

[10] This theorem also holds under the hypothesis that $\mathscr{M}$ is elementary and $\dim_{\Phi,L}X<\infty$, the proof of which uses 3.11. This case was included in the first edition, but it has been dropped in this edition from the results using this theorem because of the unlikelihood of its usefulness beyond that covered by the $(\mathscr{M},\Phi)$ elementary case, which gives stronger results than the "Φ-elementary" condition used in the first edition. E.g., if X has finite covering dimension, then X is paracompact, and so any elementary sheaf $\mathscr{M}$ is $(\mathscr{M},\Phi)$-elementary for any Φ.

[11] Also see 3.5 and 5.3.

The reader may check that the vertical homomorphism on the left is the canonical inclusion described at the beginning of this section. Applying Γ_Φ, we obtain the following commutative diagram (since $K \in \Phi$):

$$\begin{array}{ccc} C_*(K;\mathscr{M}) & \xrightarrow{f} & \Gamma(\mathscr{D}(\mathscr{L}^*|K)\otimes\mathscr{M}|K) \\ \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle g} \\ C_*^\Phi(X;\mathscr{M}) & \xrightarrow{h} & \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}). \end{array}$$

By 5.5, since $K \in \Phi$, g can be identified with the inclusion

$$\Gamma_{\Phi|K}(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}) \hookrightarrow \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}),$$

and hence it becomes an isomorphism upon passage to the direct limit over $K \in \Phi$.

Assume for the moment that for any replete $\mathscr{L}^*$, f induces an isomorphism in homology. If $\mathscr{L}^* = \mathscr{I}^*(X;L)$, then $h = 1$, and thus we would obtain part (i) of our theorem; that is, k would induce an isomorphism in the limit. But then for *any* replete $\mathscr{L}^*$, h would induce an isomorphism; that is, we would obtain part (ii) of our theorem.

Thus it suffices to prove part (ii) (only) of the theorem for X replaced by any $K \in \Phi$. That is, it suffices to prove (ii) for the case in which $\Phi = cld$.

Part (ii) clearly holds for $\mathscr{M}$ constant with finitely generated free stalks. An exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of *elementary* sheaves induces the following commutative diagram with exact rows (see 3.12):

$$\begin{array}{ccccccc} \cdots\longrightarrow & H_p(X;\mathscr{M}') & \longrightarrow & H_p(X;\mathscr{M}) & \longrightarrow & H_p(X;\mathscr{M}'') & \longrightarrow\cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots\to & H_p(\Gamma(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}')) & \to & H_p(\Gamma(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M})) & \to & H_p(\Gamma(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{M}'')) & \to\cdots \end{array}$$

The 5-lemma implies that (ii) holds for all constant sheaves $\mathscr{M}$ with finitely generated stalks, since any finitely generated L-module is the quotient of free finitely generated modules.

Now let $\mathscr{M}$ be a fixed sheaf on X with $(\mathscr{M}, cld)$ elementary. Consider the collection

$$\Omega = \{U \subset X \text{ open} \mid H_*(U;\mathscr{M}|U) \to H_*(\Gamma(\mathscr{D}(\mathscr{L}^*|U)\otimes\mathscr{M}|U)) \text{ is an isomorphism}\}.$$

We have just shown that if $\mathscr{M}|U$ is constant then $U \in \Omega$.

Now let $U = U_1 \cup U_2$ and $V = U_1 \cap U_2$, and suppose that U_1, U_2, and V are all in Ω. The exact sequence[12] $0 \to \mathscr{L}_V^* \to \mathscr{L}_{U_1}^* \oplus \mathscr{L}_{U_2}^* \to \mathscr{L}_U^* \to 0$ induces the exact sequence

$$0 \to \mathscr{D}(\mathscr{L}_U^*)\otimes\mathscr{M} \to (\mathscr{D}(\mathscr{L}_{U_1}^*)\otimes\mathscr{M}) \oplus (\mathscr{D}(\mathscr{L}_{U_2}^*)\otimes\mathscr{M}) \to \mathscr{D}(\mathscr{L}_V^*)\otimes\mathscr{M} \to 0.$$

Let us temporarily use the abbreviation $G_*(U) = \Gamma(\mathscr{D}(\mathscr{L}^*|U)\otimes\mathscr{M}|U)$ for open sets $U \subset X$. Then, applying Γ to the preceding sequence and

[12]See II-13.

using 3.10 and 5.4, we obtain the following commutative diagram of chain complexes:

$$\begin{array}{ccccccccc} 0 & \to & C_*(U;\mathscr{M}) & \to & C_*(U_1;\mathscr{M}) \oplus C_*(U_2;\mathscr{M}) & \to & C_*(V;\mathscr{M}) & \to & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \to & G_*(U) & \to & G_*(U_1) \oplus G_*(U_2) & \to & G_*(V) & \to & 0, \end{array}$$

where the first row coincides with the second when $\mathscr{L}^* = \mathscr{I}^*(U;L)$. By assumption, g and h induce isomorphisms in homology. The 5-lemma, applied to the induced homology ladder, then implies that f induces an isomorphism in homology. This shows that $U \in \Omega$.

Next, if $U = +\!\!\!\!\!\!\;\vdash U_\alpha$ and each $U_\alpha \in \Omega$, then it is obvious that $U \in \Omega$.

Therefore, Ω satisfies all the properties (a)–(c) of 3.6. Since $\Omega_{\mathscr{M}}$ is the smallest such collection, we have $\Omega_{\mathscr{M}} \subset \Omega$. Since $X \in cld$ we have $X \in \Omega_{\mathscr{M}}$ and hence $X \in \Omega$, which is the desired result.

For the last statement of the theorem, part (i) already follows from 5.3. For part (ii), when $\Phi = c$, consider the homomorphism

$$\theta : \Gamma_c(\mathscr{D}(\mathscr{I}^*(X;L)) \otimes \mathscr{A}) \to \Gamma_c(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}).$$

By 3.4, II-Exercise 18, and II-14.5, it follows that both sides are exact functors of $\mathscr{A}$ and commute with direct limits in $\mathscr{A}$. When $\mathscr{A} = L_U$, for U open, θ induces a homology isomorphism, since

$$\Gamma_c(\mathscr{D}(\mathscr{L}^*) \otimes L_U) \approx \Gamma_{c|U}(\mathscr{D}(\mathscr{L}^*)) \approx \Gamma_{c|U}(\mathscr{D}(\mathscr{L}^*)|U) \approx \Gamma_{c|U}(\mathscr{D}(\mathscr{L}^*|U))$$

[by (5) on page 290] and by that part of 5.6 already proved. It follows immediately from II-16.12 that the class of sheaves $\mathscr{A}$ for which θ yields an isomorphism in homology consists of all sheaves. □

5.7. Corollary. *Let $A \subset X$ be locally closed, let $\mathscr{M}$ be a sheaf on X, and assume that $(\mathscr{M}, \Phi|A)$ is elementary. Then the canonical map*

$$H_*^{\Phi|A}(A;\mathscr{M}|A) \to H_*^{\Phi|A}(X;\mathscr{M})$$

is an isomorphism. □

Now we shall extend the excision property to more general situations than are covered by 5.2.

5.8. Theorem. *Let $A \subset X$ be locally closed. Suppose that $\mathscr{M}$ is a sheaf on X such that $(\mathscr{M}, \Phi|A)$ is elementary. Then*

$$\boxed{H_*^{\Phi}(X,A;\mathscr{M}) \approx H_*(C_*^{\Phi}(X;\mathscr{M})/C_*^{\Phi|A}(X;\mathscr{M})).}$$

Moreover, if $X - A$ is Φ-taut and locally closed in X, this may be identified with

$$H_*(\Gamma_{\Phi\cap(X-A)}(\mathscr{C}_*(X;\mathscr{M})|X-A)).$$

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & C_*^{\Phi|A}(A;\mathscr{M}|A) & \to & C_*^{\Phi}(X;\mathscr{M}) & \longrightarrow & C_*^{\Phi}(X,A;\mathscr{M}) & \longrightarrow & 0 \\ & \downarrow & & \downarrow = & & \downarrow & & \\ 0 \to & C_*^{\Phi|A}(X;\mathscr{M}) & \to & C_*^{\Phi}(X;\mathscr{M}) & \to & C_*^{\Phi}(X;\mathscr{M})/C_*^{\Phi|A}(X;\mathscr{M}) & \to & 0 \end{array}$$

in which the rows are exact. The first two vertical maps induce isomorphisms in homology by 5.7, and the result follows from the 5-lemma. The second statement follows from the fact that the bottom of this diagram can be identified with the exact sequence

$$0 \to \Gamma_{\Phi|A}(\mathscr{C}_*(X;\mathscr{M})) \to \Gamma_{\Phi}(\mathscr{C}_*(X;\mathscr{M})) \to \Gamma_{\Phi\cap(X-A)}(\mathscr{C}_*(X;\mathscr{M})|X-A) \to 0,$$

where the exactness on the right is due to the tautness assumption and the fact that $\mathscr{C}_*(X;\mathscr{M})$ is flabby by 3.8. □

5.9. Corollary. *Let $B \subset A \subset X$, where $\overline{B} \subset A$ and where A and $X - B$ are locally closed in X. Let Φ and Ψ be families of supports on X and $X - B$ respectively, and assume that $\Phi \cap X - A = \Psi \cap X - A$ and that $X - A$ is Φ-taut in X and Ψ-taut in $X - B$. Let $\mathscr{M}$ be a sheaf on X such that $(\mathscr{M}, \Phi|A)$ is elementary and $(\mathscr{M}|X - B, \Phi|A - B)$ is elementary. Then there is a natural isomorphism*

$$\boxed{H_*^{\Phi}(X,A;\mathscr{M}) \approx H_*^{\Psi}(X - B, A - B;\mathscr{M}).}$$

Proof. We have

$$\begin{aligned} \mathscr{C}_*(X;\mathscr{M})|X - A &= (\mathscr{C}_*(X;\mathscr{M})|X - \overline{B})|X - A \\ &\approx \mathscr{C}_*(X - \overline{B};\mathscr{M})|X - A \\ &\approx (\mathscr{C}_*(X - B;\mathscr{M})|X - \overline{B})|X - A \\ &= \mathscr{C}_*(X - B;\mathscr{M})|X - A. \end{aligned}$$

Thus

$$\Gamma_{\Phi\cap(X-A)}(\mathscr{C}_*(X;\mathscr{M})|X - A) \approx \Gamma_{\Psi\cap(X-A)}(\mathscr{C}_*(X - B;\mathscr{M})|X - A),$$

and the result follows from 5.8. □

Applying this to the case in which $B = A = F$ is closed we get:

5.10. Corollary. *If $F \subset X$ is closed and if $\mathscr{M}$ is a sheaf on X such that $(\mathscr{M}, \Phi|F)$ is elementary, then*

$$\boxed{H_*^{\Phi}(X,F;\mathscr{M}) \approx H_*^{\Phi\cap(X-F)}(X - F;\mathscr{M}).}$$

□

Let $U = X - F$ with F closed and let $(\mathscr{M}, \Phi|F)$ be elementary as in 5.10. Then by 5.10 the exact homology sequence of the pair (X, F) may be written in the form

$$\begin{aligned} \cdots \to H_p^{\Phi|F}(F;\mathscr{M}) \to H_p^{\Phi}(X;\mathscr{M}) \to H_p^{\Phi\cap U}(U;\mathscr{M}) \to \\ H_{p-1}^{\Phi|F}(F;\mathscr{M}) \to \cdots \end{aligned} \tag{36}$$

If $(\mathscr{M}, \Phi)$ is elementary and if $\mathscr{L}^*$ is a replete resolution of L, then this sequence is induced, via 5.4 and 5.6(ii), by the exact sequence

$$0 \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}_F^*) \otimes \mathscr{M}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{M}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{L}_U^*) \otimes \mathscr{M}) \to 0$$

of chain complexes, which derives from the exact sequence

$$0 \to \mathscr{L}_U^* \to \mathscr{L}^* \to \mathscr{L}_F^* \to 0.$$

5.11. Corollary. $\mathscr{H}_p(X;L)_x \approx H_p(X, X - \{x\}; L)$.

Proof. Since $\{x\}$ is taut, the theorem gives

$$H_p(X, X-\{x\}; L) \approx H_p(\Gamma(\mathscr{C}_*(X;L)|\{x\}) = H_p(\mathscr{C}_*(X;L)_x) = \mathscr{H}_p(X;L)_x$$

as claimed. □

5.12. Example. This simple example shows that 5.10 does not hold for general coefficient sheaves. Let $X = [0,1]$ and $F = \{0,1\}$. Consider the sheaf L_F on X. By (34) we have that

$$H_p(X; L_{X-F}) \approx H_p^c(X - F; L) \approx \begin{cases} L, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

The exact homology sequence of X induced by the coefficient sequence $0 \to L_{X-F} \to L \to L_F \to 0$ shows that $H_p(X; L_F) = 0$ for all p. Then the exact sequence of the pair (X, F) with coefficients in L_F shows that $H_1(X, F; L_F) \approx H_0(F; L) \approx L \oplus L$. This is the left-hand side of the isomorphism in 5.10, but the right-hand side is $H_1(X - F; L_F|X - F) = H_1(X - F; 0) = 0$. Thus 5.10 fails even for closed supports on the compact space $[0,1]$ for the coefficient sheaf $L_{\{0,1\}}$. Consequently, it seems unlikely that there is a general "single space" interpretation of homology relative to a closed subspace analogous to that for cohomology or to that relative to an open subspace. ◇

Now we shall provide some criteria for H_{-1} to be zero and investigate the nature of H_0^c.

5.13. Theorem. *$H_{-1}^{\Phi}(X;\mathscr{A}) = 0$ for Φ paracompactifying and for any sheaf $\mathscr{A}$ on X and also for arbitrary Φ if $\mathscr{A}$ is elementary.*

Proof. By 5.6, $H^{\Phi}_{-1}(X;L) \approx \varprojlim H_{-1}(K;L)$, K ranging over Φ. But $H_{-1}(K;L) = \operatorname{Ext}(H^0_c(K;L), L) = 0$ since $H^0_c(K;L)$ is free by the result of Nöbeling; see II-Exercise 34. Therefore

$$H^{\Phi}_{-1}(X;L) = 0 \tag{37}$$

for *any* family Φ of supports.

Let $\mathscr{L}_p = \mathscr{C}_p(X;L)$. For $U \subset X$ open, $d : \mathscr{L}_0(U) \to \mathscr{L}_{-1}(U)$ is onto since $H_{-1}(U;L) = 0$ by (37). Let $\mathscr{K}$ be the kernel of $d : \mathscr{L}_0 \to \mathscr{L}_{-1}$, so that

$$0 \to \mathscr{K} \to \mathscr{L}_0 \to \mathscr{L}_{-1} \to 0 \tag{38}$$

is exact. The cohomology sequence with supports in Φ of the coefficient sequence (38) shows that

$$H^1_{\Phi}(X;\mathscr{K}) \approx \operatorname{Coker}\{\Gamma_{\Phi}(\mathscr{L}_0) \to \Gamma_{\Phi}(\mathscr{L}_{-1})\} = H^{\Phi}_{-1}(X;L) = 0$$

by (37). Therefore $\mathscr{K}$ is flabby by II-Exercise 21. The sheaves in (38) are torsion-free, so that (38) remains exact upon tensoring with $\mathscr{A}$. Its cohomology sequence has the segment

$$\Gamma_{\Phi}(\mathscr{L}_0 \otimes \mathscr{A}) \xrightarrow{d_0} \Gamma_{\Phi}(\mathscr{L}_{-1} \otimes \mathscr{A}) \to H^1_{\Phi}(X;\mathscr{K} \otimes \mathscr{A}),$$

and by definition

$$H^{\Phi}_{-1}(X;\mathscr{A}) = \operatorname{Coker} d_0.$$

If Φ is paracompactifying, then $\mathscr{K} \otimes \mathscr{A}$ is Φ-soft by II-9.6 and II-9.18, so that d_0 is onto, whence $H^{\Phi}_{-1}(X;\mathscr{A}) = 0$. If $\mathscr{A}$ is elementary, then $\mathscr{K} \otimes \mathscr{A}$ is flabby by 3.8, and so we have the same conclusion for arbitrary Φ. □

5.14. Theorem. *Let X be locally connected. If either L is a field or X is clc^1_L, then $H^c_0(X;L)$ is the free L-module on the components of X.*[13]

Proof. We may as well assume X to be connected. Then $H^c_0(X;L) = \varinjlim H_0(K;L)$, K ranging over the compact connected subsets of X. If L is a field, then $H_0(K;L) \approx \operatorname{Hom}(H^0(K;L), L) \approx \operatorname{Hom}(L,L) \approx L$, and the result follows.

Suppose that X is clc^1_L. By Exercise 28 of Chapter II we have that $H^1(K;L)$ is torsion-free for any $K \subset X$. If $K \subset K'$ are both compact and connected with $K \subset \operatorname{int} K'$, then it follows from II-17.5 that the homomorphism

$$H^1(K';L) \to H^1(K;L)$$

has a finitely generated image. This image is torsion-free, and hence free, so that

$$\operatorname{Ext}(H^1(K), L) \to \operatorname{Ext}(H^1(K'), L)$$

factors through zero. Therefore $\varinjlim \operatorname{Ext}(H^1(K), L) = 0$, and we deduce that $H^c_0(X;L) \approx \varinjlim H_0(K;L) \approx \varinjlim \operatorname{Hom}(H^0(K;L), L) \approx L$. □

[13] Also see Exercise 26 and VI-10.3.

5.15. Suppose that $K_1 \leftarrow K_2 \leftarrow \cdots$ is an inverse sequence of compact spaces with inverse limit $K = \varprojlim K_n$. By passing to a union of mapping cylinders, we may assume that $K_1 \supset K_2 \supset \cdots$ is a decreasing sequence of spaces and $K = \bigcap K_i$. Let M be a finitely generated L-module, and hence an elementary sheaf on K_1. By 5.5 and 5.6, if $\mathscr{L}^*$ is a replete resolution of L on K_1, then $H_p(K_n; M) \approx H_p(\Gamma_{cld|K_n}(\mathscr{D}(\mathscr{L}^*) \otimes M))$. Also, $\mathscr{D}(\mathscr{L}^*) \otimes M$ is flabby by 3.4. Therefore, a translation of IV-2.10 shows that there is an exact sequence of the form

$$\boxed{0 \to \varprojlim{}^1 H_{p+1}(K_n; M) \to H_p(K; M) \to \varprojlim H_p(K_n; M) \to 0.}$$

Now suppose that the K_n are all polyhedra, so that their homology is simplicial homology. Then the right-hand term is just the Čech homology $\check{H}_p(K; M)$ of K by the continuity property of Čech homology. Thus the sequence can be written[14]

$$\boxed{0 \to \varprojlim{}^1 H_{p+1}(K_n; M) \to H_p(K; M) \to \check{H}_p(K; M) \to 0.} \qquad (39)$$

Since $H_{-1}(K; M) = 0$, there is the consequence that

$$\varprojlim{}^1 H_0(K_n; M) = 0,$$

which is perhaps not too hard to see directly.

As an explicit instance of this, let each $K_n = \mathbb{S}^1$ and let $K_n \to K_{n-1}$ be the standard covering map of degree n. Then $\mathbb{Z} \approx H_1(K_n; \mathbb{Z}) \to H_1(K_{n-1}; \mathbb{Z}) \approx \mathbb{Z}$ is multiplication by n. The space K is a "solenoid" and has

$$H_0(K; \mathbb{Z}) \approx \operatorname{Hom}(H^0(K; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}(H^1(K; \mathbb{Z}), \mathbb{Z}) \approx \mathbb{Z} \oplus \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}),$$

by (9) on page 292. The sequence (39) takes the form

$$0 \to \varprojlim{}^1\{\cdots \xrightarrow{4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{2} \mathbb{Z}\} \to H_0(K; \mathbb{Z}) \to \mathbb{Z} \to 0.$$

It is not hard to see that the $\mathbb{Z}$ splits off uniquely, and so we conclude that

$$\varprojlim{}^1\{\cdots \xrightarrow{4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{2} \mathbb{Z}\} \approx \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}).$$

It will be shown in 14.8 that this group is uncountable. This also follows from 5.17.

5.16. Proposition. *If $\mathscr{A} = \{\cdots \to A_2 \to A_1\}$ is an inverse sequence of countable abelian groups and if $\varprojlim{}^1 \mathscr{A} \neq 0$, then $\varprojlim{}^1 \mathscr{A}$ is uncountable.*

Proof. If $\varprojlim{}^1 \mathscr{A} \neq 0$, then $\mathscr{A}$ is not Mittag-Leffler, and so by passing to a subsequence (see II-11.6), we can assume that the subgroups $I_k = \operatorname{Im}(A_k \to A_1)$ are strictly decreasing. Let $K_i = \operatorname{Ker}(A_i \to A_1)$, so that we have the exact sequence $0 \to \mathscr{K} \to \mathscr{A} \to \mathscr{I} \to 0$ of sheaves on $\mathbb{N}$.[15]

[14]This sequence is not valid without the finite-generation condition on M. For example, it does not hold when $L = \mathbb{Z}$ and $M = \mathbb{Q}$, as can be seen by computing its terms for an inverse sequence of lens spaces and $p = 1$.

[15]Here $\mathbb{N}$ is the topology on the positive integers from II-Exercise 27.

It follows that $\varprojlim^1 \mathscr{A} \to \varprojlim^1 \mathscr{I}$ is surjective. Thus it suffices to show that $\varprojlim^1 \mathscr{I}$ is uncountable. That is, it suffices to prove the following lemma. □

5.17. Lemma. *If $\mathscr{A} = \{A_1 \supset A_2 \supset \cdots\}$ is a strictly decreasing sequence of countable groups, then $\varprojlim^1 \mathscr{A}$ is uncountable.*

Proof. Let $Q_i = A_1/A_i$, and note that $\operatorname{Ker}(Q_{i+1} \to Q_i) \approx A_i/A_{i+1} \neq 0$. It follows that $\varprojlim \mathscr{Q}$ is uncountable. Consider the inverse system $\mathscr{B}$, which is constant with group A_1. Then we have the exact sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{Q} \to 0$ of sheaves on $\mathbb{N}$ with $\mathscr{B}$ flabby.[15] Therefore there is the exact sequence $A_1 = \varprojlim \mathscr{B} \to \varprojlim \mathscr{Q} \to \varprojlim^1 \mathscr{A} \to 0$. Since A_1 is countable and $\varprojlim \mathscr{Q}$ is uncountable, $\varprojlim^1 \mathscr{A}$ must be uncountable. □

5.18. Corollary. *If X is a compact metrizable space with $H^p(X;\mathbb{Z})$ and $H^{p+1}(X;\mathbb{Z})$ finitely generated, then $H_p(X;\mathbb{Z}) \approx \check{H}_p(X;\mathbb{Z})$ naturally.*

Proof. Such a space is an inverse limit of an inverse sequence of finite polyhedra K_n. By (9) on page 292, $H_p(X;\mathbb{Z})$ is finitely generated and hence countable. By (39), $\varprojlim^1 H_{p+1}(K_n;\mathbb{Z})$ is finitely generated, and by 5.16 it must be zero. □

5.19. Corollary. *If X is compact, metrizable, and $clc_{\mathbb{Z}}^{k+1}$, then we have that $H_k(X;\mathbb{Z}) \approx \check{H}_k(X;\mathbb{Z})$ naturally.*

Proof. This follows immediately from 5.18 and II-17.7. □

This last corollary is known without metrizability and holds for general constant coefficients; see VI-10.

Note that since $H_p(K;L) \approx \check{H}_p(K;L)$ naturally for compact spaces K when L is a *field* by 3.15, we must have that

$$\varprojlim^1 H_p(K_i;L) = 0$$

for any inverse sequence of compact spaces K_i when L is a field. Since $\varprojlim^1$ need not vanish over a field as base ring (see 5.20), this represents a limitation on inverse sequences of vector spaces that can arise this way.

5.20. Example. Let $X_n = \bigcup_{i=n}^{\infty} C_i$, where C_i is the circle in the plane of radius $1/i$ tangent to the real axis at the origin. We have

$$H_0(X_n;L) \approx \operatorname{Hom}(L,L) \oplus \operatorname{Ext}(\textstyle\bigoplus L, L) \approx L.$$

Then the inclusions $X_{n+1} \hookrightarrow X_n$ give an inverse sequence whose inverse limit is a point $\star = \bigcap X_n$. Now, $H^1(X_n;L)$ is the free L-module on

[15] Here $\mathbb{N}$ is the topology on the positive integers from II-Exercise 27.

the basis $\{a_n, a_{n+1}, \ldots\}$, and the restriction $H^1(X_n; L) \to H^1(X_{n+1}; L)$ takes $a_n \mapsto 0$ and $a_k \mapsto a_k$ for $k > n$. Dualizing to $H_1(X_n; L) = \operatorname{Hom}(H^1(X_n; L), L)$, we may regard $H_1(X_n; L)$ as the L-module of sequences $\alpha = \{\alpha_n, \alpha_{n+1}, \ldots\}$, where $\alpha_k \in L$ and where $\alpha(a_j) = \alpha_j$ for $j \geq n$. For the inclusion $i : X_{n+1} \hookrightarrow X_n$ we have $i_*(\alpha)(a_j) = \alpha(i^*(a_j)) = \alpha_j$ for $j > n$ and $i_*(\alpha)(a_n) = \alpha(i^*(a_n)) = \alpha(0) = 0$. Thus $i_*(\alpha_{n+1}, \alpha_{n+2}, \ldots) = (0, \alpha_{n+1}, \alpha_{n+2}, \ldots)$. That is, i_* is the inclusion $\prod_{i=n+1}^{\infty} L \hookrightarrow \prod_{i=n}^{\infty} L$. Call this inverse sequence of L-modules $\mathscr{P}$. Then we have the exact sequence

$$0 \to \varprojlim{}^1 \mathscr{P} \to H_0(\star; L) \to \varprojlim L \to 0,$$

and it follows that $\varprojlim{}^1 \mathscr{P} = 0$. This shows that the countability hypothesis in 5.17 is essential.

Note that $\mathscr{P} = \prod_n \mathscr{P}_n$, where $\mathscr{P}_n = \{L \leftarrow \cdots \leftarrow L \leftarrow 0 \leftarrow 0 \leftarrow \cdots\}$ with the last nonzero term in the nth place. Since the inverse sequence $\mathscr{P}_n$ is Mittag-Leffler, it is acyclic. Since $\mathbb{N}$ is a rudimentary space, it follows from II-Exercise 57 that $\mathscr{P}$ is acyclic, giving another, more direct, proof that $\varprojlim{}^1 \mathscr{P} = 0$. (It is also quite easy to prove this directly from the definition.)

Now let $\mathscr{S}$ be the similar inverse system obtained from $\mathscr{P}$ by replacing the direct products by direct sums. There is the monomorphism $\mathscr{S} \rightarrowtail \mathscr{P}$. Now the map $\prod_{i=n}^{\infty} L \hookrightarrow \prod_{i=n+1}^{\infty} L$ induces an isomorphism

$$\left(\prod_{i=n}^{\infty} L\right) \Big/ \left(\bigoplus_{i=n}^{\infty} L\right) \xrightarrow{\approx} \left(\prod_{i=n+1}^{\infty} L\right) \Big/ \left(\bigoplus_{i=n+1}^{\infty} L\right).$$

Call this quotient group Q, so that there is the constant inverse sequence $\mathscr{Q}$ with all terms Q. The exact sequence $0 \to \mathscr{S} \to \mathscr{P} \to \mathscr{Q} \to 0$ of sheaves on $\mathbb{N}$ induces the exact sequence

$$\varprojlim \mathscr{P} \to \varprojlim \mathscr{Q} \to \varprojlim{}^1 \mathscr{S} \to \varprojlim{}^1 \mathscr{P},$$

whose terms on the ends are both zero. This shows that

$$\varprojlim{}^1 \mathscr{S} \approx Q,$$

which is an uncountably generated L-module for any L. In particular, when L is a field, this gives an example showing that $\varprojlim{}^1$ need not vanish over a field. (Of course, the fact that $\varprojlim{}^1 \mathscr{S}$ is uncountable also follows from 5.17, provided that L is countable.) ◇

6 The Vietoris theorem, homotopy, and covering spaces

In this section we consider the homological version of the Vietoris mapping theorem and apply that to prove the homotopy invariance of homology.

We also apply the general theorem to the case of covering maps, obtaining a homological transfer map.

6.1. Theorem. *Let $f : X \to Y$ be a proper surjective map between locally compact spaces, and suppose that each $f^{-1}(y)$, $y \in Y$, is connected and has $H^p(f^{-1}(y); L) = 0$ for $0 < p \le n$. Then*

$$f_* : H_p^{f^{-1}\Phi}(X; f^*\mathscr{B}) \to H_p^{\Phi}(Y; \mathscr{B})$$

is an isomorphism for $p < n$ and an epimorphism for $p = n$, for all sheaves $\mathscr{B}$ on Y and all families Φ of supports on Y. If L is a field, then f_* is an isomorphism for $p < n+1$ and an epimorphism for $p = n+1$.

Proof. Let $\mathscr{I}^* = \mathscr{I}^*(X; L)$. By 2.6 we have $\mathscr{D}(f\mathscr{I}^*) = f\mathscr{D}(\mathscr{I}^*)$ since f is proper.[16] By (21) on page 301 we also have the natural isomorphism

$$\mathscr{D}(f\mathscr{I}^*) \otimes \mathscr{B} \approx f(\mathscr{D}(\mathscr{I}^*) \otimes f^*\mathscr{B}) = f\mathscr{C}_*(X; f^*\mathscr{B}). \tag{40}$$

Applying $H_*(\Gamma_\Phi(\bullet))$, this yields the isomorphism

$$H_p(\Gamma_\Phi(\mathscr{D}(f\mathscr{I}^*) \otimes \mathscr{B})) \approx H_p^{f^{-1}\Phi}(X; f^*\mathscr{B}). \tag{41}$$

Now, f_* is clearly induced from (41) via the homomorphism $h : \mathscr{L}^* \to f\mathscr{I}^*$ of differential sheaves associated with the canonical f-cohomomorphism $\mathscr{L}^* \rightsquigarrow \mathscr{I}^*$, where $\mathscr{L}^* = \mathscr{I}^*(Y; L)$.

The derived sheaf $\mathscr{H}^*(f\mathscr{I}^*)$ is just the Leray sheaf $\mathscr{H}^*(f; L)$ of f. Since f is proper and since each $f^{-1}(y)$ is connected and acyclic through degree n, we have that $\mathscr{H}^0(f; L) \approx L$ and $\mathscr{H}^p(f; L) = 0$ for $0 < p \le n$. Moreover, $f\mathscr{I}^*$ is injective by II-3.1. Therefore there exists an injective resolution $\mathscr{J}^*$ of L on Y with $\mathscr{J}^p = f\mathscr{I}^p$ for $p \le n+1$, and there is a map $j : \mathscr{J}^* \to f\mathscr{I}^*$ extending the identity in degrees at most $n+1$. There also exist homomorphisms $g : \mathscr{L}^* \to \mathscr{J}^*$ and $k : \mathscr{J}^* \to \mathscr{L}^*$ of resolutions. Then kg and gk are chain homotopic to the identity maps, and j is chain homotopic to hk. This situation persists upon passing to duals, so that

$$k' : C_*^\Phi(Y; \mathscr{B}) = \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{B}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{J}^*) \otimes \mathscr{B})$$

is a chain equivalence. Also,

$$j' : \Gamma_\Phi(\mathscr{D}(f\mathscr{I}^*) \otimes \mathscr{B}) \to \Gamma_\Phi(\mathscr{D}(\mathscr{J}^*) \otimes \mathscr{B})$$

is the identity map in degrees at most n, whence

$$j_* : H_p(\Gamma_\Phi(\mathscr{D}(f\mathscr{I}^*) \otimes \mathscr{B})) \to H_p(\Gamma_\Phi(\mathscr{D}(\mathscr{J}^*) \otimes \mathscr{B}))$$

is an isomorphism for $p < n$ and an epimorphism for $p = n$. But $j' \simeq k'h'$, whence $j_* = k_*h_*$. Since k_* is an isomorphism in all degrees, the result

[16] f being proper implies that $f_c\mathscr{I}^* = f\mathscr{I}^*$.

follows. If L is a field, then we can take $L^1 = 0$, whence j' is the identity in degrees at most $n+1$, giving the one-degree improvement. □

For coefficients in L one can prove much stronger results from the corresponding facts for cohomology:

6.2. Theorem. *Let $f : X \to Y$ be a proper surjective map between locally compact spaces. Let*

$$S_k = \{y \in Y \mid \tilde{H}^k(f^{-1}(y); L) \neq 0\}, \quad b_k = \underline{\dim}_{c,L}^{Y} S_k.$$

For a given integer (or ∞) N let

$$n = 1 + \sup\{k + b_k \mid 0 \le k < N\}.$$

Then for any support family Φ on Y consisting of paracompact sets,

$$f_* : H_p^{f^{-1}\Phi}(X; L) \to H_p^{\Phi}(Y; L)$$

is an isomorphism for $n < p < N-1$, a monomorphism for $p = n$, and an epimorphism for $p = N-1$. If L is a field, then f_* is also an isomorphism for $p = N-1$ and an epimorphism for $p = N$.

Proof. By 5.6 it suffices to prove the same thing for $f|f^{-1}K : f^{-1}K \to K$, where $K \in \Phi$. But that follows immediately from IV-8.21 and 4.1. □

Let us call a space T *acyclic* if $\tilde{H}^*(T; L) = 0$. It should be noted that if L is a field or the integers, then by (9) on page 292, cohomological acyclicity for a compact space [e.g., $f^{-1}(y)$] is equivalent to homological acyclicity, since $\operatorname{Hom}(M, L) = 0 = \operatorname{Ext}(M, L)$ implies $M = 0$ in these cases; see [64] and 14.7.

6.3. Theorem. *Let T be a compact acyclic space. Let Φ be a family of supports on X and let $\mathscr{A}$ be a sheaf on X. Then the inclusion $i^t : X \to X \times T$ [where $i^t(x) = (x, t)$] induces an isomorphism*

$$i_*^t : H_*^{\Phi}(X; \mathscr{A}) \xrightarrow{\approx} H_*^{\Phi \times T}(X \times T; \mathscr{A} \times T),$$

which is independent of $t \in T$.

Proof. If $\pi : X \times T \to X$ is the projection, then $\pi_* i_*^t = 1$. But π_* is an isomorphism by 6.1, and so $i_*^t = \pi_*^{-1}$. □

6.4. Corollary. *If $f, g : X \to Y$ are c-properly homotopic maps, with respect to the families Φ and Ψ of supports, then*

$$f_* = g_* : H_*^{\Phi}(X; L) \to H_*^{\Psi}(Y; L).$$ □

6.5. Corollary. *If $f, g : X \to Y$ are homotopic maps, then*

$$f_* = g_* : H_*^c(X; L) \to H_*^c(Y; L).$$

If f and g are properly homotopic, then also

$$f_* = g_* : H_*(X; L) \to H_*(Y; L). \qquad \square$$

6.6. It is important to notice that the analogue of II-11.1 does not hold for homology. (Indeed, such a homomorphism is not even generally defined.) For example, if f projects the unit circle onto the interval $I = [-1, 1]$, then $fL \approx L \oplus L_U$, where $U = I - \partial I$, and it is easily seen from (34) on page 306 and 6.5 that $H_1(I; fL) = 0$; whence $H_1(I; fL) \not\approx H_1(\mathbb{S}^1; L) \approx L$.

However, such a result does hold for covering maps, and more generally for local homeomorphisms. We shall proceed to establish this contention and to indicate some further facts. In this connection, also see Section 19.

Suppose from now on that $\pi : X \to Y$ is a *local homeomorphism* (not necessarily proper). Since the definition of $\mathscr{C}_*(\bullet\,; L)$ is of a "local" character, we see easily that there is a natural isomorphism

$$\mathscr{C}_*(X; L) \xrightarrow{\approx} \pi^* \mathscr{C}_*(Y; L). \tag{42}$$

By 4.3 there is also a natural isomorphism

$$\mathscr{C}_*(Y; L) \otimes \pi_c \mathscr{A} \xrightarrow{\approx} \pi_c(\pi^* \mathscr{C}_*(Y; L) \otimes \mathscr{A}) \tag{43}$$

for any sheaf $\mathscr{A}$ on X. Thus we have the isomorphisms

$$\pi_c(\mathscr{C}_*(X; L) \otimes \mathscr{A}) \xrightarrow{\approx} \pi_c(\pi^* \mathscr{C}_*(Y; L) \otimes \mathscr{A}) \xleftarrow{\approx} \mathscr{C}_*(Y; L) \otimes \pi_c \mathscr{A}. \tag{44}$$

Therefore

$$\pi_c \mathscr{C}_*(X; \mathscr{A}) \approx \mathscr{C}_*(Y; \pi_c \mathscr{A}),$$

$$C_*^{\Phi(c)}(X; \mathscr{A}) \approx C_*^{\Phi}(Y; \pi_c \mathscr{A}),$$

and

$$\boxed{H_*^{\Phi(c)}(X; \mathscr{A}) \approx H_*^{\Phi}(Y; \pi_c \mathscr{A}).}$$

It can be shown that the diagram

$$\begin{array}{ccc} H_p^{\Phi}(Y; \pi_c \mathscr{A}) & \xrightarrow{\approx} & H_p^{\Phi(c)}(X; \mathscr{A}) \\ & {\scriptstyle \pi_*} \nwarrow \quad \nearrow {\scriptstyle h_*} & \\ & H_p^{\Phi(c)}(X; \pi^* \pi_c \mathscr{A}) & \end{array}$$

commutes, where h_* is induced from the canonical homomorphism $h : \pi^* \pi_c \mathscr{A} \to \pi^* \pi \mathscr{A} \to \mathscr{A}$. Since this will not be used, we leave the verification of it to the reader.

Now let $\mathscr{B}$ be a sheaf on Y, and note that for $y \in Y$,

$$(\pi_c \pi^* \mathscr{B})_y = \bigoplus_{\pi(x)=y} (\pi^* \mathscr{B})_x = \bigoplus \mathscr{B}_y$$

[one copy for each $x \in \pi^{-1}(y)$]. Summation produces a natural map $(\pi_c \pi^* \mathscr{B})_y \xrightarrow{\sigma} \mathscr{B}_y$, and it is not hard to see that this gives a (continuous) homomorphism

$$\sigma : \pi_c \pi^* \mathscr{B} \to \mathscr{B}.$$

[Note that the homomorphism

$$\pi_c \pi^* \mathscr{C}_*(Y; L) \approx \pi_c \mathscr{C}_*(X; L) \to \mathscr{C}_*(Y; L),$$

obtained from (42) and (20) on page 300, is of this type.]

It can be checked that the diagram

$$\begin{array}{ccc} \pi_c(\mathscr{C}_*(X;L) \otimes \pi^* \mathscr{B}) & \xleftarrow{\approx} & \mathscr{C}_*(Y;L) \otimes \pi_c \pi^* \mathscr{B} \\ \uparrow \approx & & \downarrow 1 \otimes \sigma \\ \pi_c(\mathscr{C}_*(X;L)) \otimes \mathscr{B} & \xrightarrow{\pi_* \otimes 1} & \mathscr{C}_*(Y;L) \otimes \mathscr{B} \end{array}$$

commutes and hence so does the diagram

$$\begin{array}{ccc} H_p^{\Phi}(Y; \pi_c \pi^* \mathscr{B}) & \xrightarrow{\approx} & H_p^{\Phi(c)}(X; \pi^* \mathscr{B}) \\ & {\scriptstyle \sigma_*} \searrow \quad \swarrow {\scriptstyle \pi_*} & \\ & H_p^{\Phi}(Y; \mathscr{B}). & \end{array}$$

Now, if π is proper, so that $\pi_c \pi^* \mathscr{B} = \pi \pi^* \mathscr{B}$, then the canonical map $\beta : \mathscr{B} \to \pi \pi^* \mathscr{B}$ has the property that $\sigma\beta$ is multiplication by the number k of sheets of π (in this case, π is necessarily a covering map, and k is constant on components of Y). Thus β induces a *transfer* homomorphism

$$\boxed{\mu : H_p^{\Phi}(Y; \mathscr{B}) \to H_p^{\pi^{-1}\Phi}(X; \pi^* \mathscr{B})}$$

(when π is proper) with $\pi_* \mu = k$ (when k is constant). If π is a regular covering map (i.e., if the group G of deck transformations is transitive on each fiber), then we also have that $\mu \pi_* = \sum_{g \in G} g_*$.

Also, see Section 19 for a different, and essentially more general, treatment of the transfer map. Also, with a niceness assumption, an even closer analogue of II-11.1 can be proved for infinite coverings; see Exercise 37.

We note that the discussion in 6.6 is also valid for singular homology with $\mathscr{S}_*(\bullet; L)$ replacing $\mathscr{C}_*(\bullet; L)$. This is not generally the case for the method utilized in Section 19, and indeed, transfer maps in singular homology (or cohomology) do not generally exist for ramified coverings as is shown by the following example: Let X be the union in $\mathbb{R}^3$ of 2-spheres X_n

of radius $1/n$ that are all tangent to one another at a given common point x. Let $g : X \to X$ be the reflection through some plane through x that intersects each X_n in a great circle. Let $G = \{e, g\}$. In [3] a nontrivial element $\gamma \in {}_SH_3(X;\mathbb{Q})$ is constructed, and it is easy to check that $g_*(\gamma) = \gamma$. Hence, if there were a transfer map $\mu : {}_SH_3(X/G;\mathbb{Q}) \to {}_SH_3(X;\mathbb{Q})$ with $\mu\pi_* = 1 + g_*$, we would have that $0 \neq 2\gamma = \mu\pi_*(\gamma) = 0$, since X/G is contractible.

7 The homology sheaf of a map

In this section, $f : X \to Y$ is a map between locally compact spaces, $\mathscr{A}$ is a sheaf on X, and Ψ is a family of supports on X. For $B \subset Y$, $B^\bullet$ denotes $f^{-1}B \subset X$.

7.1. Definition. *The "homology sheaf" of the map f, with coefficients in $\mathscr{A}$ and supports in Ψ, is defined to be the derived sheaf $\mathscr{H}_*^\Psi(f;\mathscr{A})$ of the differential sheaf $f_\Psi \mathscr{C}_*(X;\mathscr{A})$ on Y. Similarly, if $A \subset X$ is locally closed, then the derived sheaf of $f_\Psi \mathscr{C}_*(X, A;\mathscr{A})$ is denoted by $\mathscr{H}_*^\Psi(f, f|A;\mathscr{A})$.*

Recall that

$$f_\Psi \mathscr{C}_*(X;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}\,(U \mapsto \Gamma_{\Psi\cap U^\bullet}(\mathscr{C}_*(X;\mathscr{A})|U^\bullet)) = C_*^{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}).$$

Thus

$$\mathscr{H}_p^\Psi(f;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}\left(U \mapsto H_p^{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A})\right).$$

Note that $\mathscr{C}_*(X, A;\mathscr{A})$ vanishes on $\operatorname{int} A$ and coincides with $\mathscr{C}_*(X;\mathscr{A})$ on $X - \bar{A}$. Hence, for $V \subset U$ open in Y with $\overline{V} \subset U$ and for Ψ paracompactifying, there are the natural maps (coefficients in $\mathscr{A}$)

$$C_*^\Psi(X, X - \overline{U}^\bullet) \to C_*^{\Psi\cap U^\bullet}(U^\bullet) \to C_*^\Psi(X, X - \overline{V}^\bullet) \to C_*^{\Psi\cap V^\bullet}(V^\bullet).$$

It follows that

$$\mathscr{H}_p^\Psi(f;\mathscr{A}) = \mathscr{S}\!\mathit{heaf}\left(U \mapsto H_p^\Psi(X, X - \overline{U}^\bullet;\mathscr{A})\right). \tag{45}$$

Clearly, $\overline{U}^\bullet$ may be replaced by $U^\bullet$ in (45) without changing the associated sheaf.

By (45) there is a canonical homomorphism, for $y \in Y$,

$$\boxed{r_*^y : \mathscr{H}_p^\Psi(f;\mathscr{A})_y \to H_p^\Psi(X, X - y^\bullet;\mathscr{A}).}$$

7.2. Proposition. *If f is Ψ-closed and Ψ is paracompactifying, then the homomorphism r_*^y is an isomorphism for each $y \in Y$.*

Proof. If follows immediately from 5.3 that $H_*^{\Psi|X-y^\bullet}(X-y^\bullet;\mathscr{A}) = \varinjlim H_*^{\Psi|X-\overline{U}^\bullet}(X-\overline{U}^\bullet;\mathscr{A})$, U ranging over the neighborhoods of y in Y, and the result follows upon consideration of the homology sequences of the pairs $(X, X-y^\bullet)$ and $(X, X-\overline{U}^\bullet)$. □

Note that if $1_X : X \to X$ is the identity map and if Ψ contains a neighborhood of each point, then

$$\boxed{\mathscr{H}_*^\Psi(1_X;\mathscr{A}) = \mathscr{H}_*(X;\mathscr{A}).}$$

We now consider the case in which $\Psi = c$ and $\mathscr{A} = f^*\mathscr{B}$ for some sheaf $\mathscr{B}$ on Y. By (21) on page 301 there is the natural isomorphism

$$f_c(\mathscr{C}_*(X;f^*\mathscr{B})) \approx f_c(\mathscr{C}_*(X;L))\otimes\mathscr{B}.$$

The right-hand side is, by definition, generated by the presheaf

$$U \mapsto (f_c\mathscr{C}_*(X;L))(U)\otimes\mathscr{B}(U) = C_*^{c(U)}(U^\bullet;L)\otimes\mathscr{B}(U).$$

Since $C_*^{c(U)}(U^\bullet;L)$ is torsion-free, we have the natural exact sequence

$$0\to H_p^{c(U)}(U^\bullet;L)\otimes\mathscr{B}(U)\to H_p(C_*^{c(U)}(U^\bullet;L)\otimes\mathscr{B}(U))\to H_{p-1}^{c(U)}(U^\bullet;L)*\mathscr{B}(U)\to 0$$

and hence the exact sequence of associated sheaves

$$\boxed{0\to\mathscr{H}_p^c(f;L)\otimes\mathscr{B}\to\mathscr{H}_p^c(f;f^*\mathscr{B})\to\mathscr{H}_{p-1}^c(f;L)*\mathscr{B}\to 0,} \tag{46}$$

which generalizes (13) on page 294.

Note that the map $f_c\mathscr{C}_*(X;f^*\mathscr{B}) \to \mathscr{C}_*(Y;\mathscr{B})$ of (35) on page 306 induces a natural map

$$\mathscr{H}_*^c(f;f^*\mathscr{B}) \to \mathscr{H}_*(Y;\mathscr{B}) \tag{47}$$

(the reader is urged to consider generalizations of this). By definition, (47) is induced by the homomorphism

$$f_* : H_*^{c(U)}(U^\bullet;f^*\mathscr{B}) \to H_*(U;\mathscr{B}),$$

from (26), of presheaves on Y.

If f is *proper*, then for $\Psi\supset c$, $\mathscr{H}_*(f;\mathscr{A}) = \mathscr{H}_*^\Psi(f;\mathscr{A}) = \mathscr{H}_*^c(f;\mathscr{A})$, and we note that trivially, (47) is induced by the homomorphism

$$f_* : H_*(U^\bullet;f^*\mathscr{B}) \to H_*(U;\mathscr{B}).$$

If f is a *Vietoris map* [i.e., proper, surjective, and with acyclic point inverses] then it follows that (47) provides an isomorphism

$$\boxed{\mathscr{H}_*(f;f^*\mathscr{B}) \approx \mathscr{H}_*(Y;\mathscr{B}) \quad \text{for } f \text{ a Vietoris map.}} \tag{48}$$

8 The basic spectral sequences

Here we produce the fundamental spectral sequence of a map that will be used in the next section to prove a very general Poincaré duality theorem.

8.1. Theorem. *Let $f : X \to Y$, let Ψ and Φ be families of supports on X and Y respectively, and assume that* $\dim_\Phi Y < \infty$. *Let $A \subset X$ be locally closed and let $\mathscr{A}$ be a sheaf on X. Assume that at least one of the following three conditions is satisfied:*

(a) *Φ and $\Phi(\Psi)$ are paracompactifying;*

(b) *A is closed, $\mathscr{A}$ is elementary, and either Φ is paracompactifying or $\Psi = cld$;*

(c) *$A = \varnothing$ and Φ is paracompactifying.*

Then there is a spectral sequence with

$$\boxed{E_2^{p,q} = H_\Phi^p(Y; \mathscr{H}_{-q}^\Psi(f, f|A; \mathscr{A}) \Longrightarrow H_{-p-q}^{\Phi(\Psi)}(X, A; \mathscr{A}).}$$

Proof. Consider the differential sheaf $\mathscr{L}^q = f_\Psi \mathscr{C}_{-q}(X, A; \mathscr{A})$. Now $\mathscr{L}^q$ is a $\mathscr{C}^0(Y; L)$-module, by 5.1 and IV-3.4, whence it is Φ-fine for Φ paracompactifying. If $\Psi = cld$, $\mathscr{A}$ is elementary, and A is closed, then $\mathscr{L}^q$ is flabby by 5.1 and II-5.7. Thus, the spectral sequence is given by IV-2.1, since $\dim_\Phi Y < \infty$. (The hypotheses ensure that the isomorphism (30) on page 305 holds.) □

8.2. We shall describe some important special cases of 8.1. Let $B \subset Y$ be *closed*. Put $U = Y - B$. Let Φ be a *paracompactifying* family of supports on Y and assume that $\dim_\Phi Y < \infty$. Also assume that f is Ψ-closed. Then there is a spectral sequence with

$$\boxed{{}'E_2^{p,q} = H_\Phi^p(Y, B; \mathscr{H}_{-q}^\Psi(f; \mathscr{A})) \Longrightarrow H_{-p-q}^{\Phi(\Psi)|U^\bullet}(U^\bullet; \mathscr{A}),}$$

where $U^\bullet = f^{-1}(U)$ as usual. To obtain this, apply 8.1 to the case in which A is empty and with supports on Y being $\Phi|U$ [use II-12.1 and note that $(\Phi|U)(\Psi) = \Phi(\Psi)|U^\bullet$ by IV-5.4(8) since f is Ψ-closed].

If Φ, Ψ, and $\Phi(\Psi)$ are all paracompactifying and if f is Ψ-closed, then we have a spectral sequence with

$$\boxed{{}''E_2^{p,q} = H_{\Phi|B}^p(B; \mathscr{H}_{-q}^\Psi(f; \mathscr{A})|B) \Longrightarrow H_{-p-q}^{\Phi(\Psi)}(X, U^\bullet; \mathscr{A}),}$$

which is obtained from 8.1 by taking $A = U^\bullet$, noting that

$$\mathscr{H}_*^\Psi(f, f|U^\bullet; \mathscr{A}) \approx \mathscr{H}_*^\Psi(f; \mathscr{A})_B$$

(see Exercise 11 of Chapter IV) and using II-10.2.

8.3. We note that the spectral sequences $'E_r$, E_r (with A empty) and $''E_r$ are respectively the spectral sequences of the double complexes

$$\boxed{'M^{p,q} = C^p_\Phi(Y; (f_\Psi \mathscr{C}_{-q}(X;\mathscr{A}))_U),}$$

$$\boxed{M^{p,q} = C^p_\Phi(Y; f_\Psi \mathscr{C}_{-q}(X;\mathscr{A})),}$$

and

$$\boxed{''M^{p,q} = C^p_\Phi(Y; (f_\Psi \mathscr{C}_{-q}(X;\mathscr{A}))_B).}$$

Thus, the exact sequence $0 \to {'M^{p,q}} \to M^{p,q} \to {''M^{p,q}} \to 0$ yields a relationship among them. This will be applied in Section 9.

8.4. Unlike the case of the Leray spectral sequence, the main case of importance of 8.1 is that for which f is the identity map $X \to X$. We shall now reformulate 8.1 for this important case.

Let Φ be a family of supports on X with $\dim_\Phi X < \infty$. Let $A \subset X$ be locally closed, and let $\mathscr{A}$ be a sheaf on X. Assume either that A is closed and $\mathscr{A}$ is elementary, or that Φ is paracompactifying. Then there is a spectral sequence with

$$\boxed{E_2^{p,q} = H^p_\Phi(X; \mathscr{H}_{-q}(X, A;\mathscr{A})) \Longrightarrow H^\Phi_{-p-q}(X, A;\mathscr{A}).}$$

In this case, 8.2 becomes the following: Let $F \subset X$ be closed with $U = X - F$, and assume that Φ is paracompactifying with $\dim_\Phi X < \infty$. Then there are the spectral sequences

$$\boxed{'E_2^{p,q} = H^p_\Phi(X, F; \mathscr{H}_{-q}(X;\mathscr{A})) \Longrightarrow H^{\Phi|U}_{-p-q}(U;\mathscr{A})}$$

and

$$\boxed{''E_2^{p,q} = H^p_{\Phi|F}(F; \mathscr{H}_{-q}(X;\mathscr{A})|F) \Longrightarrow H^\Phi_{-p-q}(X, U;\mathscr{A}).}$$

(Here, the family Ψ in 8.2 may be taken to be c.)

8.5. For nonopen subsets (U as in 8.4) or nonparacompactifying supports Φ, we can still obtain these spectral sequences provided we consider only elementary coefficient sheaves. Thus let us assume $A \subset X$ is locally closed, that $\dim_\Phi X < \infty$, and $\dim_{\Phi|A} X < \infty$, and let $(\mathscr{M}, \Phi|A)$ be elementary, where $X - A$ is Φ-taut. Then there is the spectral sequence

$$\boxed{'E_2^{p,q} = H^p_\Phi(X, X - A; \mathscr{H}_{-q}(X;\mathscr{M})) \Longrightarrow H^{\Phi|A}_{-p-q}(A;\mathscr{M})}$$

(use 5.7 and II-12.1), and we also have the spectral sequence

$$\boxed{''E_2^{p,q} = H^p_{\Phi\cap X-A}(X - A; \mathscr{H}_{-q}(X;\mathscr{M})) \Longrightarrow H^\Phi_{-p-q}(X, A;\mathscr{M}).}$$

[The latter spectral sequence is obtained from 5.8 using the double complex $C^p_{\Phi\cap X-A}(X-A; \mathscr{C}_{-q}(X;\mathscr{M})|X-A)$.] Again, as in 8.3, these spectral sequences are linked together through a short exact sequence of their defining double complexes.

Note that when $X-A$ is Φ-taut and $\dim_\Phi X<\infty$, it follows from the cohomology sequence of the pair $(X, X-A)$ that the condition $\dim_{\Phi|A} X<\infty$ is equivalent to $\dim_{\Phi\cap(X-A)}(X-A)<\infty$, whence the convergence of the spectral sequence $''E_r$.

8.6. Similar considerations apply to the case of singular homology. We shall remark on the case of one space and leave the extension to maps to the reader. (See, however, IV-Exercise 1.)

Let Φ be a paracompactifying family of supports on X and let $\mathscr{B}$ be any sheaf on X and let $A\subset X$ be locally closed. Consider the differential sheaf $\mathscr{L}^q=\mathscr{S}_{-q}(X,A;\mathscr{B})$, which is c-soft and hence Φ-soft. It follows from IV-2.1 that if $\dim_\Phi X<\infty$, then there is a spectral sequence with

$$\boxed{E_2^{p,q}=H^p_\Phi(X; {}_S\mathscr{H}_{-q}(X,A;\mathscr{B}))\Longrightarrow {}_SH^\Phi_{-p-q}(X,A;\mathscr{B}).}$$

Remarks analogous to 8.2 and 8.3 also apply in this situation. Also see IV-2.9.

As an immediate application of 8.1 we have the following.

8.7. Proposition. *Let $f:X\to Y$ be a map between locally compact spaces with $\dim_L Y<\infty$. Let $n=\max\{k\,|\,\mathscr{H}_k(f;L)\neq 0\}$, assuming that this exists. Then*

$$\boxed{\Gamma(\mathscr{H}_n(f;L))\approx H_n(X;L).}$$

Proof. In the spectral sequence of 8.1 we have that $E_2^{p,q}=0$ for $q<-n$, whence $\Gamma(\mathscr{H}_n(f;L))\approx H^0(Y;\mathscr{H}_n(f;L))=E_2^{0,-n}\approx E_\infty^{0,-n}\approx H_n(X;L)$. □

8.8. Example. As a simple example of 8.7 consider the case in which $X=\mathbb{S}^1$, and Y is the figure eight with $f:X\to Y$ the obvious map with one double point at $y\in Y$. Take coefficients in $L=\mathbb{Z}$ throughout. Then $\mathscr{H}_p(f)=0$ for $p\neq 1$, and $\mathscr{H}_1(f)$ has stalks $\mathbb{Z}$ except at y, where the stalk is $\mathbb{Z}\oplus\mathbb{Z}$. From 8.7 we have $\Gamma(\mathscr{H}_1(f))\approx H_1(X)\approx\mathbb{Z}$. Also, consider the space $W=\mathbb{S}^1+\mathbb{S}^1$, the topological sum of two circles, and the map $g:W\to Y$ with one double point. Then $\mathscr{H}_1(g)$ has the same stalks as does $\mathscr{H}_1(f)$, but these are not isomorphic sheaves on the figure eight since by the same reasoning, $\Gamma(\mathscr{H}_1(g))\approx H_1(W)\approx\mathbb{Z}\oplus\mathbb{Z}$. The sheaf $\mathscr{H}_1(Y)$ differs from both since $\mathscr{H}_1(Y)_y\approx H_1(Y,Y-\{y\})\approx\tilde{H}_0(4\text{ points})\approx\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. The reader might try to understand just how these sheaves look as spaces as well as to understand the homomorphisms $\mathscr{H}_1(f)\to\mathscr{H}_1(Y)\leftarrow\mathscr{H}_1(g)$, which are isomorphisms away from the point y. ◇

8.9. Example. As an example of 8.1 consider the bundle projection $f : K \to Y = \mathbb{S}^1$, where K is the Klein bottle. Use coefficients in $\mathbb{Z}$ throughout. The stalks of $\mathscr{H}_i(f)$ are

$$\mathscr{H}_i(f)_y \approx H_i(\mathbb{S}^1 \times (\mathbb{I}, \partial\mathbb{I})) \approx H_{i-1}(\mathbb{S}^1) \approx \begin{cases} \mathbb{Z}, & \text{for } i = 1, 2, \\ 0, & \text{for } i \neq 1, 2. \end{cases}$$

It is clear that $\mathscr{H}_1(f)$ is the constant sheaf $\mathbb{Z}$ and that $\mathscr{H}_2(f)$ is the twisted sheaf $\mathbb{Z}^t$ of II-11.3. In the spectral sequence we have $E_2^{0,-1} = H^0(Y;\mathbb{Z}) \approx \mathbb{Z}$, $E_2^{1,-1} = H^1(Y;\mathbb{Z}) \approx \mathbb{Z}$, $E_2^{0,-2} = H^0(Y;\mathbb{Z}^t) = \Gamma(\mathbb{Z}^t) = 0$, $E_2^{1,-2} = H^1(Y;\mathbb{Z}^t) \approx \mathbb{Z}_2$ (by II-11.3), and $E_2^{p,q} = 0$ otherwise. Thus the spectral sequence degenerates into the isomorphisms $H_2(K) \approx E_\infty^{0,-2} \approx E_2^{0,-2} \approx 0$, $H_0(K) \approx E_\infty^{1,-1} \approx E_2^{1,-1} \approx H^1(Y;\mathbb{Z}) \approx \mathbb{Z}$, and the exact sequence $0 \to E_2^{1,-2} \to H_1(K) \to E_2^{0,-1} \to 0$, which is $0 \to \mathbb{Z}_2 \to H_1(K) \to \mathbb{Z} \to 0$. ◇

8.10. Example. Consider the map $f : \mathbb{S}^n \to [-1, 1]$, the projection to a coordinate axis, and take coefficients in $\mathbb{Z}$. At a point $y \in (-1, 1)$ the stalk of $\mathscr{H}_i(f)$ is

$$\mathscr{H}_i(f)_y \approx H_i(\mathbb{S}^{n-1} \times (\mathbb{I}, \partial\mathbb{I})) \approx H_{i-1}(\mathbb{S}^{n-1}) \approx \begin{cases} \mathbb{Z}, & \text{for } i = 1, n, \\ 0, & \text{for } i \neq 1, n. \end{cases}$$

At an end point, $\mathscr{H}_i(f)_y \approx H_i(\mathbb{D}^n, \mathbb{S}^{n-1})$, which is $\mathbb{Z}$ for $i = n$ and is zero otherwise. It is clear that $\mathscr{H}_n(f) \approx \mathbb{Z}$, the constant sheaf, and $\mathscr{H}_1(f) \approx \mathbb{Z}_{(-1,1)}$. Thus the spectral sequence has

$$E_2^{p,-1} = H^p([-1,1];\mathbb{Z}_{(-1,1)}) \approx H_c^p((-1,1);\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } p = 1, \\ 0, & \text{for } p \neq 1 \end{cases}$$

and

$$E_2^{p,-n} = H^p([-1,1];\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{for } p = 0, \\ 0, & \text{for } p \neq 0. \end{cases}$$

Thus it degenerates into the isomorphisms $H_0(\mathbb{S}^n) \approx E_2^{1,-1} \approx \mathbb{Z}$ and $H_n(\mathbb{S}^n) \approx E_2^{0,-n} \approx \mathbb{Z}$. ◇

8.11. Example. Consider $X = \mathbb{RP}^2$ with the usual action of the circle group G on it. (Thinking of X as the 2-disk with antipodal points on the boundary identified, the action is by rotations of the disk.) Let $f : X \to X/G = Y$ be the orbit map. Now Y can be thought of as the interval $[0, 1]$ with 0 corresponding to the fixed orbit, a point called the singular orbit; 1 corresponding to the boundary of the disk, called the exceptional orbit; and points $0 < y < 1$ corresponding to the other circles about the origin, called principal orbits. Let us compute the stalks of the sheaves $\mathscr{H}_p(f)$. We have, with integer coefficients,

$$\mathscr{H}_p(f)_0 \approx H_p(\mathbb{D}^2, \mathbb{S}^1) \approx \begin{cases} 0, & p = 0, \\ 0, & p = 1, \\ \mathbb{Z}, & p = 2. \end{cases}$$

Also, for $0 < y < 1$,

$$\mathscr{H}_p(f)_y \approx H_p(\mathbb{S}^1 \times (\mathbb{I}, \partial\mathbb{I})) \approx \begin{cases} 0, & p = 0, \\ \mathbb{Z}, & p = 1, \\ \mathbb{Z}, & p = 2, \end{cases}$$

and

$$\mathscr{H}_p(f)_1 \approx H_p(M, \partial M) \approx \begin{cases} 0, & p = 0, \\ \mathbb{Z}_2, & p = 1, \\ 0, & p = 2 \end{cases}$$

where M is a neighborhood of the exceptional orbit and is a Möbius strip. It is clear that

$$\mathscr{H}_2(f) \approx \mathbb{Z}_{[0,1)}.$$

Similarly, it is clear that $\mathscr{H}_1(f)$ is constant over $(0,1)$, and of course it vanishes at 0. To see its topology at 1, consider a local section s about 1 giving the nonzero element of $\mathscr{H}_1(f)_1 \approx \mathbb{Z}_2$. Since $2s$ vanishes at 1, it must vanish over a neighborhood of 1. Since the stalks of $\mathscr{H}_1(f)$ are free at points other than 1, s itself must vanish on a neighborhood of 1, except at 1 itself.[17] From this it is clear that

$$\mathscr{H}_1(f) \approx \mathbb{Z}_{(0,1)} \oplus (\mathbb{Z}_2)_{\{1\}}.$$

Thus we compute

$$H^p(Y; \mathscr{H}_2(f)) \approx H^p([0,1]; \mathbb{Z}_{[0,1)}) \approx H^p([0,1], \{1\}; \mathbb{Z}) = 0$$

for all p. Also,

$$H^p(Y; \mathscr{H}_1(f)) \approx H^p([0,1]; \mathbb{Z}_{(0,1)}) \oplus H^p(\{1\}; \mathbb{Z}_2) \approx \begin{cases} \mathbb{Z}_2, & p = 0, \\ \mathbb{Z}, & p = 1, \\ 0, & p \neq 0, 1. \end{cases}$$

Therefore $E_2^{0,-1} = H^0(Y; \mathscr{H}_1(f)) \approx \mathbb{Z}_2$, $E_2^{1,-1} = H^1(Y; \mathscr{H}_1(f)) \approx \mathbb{Z}$, and all other terms are zero. Thus the spectral sequence degenerates to give the familiar homology groups $H_0(X) \approx E_2^{1,-1} \approx \mathbb{Z}$ and $H_1(X) \approx E_2^{0,-1} \approx \mathbb{Z}_2$ of $\mathbb{RP}^2$. ◇

8.12. Example. Let $X = \mathbb{I}^\infty$, the Hilbert cube. Then $\mathscr{H}_p(X; L) = 0$ for all p since every point has an open neighborhood basis consisting of sets homeomorphic to $Y \times (0,1]$ for some locally compact Y and since $H_c^p((0,1]; L) \approx H^p([0,1], \{0\}; L) = 0$ for all p. If the spectral sequence of 8.4 were valid in this case, we would have $E_2^{p,q} = 0$ for all p, q whence $H_0^c(X; L) = 0$, which is false. Consequently, the dimensional hypothesis of 8.1 is essential. ◇

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[17] A more direct way to see this is to note that the generator of the stalk at 1 is given by the singular cycle made up of the exceptional orbit, as a cycle of $(M, \partial M)$, and that this induces the zero homology class at a nearby principal orbit.

8.13. Example. Let $X = \mathbb{I}^\infty$, $Y = \mathbb{I}$, and $f : X \to Y$ a projection. Then for $y = 0, 1$ we have $\mathscr{H}_q(f; L)_y \approx H_q([0,1) \times \mathbb{I}^\infty; L) \approx H_q([0,1); L) = 0$ for all q. For $y \neq 0, 1$ we have

$$\mathscr{H}_q(f; L)_y \approx H_q((0,1) \times \mathbb{I}^\infty; L) \approx H_q((0,1); L) \approx \begin{cases} L, & \text{for } q = 1, \\ 0, & \text{for } q \neq 1. \end{cases}$$

Clearly, $\mathscr{H}_1(f; L) \approx L_{(0,1)}$ and $\mathscr{H}_q(f; L) = 0$ for $q \neq 1$. The basic spectral sequence 8.1 is valid here even though $\dim_L X = \infty$. For it, we have $E_2^{1,-1} \approx H^1([0,1]; L_{(0,1)}) \approx L$, and $E_2^{p,q} = 0$ in all other cases. Thus the spectral sequence degenerates into the uninteresting isomorphism $L \approx E_2^{1,-1} \approx H_{-1+1}(\mathbb{I}^\infty; L)$. ◇

9 Poincaré duality

In this section we will establish Poincaré duality for spaces having the local homological properties of n-manifolds with or without boundary. A general class of such spaces is given by the following definition:

9.1. Definition. *A locally compact space X is called an "n-dimensional weak homology manifold over L" (abbreviated as n-whm$_L$) if* $\dim_L X < \infty$ *and $\mathscr{H}_p(X; L) = 0$ for $p \neq n$ and is torsion-free if $p = n$. $\mathscr{H}_n(X; L)$ is called the "orientation sheaf" of X and will be denoted by $\mathscr{O} = \mathscr{O}_X$. An n-whm$_L$ X is said to be an "n-dimensional homology manifold over L" (abbreviated as n-hm$_L$) if $\mathscr{O}$ is locally constant*[18] *with stalks isomorphic to L. It is said to be "orientable" if $\mathscr{O}$ is constant.*

Note that for any *n-whm$_L$* X,

$$\mathscr{H}_p(X; \mathscr{A}) = \begin{cases} \mathscr{O} \otimes \mathscr{A}, & p = n, \\ 0, & p \neq n \end{cases}$$

by (13) on page 294. Also note that a topological n-manifold, with or without boundary, is an *n-whm$_L$*, and it is an *n-hm$_L$* if the boundary is empty. (We shall study homology manifolds with boundary in Section 16.)

From 8.4 we immediately obtain:

9.2. Theorem. *If X is an n-whm$_L$ and Φ is a paracompactifying family of supports on X, then there is a natural isomorphism*

$$\boxed{\Delta : H^p_\Phi(X; \mathscr{O} \otimes \mathscr{A}) \xrightarrow{\approx} H^\Phi_{n-p}(X; \mathscr{A})}$$

for sheaves $\mathscr{A}$ on X. Moreover, Δ is an isomorphism of connected sequences of functors of $\mathscr{A}$; that is, $\partial\Delta = (-1)^n \Delta\delta$ in the diagram

$$\begin{array}{ccc} H^p_\Phi(X; \mathscr{O} \otimes \mathscr{A}'') & \xrightarrow{\Delta} & H^\Phi_{n-p}(X; \mathscr{A}'') \\ \downarrow \delta & & \downarrow \partial \\ H^{p+1}_\Phi(X; \mathscr{O} \otimes \mathscr{A}') & \xrightarrow{\Delta} & H^\Phi_{n-p-1}(X; \mathscr{A}') \end{array}$$

[18] See, however, 16.8.

induced by the exact coefficient sequence $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$, *in addition to naturality in the coefficient sheaf* $\mathscr{A}$.

Proof. In the spectral sequence $E_r^{p,q}$ of 8.4 we have $E_2^{p,q} = 0$ for $q \neq -n$. Thus $H_\Phi^p(X; \mathscr{O} \otimes \mathscr{A}) \approx E_2^{p,-n} \approx E_\infty^{p,-n} \approx H_{-p+n}^\Phi(X; \mathscr{A})$ as claimed. The last statement follows from the compatibility relations proved in IV-1, the sign $(-1)^n$ being achieved by redefinition of Δ by induction on p if necessary. □

If $F \subset X$ is closed and $U = X - F$, then we also obtain from 8.4 the isomorphisms

$$\boxed{H_{\Phi|U}^p(U; \mathscr{O} \otimes \mathscr{A}) \approx H_\Phi^p(X, F; \mathscr{O} \otimes \mathscr{A}) \approx H_{n-p}^{\Phi|U}(U; \mathscr{A})}$$

and

$$\boxed{H_{\Phi|F}^p(F; \mathscr{O} \otimes \mathscr{A}) \approx H_{n-p}^\Phi(X, U; \mathscr{A}).}$$

However, these also follow directly from 9.2 by replacing $\mathscr{A}$ by $\mathscr{A}_U$ and $\mathscr{A}_F$ respectively.

It follows easily from 8.3 that the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H_{\Phi|U}^p(U; \mathscr{A} \otimes \mathscr{O}) & \to & H_\Phi^p(X; \mathscr{A} \otimes \mathscr{O}) & \to & H_{\Phi|F}^p(F; \mathscr{A} \otimes \mathscr{O}) & \to \cdots \\
(-1)^n & \Delta \downarrow \approx & 1 & \Delta \downarrow \approx & 1 & \Delta \downarrow \approx & (-1)^n \\
\cdots \to & H_{n-p}^{\Phi|U}(U; \mathscr{A}) & \to & H_{n-p}^\Phi(X; \mathscr{A}) & \to & H_{n-p}^\Phi(X, U; \mathscr{A}) & \to \cdots
\end{array}$$

commutes up to the indicated signs, where the first row is the cohomology sequence of (X, F) and the second row is the homology sequence of (X, U).

For elementary coefficient sheaves we obtain, from 8.5, the following generalization of 9.2 valid for nonparacompactifying families of supports and more general subspaces:

9.3. Theorem. *Let X be an n-whm$_L$, $\mathscr{M}$ an elementary sheaf on X, and Φ a family of supports on X with* $\dim_\Phi X < \infty$. *Then there is an isomorphism*

$$\boxed{\Delta : H_\Phi^p(X; \mathscr{O} \otimes \mathscr{M}) \xrightarrow{\approx} H_{n-p}^\Phi(X; \mathscr{M})}$$

natural in $\mathscr{M}$. Assume, moreover, that A is a locally closed subspace of X such that $X - A$ is Φ-taut in X, and that $(\mathscr{M}, \Phi|A)$ is elementary. If, in addition, $\dim_{\Phi|A} X < \infty$, *then*

$$\boxed{H_\Phi^p(X, X - A; \mathscr{O} \otimes \mathscr{M}) \approx H_{n-p}^{\Phi|A}(A; \mathscr{M})}$$

and

$$\boxed{H_{\Phi \cap (X-A)}^p(X - A; \mathscr{O} \otimes \mathscr{M}) \approx H_{n-p}^\Phi(X, A; \mathscr{M}).}$$

□

If all the conditions of 9.3 are satisfied, then as in 9.2 the following diagram commutes up to the indicated sign:

$$\begin{array}{ccccccc}
\cdots \to H^p_\Phi(X, X-A; \mathcal{O}\otimes\mathcal{M}) & \to & H^p_\Phi(X; \mathcal{O}\otimes\mathcal{M}) & \to & H^p_{\Phi\cap(X-A)}(X-A; \mathcal{O}\otimes\mathcal{M}) \to \cdots \\
(-1)^n \quad \Delta\downarrow\approx & 1 & \Delta\downarrow\approx & 1 & \Delta\downarrow\approx \quad (-1)^n \\
\cdots \to H^{\Phi|A}_{n-p}(A; \mathcal{M}) & \to & H^\Phi_{n-p}(X; \mathcal{M}) & \to & H^\Phi_{n-p}(X, A; \mathcal{M}) \to \cdots
\end{array}$$

Danger: When Φ is not paracompactifying, then as far as the author knows, unless X is locally hereditarily paracompact, one cannot draw the conclusion that $\dim_\Phi X < \infty$ from the assumption that $\dim_L X < \infty$; see II-16 and II-Exercise 25. Note, however, II-16.10.

9.4. As an example of the use of 9.3, let X be an orientable n-manifold with boundary consisting of the disjoint union $A \cup B$ of closed sets. (A or B can be empty.) Then $\mathcal{O} = L_{X-A-B}$, and we have

$$\begin{aligned} H^c_{n-p}(X, A; L) &\approx H^p_{c\cap(X-A)}(X - A; L_{X-A-B}) \\ &\approx H^p_{c\cap(X-A)}(X - A, B; L) \approx H^p_c(X, B; L), \end{aligned}$$

the last isomorphism following from invariance under homotopy. Also see 16.30 for a generalization of this.

9.5. If Φ consists of all closed sets in 9.3 and $\mathcal{M} = L$, we can substitute $H^{n-p}(U; \mathcal{O})$ for $H_p(U; L)$ in (9) on page 292 and obtain the exact sequence

$$\boxed{0 \to \operatorname{Ext}(H^{p+1}_c(U; L), L) \to H^{n-p}(U; \mathcal{O}) \to \operatorname{Hom}(H^p_c(U; L), L) \to 0}$$

natural in U. In particular, if L is a *field*, then there is a natural isomorphism $H^{n-p}(U; \mathcal{O}) \approx \operatorname{Hom}(H^p_c(U; L), L)$ for U open in an n-whm_L X with $\dim_{cld} U < \infty$. Also see 10.6.

9.6. If X is an n-hm_L, then $\mathcal{O}$ is a "bundle" of coefficients locally isomorphic to L. This implies that there is an "inverse sheaf" $\mathcal{O}^{-1}$ such that $\mathcal{O} \otimes \mathcal{O}^{-1} \approx L$ and $\mathcal{O}^{-1}$ is unique up to isomorphism. In this case, 9.2 shows that

$$H^p_\Phi(X; \mathcal{A}) \approx H^p_\Phi(X; \mathcal{O} \otimes \mathcal{O}^{-1} \otimes \mathcal{A}) \approx H^\Phi_{n-p}(X; \mathcal{O}^{-1} \otimes \mathcal{A})$$

for Φ paracompactifying. Similarly 9.3 can be used to obtain the isomorphism

$$H^p_\Phi(X; \mathcal{M}) \approx H^\Phi_{n-p}(X; \mathcal{O}^{-1} \otimes \mathcal{M})$$

when $\mathcal{M}$ is elementary and $\dim_\Phi X < \infty$.

In [11] it is claimed that $\mathcal{O} \approx \mathcal{O}^{-1}$ in general. Unfortunately, the proof is irretrievably incorrect, and this remains an unsolved conjecture.

In particular, we have $H^{n+1}_c(U; L) \approx H^c_{-1}(U; \mathcal{O}^{-1}) = 0$ by 5.13. Then it follows from II-16.14 that

$$\boxed{\dim_L X = n \quad \text{for an } n\text{-}hm_L\ X.}$$

9.7. For a topological n-manifold X (or a "singular homology manifold") we obtain the Poincaré duality

$$H^p_\Phi(X; \mathscr{B} \otimes \mathscr{O}) \approx {}_sH^\Phi_{n-p}(X; \mathscr{B})$$

(for Φ paracompactifying and $\mathscr{B}$ arbitrary) directly from 8.6. However, this also follows from the general Theorem 9.2 and from the uniqueness theorems that we shall prove in Section 12 showing that singular homology coincides with Borel-Moore homology in this case. Also see IV-2.9.

9.8. Example. Duality, and indeed Borel-Moore homology itself, is useful mostly for reasonably locally well behaved spaces such as clc_L spaces. However, let us illustrate duality in the case of a "bad" space, the solenoid, i.e., an inverse limit $\Sigma = \varprojlim C_n$, where C_n is a circle and $C_n \to C_{n-1}$ is the usual covering map of degree n. Coefficients will be in $L = \mathbb{Z}$ if not otherwise specified. By continuity, $H^0(\Sigma) \approx \mathbb{Z}$, $H^1(\Sigma) \approx \mathbb{Q}$, and $H^i(\Sigma) = 0$ for $i > 1$. Also, $\dim_{\mathbb{Z}} \Sigma = 1$. By Exercise 6 the precosheaf $\mathfrak{H}^1_c(\Sigma) : U \mapsto H^1_c(U)$ is a cosheaf, and by 1.10 and (9) on page 292, the presheaf $U \mapsto \operatorname{Hom}(H^1_c(U), \mathbb{Z}) \approx H_1(U)$ is a sheaf.[19] It is, of course, the homology sheaf $\mathscr{H}_1(\Sigma)$. Now, every point in Σ has a fundamental sequence of neighborhoods U of the form $U \approx K \times (0,1)$, where K is a Cantor set. For such a set U, $H^1_c(U) \approx H^0(K) \otimes \mathbb{Z} \approx C(K)$, the group of continuous (i.e., locally constant) functions $K \to \mathbb{Z}$. This is the free abelian group on a countably infinite set of generators; see II-Exercise 34. Similarly, $H^0_c(U) = 0$, so that $\mathscr{H}_0(U) = 0$, and $\mathscr{H}_i(U) = 0$ for $i > 1$ trivially. Thus X is a 1-$whm_{\mathbb{Z}}$ with orientation sheaf $\mathscr{O} = \mathscr{H}_1(\Sigma) : U \mapsto \operatorname{Hom}(H^1_c(U), \mathbb{Z})$.

The sheaf $\mathscr{O}$ is not readily understood, but duality gives some easy information about it. We have $H^p(\Sigma; \mathscr{O}) \approx H_{1-p}(\Sigma)$. In particular,

$$\Gamma(\mathscr{O}) = H^0(\Sigma; \mathscr{O}) \approx H_1(\Sigma) \approx \operatorname{Hom}(H^1_c(\Sigma), \mathbb{Z}) \approx \operatorname{Hom}(\mathbb{Q}, \mathbb{Z}) = 0,$$

so that $\mathscr{O}$ has no nonzero global sections. On the other hand, $\mathscr{O}$ is a very big sheaf because

$$H^1(\Sigma; \mathscr{O}) \approx H_0(\Sigma) \approx \operatorname{Hom}(H^0_c(\Sigma), \mathbb{Z}) \oplus \operatorname{Ext}(H^1_c(\Sigma), \mathbb{Z}) \approx \mathbb{Z} \oplus \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$$

and $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ is uncountable; see 14.8. ◇

9.9. Example. Again, let Σ be the solenoid. We will show that 9.3 cannot hold for $L = \mathbb{Z}$ and $\mathscr{M} = \mathbb{Q}$ (which is not an elementary sheaf since $\mathbb{Q}$ is not finitely generated over $\mathbb{Z}$). By classical dimension theory, Σ can be embedded in $\mathbb{S}^3$; also, see the next example for an explicit embedding. The exact homology sequence of the pair $(\mathbb{S}^3, \Sigma)$ shows that

$$H_2(\mathbb{S}^3, \Sigma; \mathbb{Q}) \approx H_1(\Sigma; \mathbb{Q}) \approx H_1(\Sigma; \mathbb{Z}) \otimes \mathbb{Q} = 0$$

[19]These facts are not important for the example but are helpful in understanding the sheaf $\mathscr{O}$.

by 3.13 and the computations in the previous example. If 9.3 held in this case, we would have that this is isomorphic to $H^1(\mathbb{S}^3 - \Sigma; \mathbb{Q})$. However, *using* $\mathbb{Q}$ *as base ring*,[20] we compute

$$\begin{aligned} H^1(\mathbb{S}^3 - \Sigma; \mathbb{Q}) &\approx H_2(\mathbb{S}^3, \Sigma; \mathbb{Q}) && \text{by 9.3} \\ &\approx H_1(\Sigma; \mathbb{Q}) && \text{exact sequence of } (\mathbb{S}^3, \Sigma) \\ &\approx \operatorname{Hom}_{\mathbb{Q}}(H^1(\Sigma; \mathbb{Q}), \mathbb{Q}) && \text{by (9) on page 292} \\ &\approx \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) \approx \mathbb{Q}. \end{aligned}$$

◇

9.10. Example. Consider a standard solid torus T_1 in $\mathbb{S}^3$. Inside T_1 we can embed another solid torus T_2 that winds around T_1 twice. Inside that we can embed another solid torus T_3 that winds around T_2 three times, and so on. If we make the diameters of the disk cross sections of T_n go to zero as $n \to \infty$, then the intersection of the tori is a solenoid $\Sigma = \bigcap T_n$. With integer coefficients, we have that $\mathbb{Z} \approx H_1(T_{n+1}) \to H_1(T_n) \approx \mathbb{Z}$ is multiplication by n. By duality 9.3, this is equivalent to $H^2(\mathbb{S}^3, \mathbb{S}^3 - T_{n+1}) \to H^2(\mathbb{S}^3, \mathbb{S}^3 - T_{n+1})$, and by the exact sequences of the pairs $(\mathbb{S}^3, \mathbb{S}^3 - T_i)$ this is equivalent to $H^1(\mathbb{S}^3 - T_{n+1}) \to H^1(\mathbb{S}^3 - T_n)$. Thus the latter maps give the inverse sequence $\cdots \xrightarrow{4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Also, $H^2(\mathbb{S}^3 - T_n) \approx H_1(\mathbb{S}^3, T_n) \approx \operatorname{Ker}\left(H_1(T_n) \to H_1(\mathbb{S}^3)\right) = 0$. It follows from this and IV-2.11 that

$$H^2(\mathbb{S}^3 - \Sigma; \mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}).$$

This can also be seen by the isomorphism $H^2(\mathbb{S}^3 - \Sigma) \approx H_1(\mathbb{S}^3, \Sigma)$ of 9.3, the exact homology sequence of the pair $(\mathbb{S}^3, \Sigma)$, and the calculation in 9.8 that $H_0(\Sigma) \approx \mathbb{Z} \oplus \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$. ◇

9.11. Example. Suppose that $X \subset \mathbb{R}^n$ is compact, and let M be a *finitely generated* L-module. By 9.3 we have the natural isomorphism

$$H_p(X; M) \approx H^{n-p}(\mathbb{R}^n, \mathbb{R}^n - X; M),$$

and the cohomology group here can be taken to be singular cohomology by III-2.1. Thus Borel-Moore homology for such spaces can be computed from ordinary singular cohomology. This is not true for coefficient groups M that are not finitely generated over L, as is shown by taking X to be a solenoid and computing these groups for $L = \mathbb{Z}$ and $M = \mathbb{Q}$. (This is the failure of "change of rings"; see Section 18.) ◇

9.12. Example. Another example of an n-whm_L that is not an n-hm_L is the mapping cone C_f of the covering map $f : X = \mathbb{S}^1 \to \mathbb{S}^1 = Y$ of degree d. This is a 2-$whm_{\mathbb{Z}}$. Its orientation sheaf $\mathscr{O}$ is constant with stalks $\mathbb{Z}$ over $C_f - Y$. Over Y, $\mathscr{O}$ has stalks $\mathbb{Z}^{d-1}$, and it is not too hard to see that $\mathscr{O}|Y \approx f\mathbb{Z}/\mathbb{Z}$, where $\mathbb{Z}$ sits in $f\mathbb{Z}$ as the "diagonal." It is somewhat

[20] See Section 18.

harder to understand how these two portions of $\mathscr{O}$ fit together, and we will not attempt to describe that. Duality gives

$$H^p(C_f;\mathscr{O}) \approx H_{2-p}(C_f;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & p=2, \\ \mathbb{Z}_d, & p=1, \\ 0, & p\neq 1,2 \end{cases}$$

as follows easily from the homology sequence of (M_f, X). One can compute these cohomology groups directly by looking at the exact sequence

$$\cdots \to H^p_c(C_f - Y;\mathbb{Z}) \to H^p(C_f;\mathscr{O}) \to H^p(Y;\mathscr{O}|Y) \to \cdots$$

as follows: $H^p_c(C_f - Y;\mathbb{Z}) \approx H^p_c(\mathbb{R}^2;\mathbb{Z}) \approx \mathbb{Z}$ for $p=2$ and is zero otherwise. The exact cohomology sequence induced on Y by $0 \to \mathbb{Z} \to f\mathbb{Z} \to \mathscr{O}|Y \to 0$ has the portion $H^p(Y;\mathbb{Z}) \to H^p(Y;f\mathbb{Z}) \approx H^p(X;\mathbb{Z})$ identified with f^*, so that $H^p(Y;\mathscr{O}|Y) \approx \mathbb{Z}_d$ for $p=1$ and is zero otherwise, yielding the desired calculation. ◇

9.13. Example. The purpose of this example is to show how Borel-Moore homology and duality can be used to understand a particular cohomology group (closed supports) of a certain space X. The space X is perhaps the simplest space that is locally compact but not compact, and connected but not locally connected. It is the "comb space" with the "base accumulation point" removed. Precisely,

$$X = \{(a,0) \in \mathbb{R}^2 \mid 0 < a \le 1\} \cup \{(a,b) \in \mathbb{R}^2 \mid 0 < b \le 1,\ a = 0, 1/2, 1/4, \ldots\}.$$

Note that the point $(0,0)$ is missing. Regarding $\mathbb{S}^2$ as the compactified $\mathbb{R}^2$, let $M = \mathbb{S}^2 - \{(0,0)\}$, a euclidean plane containing X as a closed subspace.

The group we wish to investigate is $H^1(X;\mathbb{Z})$.[21] One's first guess concerning this group might be that it is zero. The singular cohomology group ${}_SH^1(X;\mathbb{Z})$ is clearly zero. Also, the closure of X in $\mathbb{R}^2$, the comb space itself, is contractible and so has the cohomology of a point. However, we will show that this guess is about as far from true as it can be.

The exact sequence (coefficients are in $\mathbb{Z}$ when omitted)

$$0 = H^1(M) \to H^1(X) \to H^2(M,X) \to H^2(M) = 0$$

shows that $H^1(X) \approx H^2(M,X)$. By duality 9.2, $H^2(M,X) \approx H_0^\Psi(U)$, where $U = M - X$ and $\Psi = \{K \subset U \mid K \text{ is closed in } M\}$. Thus, by 5.3,

$$H^1(X) \approx H_0^\Psi(U) \approx \varinjlim_{K\in\Psi} H_0(K).$$

Also, there is the exact sequence

$$0 \to \varinjlim_{K\in\Psi} \operatorname{Ext}(H^1_c(K),\mathbb{Z}) \to \varinjlim_{K\in\Psi} H_0(K) \to \varinjlim_{K\in\Psi} \operatorname{Hom}(H^0_c(K),\mathbb{Z}) \to 0.$$

[21]This example arose in an attempt to prove, for homology manifolds, a certain dimensionality property that is a standard result for topological manifolds.

The term in Ext is surely zero, but we shall ignore it, as it is unimportant for the main conclusion. One set in Ψ is

$$K_0 = \{(a, a/2) \in \mathbb{R}^2 \mid a = 3/2^n,\ n = 2, 3, 4, \ldots\}.$$

We have $H_c^0(K_0) \approx \bigoplus_{i=1}^{\infty} \mathbb{Z}$, so that $\operatorname{Hom}(H_c^0(K_0), \mathbb{Z}) \approx \prod_{i=1}^{\infty} \mathbb{Z}$, which is uncountable. Note that for *any* set $K \supset K_0$ in Ψ, all but a finite number of the points of K_0 must be in separate pseudo-components of K. This implies that

$$\operatorname{Ker}\{\operatorname{Hom}(H_c^0(K_0), \mathbb{Z}) \to \operatorname{Hom}(H_c^0(K), \mathbb{Z})\} \subset \bigoplus_{i=1}^{\infty} \mathbb{Z},$$

which is countable. It follows that $\varinjlim_{K \in \Psi} \operatorname{Hom}(H_c^0(K), \mathbb{Z})$, and hence $H^1(X) \approx H_0^{\Psi}(U)$, is uncountable. It also seems likely that this group is not free, although it is torsion-free by II-Exercise 28. ◇

10 The cap product

For sheaves $\mathscr{A}$ and $\mathscr{B}$ on the locally compact space X, consider the canonical map

$$\Gamma_\Phi(\mathscr{C}_*(X; L) \otimes \mathscr{A}) \otimes \Gamma_\Psi(\mathscr{B}) \to \Gamma_{\Phi \cap \Psi}(\mathscr{C}_*(X; L) \otimes \mathscr{A} \otimes \mathscr{B})$$

of I-5, that is,

$$C_*^\Phi(X; \mathscr{A}) \otimes H_\Psi^0(X; \mathscr{B}) \to C_*^{\Phi \cap \Psi}(X; \mathscr{A} \otimes \mathscr{B}), \tag{49}$$

which is a chain map. For any integer m, we have the induced map

$$\cap : H_m^\Phi(X; \mathscr{A}) \otimes H_\Psi^0(X; \mathscr{B}) \to H_m^{\Phi \cap \Psi}(X; \mathscr{A} \otimes \mathscr{B}). \tag{50}$$

If $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ is exact and is such that $0 \to \mathscr{A}' \otimes \mathscr{B} \to \mathscr{A} \otimes \mathscr{B} \to \mathscr{A}'' \otimes \mathscr{B} \to 0$ is also exact, then the fact that (49) is a chain map implies that for $\alpha \in H_m^\Phi(X; \mathscr{A}'')$ and $s \in H_\Psi^0(X; \mathscr{B})$ we have

$$\partial(\alpha \cap s) = (\partial \alpha) \cap s \tag{51}$$

(defined when Φ and $\Phi \cap \Psi$ are paracompactifying).

Now let m and $\alpha \in H_m^\Phi(X; \mathscr{A})$ be fixed. If $\Phi \cap \Psi$ is paracompactifying, the functors

$$F^q(\mathscr{B}) = H_{m-q}^{\Phi \cap \Psi}(X; \mathscr{A} \otimes \mathscr{B}),$$

with connecting homomorphism $(-1)^m$ times that in $H_*^{\Phi \cap \Psi}(X; \bullet)$, form an $\mathscr{E}$-connected sequence of functors, where $\mathscr{E}$ is the class of pointwise split short exact sequences. By II-6.2 the natural transformation $\beta \mapsto \alpha \cap \beta$ of

$$H_\Psi^0(X; \mathscr{B}) \to F^0(\mathscr{B}) = H_m^{\Phi \cap \Psi}(X; \mathscr{A} \otimes \mathscr{B})$$

has a unique extension

$$\alpha\cap : H^p_\Psi(X;\mathscr{B}) \to H^{\Phi\cap\Psi}_{m-p}(X;\mathscr{A}\otimes\mathscr{B})$$

commuting with connecting homomorphisms. The uniqueness of $\alpha\cap$ implies immediately that $\alpha\cap\beta$ is linear in α as well as in β, so that we obtain the "cap product"

$$\boxed{\cap : H^\Phi_m(X;\mathscr{A})\otimes H^p_\Psi(X;\mathscr{B}) \to H^{\Phi\cap\Psi}_{m-p}(X;\mathscr{A}\otimes\mathscr{B})} \tag{52}$$

when $\Phi\cap\Psi$ is paracompactifying. By definition and II-6.2, we have that if $0\to\mathscr{B}'\to\mathscr{B}\to\mathscr{B}''\to 0$ and $0\to\mathscr{A}\otimes\mathscr{B}'\to\mathscr{A}\otimes\mathscr{B}\to\mathscr{A}\otimes\mathscr{B}''\to 0$ are both exact, then

$$\boxed{\partial(\alpha\cap\beta) = (-1)^m\alpha\cap\delta\beta,} \tag{53}$$

where $\alpha\in H^\Phi_m(X;\mathscr{A})$, $\beta\in H^p_\Psi(X;\mathscr{B}'')$, and ∂ and δ are the connecting homomorphisms in $H^{\Phi\cap\Psi}_*(X;\bullet)$ and $H^*_\Psi(X;\bullet)$ respectively.

By the uniqueness portion of II-6.2 we obtain, in the situation of (51) and for $\beta\in H^p_\Psi(X;\mathscr{B})$,

$$\boxed{\partial(\alpha\cap\beta) = (\partial\alpha)\cap\beta.} \tag{54}$$

That is, the two homomorphisms $H^p_\Psi(X;\mathscr{B})\to H^{\Phi\cap\Psi}_{m-p-1}(X;\mathscr{A}'\otimes\mathscr{B})$ defined by (54) for fixed $\alpha\in H^\Phi_m(X;\mathscr{A}'')$ are identical.

If $\alpha\in H^\Phi_m(X;\mathscr{A})$, $s\in\Gamma_\Psi(\mathscr{B})$, $t\in\Gamma_\Theta(\mathscr{C})$, then it follows immediately from the definition of (49) and (50) that

$$(\alpha\cap s)\cap t = \alpha\cap(s\cup t), \tag{55}$$

where $s\cup t\in\Gamma_{\Phi\cap\Theta}(\mathscr{B}\otimes\mathscr{C})$. The uniqueness part of II-6.2 applied, in the appropriate way, to the functors

$$H^*_\Psi(X;\mathscr{B}),\quad H^*_\Theta(X;\mathscr{C}),\quad \text{and}\quad H^{\Phi\cap\Psi\cap\Theta}_*(X;\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{C})$$

($\mathscr{A}$ being fixed) shows immediately that if $\Phi\cap\Psi$ and $\Phi\cap\Psi\cap\Theta$ are paracompactifying, then for $\alpha\in H^\Phi_m(X;\mathscr{A})$, $\beta\in H^p_\Psi(X;\mathscr{B})$, and $\gamma\in H^q_\Theta(X;\mathscr{C})$ we have

$$\boxed{(\alpha\cap\beta)\cap\gamma = \alpha\cap(\beta\cup\gamma)} \tag{56}$$

(given by (55) in degree zero).

Let $f : X\to Y$ be a map between locally compact spaces and let Φ and Ψ be families of supports on Y with $\Phi\cap\Psi$ paracompactifying. Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on Y. By definition of the isomorphism (21) and of the homomorphism (23) on page 23, we see that the following diagram commutes:

$$\begin{array}{ccc} f_c(\mathscr{C}_*(X;f^*\mathscr{A}))\otimes\mathscr{B} & \xrightarrow{\approx} & f_c\mathscr{C}_*(X;f^*(\mathscr{A}\otimes\mathscr{B})) \\ \downarrow & & \downarrow \\ \mathscr{C}_*(Y;\mathscr{A})\otimes\mathscr{B} & \xrightarrow{\approx} & \mathscr{C}_*(Y;\mathscr{A}\otimes\mathscr{B}). \end{array}$$

It follows that we also have the commutative diagram

$$\begin{array}{ccc}
\Gamma_{\Phi(c)}(\mathscr{C}_*(X;f^*\mathscr{A}))\otimes\Gamma_{f^{-1}\Psi}(f^*\mathscr{B}) & \longrightarrow & \Gamma_{(\Phi\cap\Psi)(c)}(\mathscr{C}_*(X;f^*(\mathscr{A}\otimes\mathscr{B}))) \\
\uparrow{\scriptstyle 1\otimes f^*} & & \uparrow \\
\Gamma_\Phi(f_c\mathscr{C}_*(X;f^*\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{B}) & \longrightarrow & \Gamma_{\Phi\cap\Psi}(f_c\mathscr{C}_*(X;f^*(\mathscr{A}\otimes\mathscr{B}))) \\
\downarrow & & \downarrow \\
\Gamma_\Phi(\mathscr{C}_*(Y;\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{B}) & \longrightarrow & \Gamma_{\Phi\cap\Psi}(\mathscr{C}_*(Y;\mathscr{A}\otimes\mathscr{B}))
\end{array}$$

[recalling from IV-5.4(7) that $\Phi(c)\cap f^{-1}(\Psi)=(\Phi\cap\Psi)(c)$].

Therefore the following diagram commutes for $p=0$:

$$\begin{array}{ccc}
H_n^{\Phi(c)}(X;f^*\mathscr{A})\otimes H^p_{f^{-1}\Psi}(X;f^*\mathscr{B}) & \xrightarrow{\ \cap\ } & H_{n-p}^{(\Phi\cap\Psi)(c)}(X;f^*(\mathscr{A}\otimes\mathscr{B})) \\
\uparrow{\scriptstyle 1\otimes f^*} & & \\
H_n^{\Phi(c)}(X;f^*\mathscr{A})\otimes H^p_\Psi(Y;\mathscr{B}) & & \downarrow{\scriptstyle f_*} \\
\downarrow{\scriptstyle f_*\otimes 1} & & \\
H_n^{\Phi}(Y;\mathscr{A})\otimes H^p_\Psi(Y;\mathscr{B}) & \xrightarrow{\ \cap\ } & H_{n-p}^{\Phi\cap\Psi}(Y;\mathscr{A}\otimes\mathscr{B}).
\end{array}$$

For $\alpha\in H_n^{\Phi(c)}(X;f^*\mathscr{A})$ it follows that the natural transformations $H^n_\Psi(Y;\mathscr{B})\to H_{n-p}^{\Phi\cap\Psi}(Y;\mathscr{A}\otimes\mathscr{B})$ of functors of $\mathscr{B}$ defined by $\beta\mapsto f_*(\alpha\cap f^*\beta)$ and $\beta\mapsto f_*(\alpha)\cap\beta$ coincide for $p=0$. Using (53), it follows from II-6.2 that these transformations are identical. Thus, for $\alpha\in H_n^{\Phi(c)}(X;f^*\mathscr{A})$ and $\beta\in H^n_\Psi(Y;\mathscr{B})$, we always have that

$$\boxed{f_*(\alpha\cap f^*\beta)=f_*(\alpha)\cap\beta.}\tag{57}$$

10.1. Theorem. *Let X be such that $\dim_L X<\infty$ and $\mathscr{H}_p(X;L)=0$ for $p>n$ and with $\mathscr{O}=\mathscr{H}_n(X;L)$ torsion-free for $p=n$ (as is always the case if $n=\dim_L X$; see Exercise 32). Then, for Φ paracompactifying, there is a natural homomorphism*

$$\boxed{\Delta: H^p_\Phi(X;\mathscr{O}\otimes\mathscr{A})\to H^\Phi_{n-p}(X;\mathscr{A})}$$

generalizing that of 9.2, which is an isomorphism for $p=0$ and satisfies $\partial\Delta=(-1)^n\Delta\delta$, where ∂ and δ are the connecting homomorphisms for the variable $\mathscr{A}$. If Φ and $\Phi\cap\Psi$ are both paracompactifying families of supports on X, then, for $\alpha\in H^r_\Phi(X;\mathscr{O}\otimes\mathscr{A})$ and $\beta\in H^p_\Psi(X;\mathscr{B})$, we have

$$\boxed{\Delta(\alpha\cup\beta)=(\Delta\alpha)\cap\beta.}$$

Proof. The homomorphism Δ is, up to sign, just the edge homomorphism

$$H^p_\Phi(X;\mathscr{O}\otimes\mathscr{A})=E_2^{p,-n}\twoheadrightarrow E_\infty^{p,-n}\rightarrowtail H^\Phi_{n-p}(X;\mathscr{A})$$

in the basic spectral sequence. The sign can be modified by induction on p, with no change when $p=0$, to satisfy the stated formula with ∂ and δ.

The fact that this generalizes the isomorphism Δ of 9.2 is immediate, as is the fact that it is always an isomorphism when $p = 0$.

The map

$$\Gamma_\Phi(\mathscr{C}_*(X;L)\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) \to \Gamma_{\Phi\cap\Psi}(\mathscr{C}_*(X;L)\otimes\mathscr{A}\otimes\mathscr{B})$$

induces the commutative diagram

$$\begin{array}{ccc} H_n(\Gamma_\Phi(\mathscr{C}_*(X;L)\otimes\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{B}) & \xrightarrow{\cap} & H_n(\Gamma_{\Phi\cap\Psi}(\mathscr{C}_*(X;L)\otimes\mathscr{A}\otimes\mathscr{B})) \\ \downarrow & & \downarrow \\ \Gamma_\Phi(\mathscr{H}_n(X;L)\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) & \xrightarrow{\cup} & \Gamma_{\Phi\cap\Psi}(\mathscr{H}_n(X;L)\otimes\mathscr{A}\otimes\mathscr{B})), \end{array}$$

that is,

$$\begin{array}{ccc} H_n^\Phi(X;\mathscr{A}))\otimes\Gamma_\Psi(\mathscr{B}) & \xrightarrow{\cap} & H_n^{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B}) \\ \downarrow & & \downarrow \\ H^0_\Phi(X;\mathscr{O}\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) & \xrightarrow{\cup} & H^0_{\Phi\cap\Psi}(X;\mathscr{O}\otimes\mathscr{A}\otimes\mathscr{B}), \end{array}$$

and it is clear that the vertical maps are precisely $\Delta^{-1}\otimes 1$ and Δ^{-1}.

Thus, we see that the formula $\Delta(\alpha\cup\beta) = (\Delta\alpha)\cap\beta$ of 10.1 holds when $p = 0 = r$. However, both sides of this formula are natural transformations $F_1^r(\mathscr{A})\otimes F_2^p(\mathscr{B}) \to F^{r,p}(\mathscr{A},\mathscr{B})$ commuting with connecting homomorphisms, as the reader may show, and where $F_1^r(\mathscr{A}) = H^r_\Phi(X;\mathscr{O}\otimes\mathscr{A})$, $F_2^p(\mathscr{B}) = H^p_\Psi(X;\mathscr{B})$, and $F^{r,p}(\mathscr{A},\mathscr{B}) = H^{\Phi\cap\Psi}_{n-r-p}(X;\mathscr{A}\otimes\mathscr{B})$. Thus the result follows from II-6.2. □

10.2. Corollary. *In the situation of* 10.1 *assume that* $\mathscr{O}$ *is locally constant with stalks* L *and that* X *is paracompact, e.g.,* X *a paracompact* n-hm_L. *Put* $[X] = \Delta(1) \in H_n(X;\mathscr{O}^{-1})$, *which is called the "fundamental homology class" of* X, *where* $1 \in L \subset \Gamma(L) = H^0(X;L)$ *is the section with constant value* 1. *Then for* Φ *paracompactifying, the homomorphism* $\Delta : H^p_\Phi(X;\mathscr{A}) \to H^\Phi_{n-p}(X;\mathscr{O}^{-1}\otimes\mathscr{A})$ *is given by*

$$\boxed{\Delta(\alpha) = [X]\cap\alpha.}$$

Proof. We have $[X]\cap\alpha = \Delta(1)\cap\alpha = \Delta(1\cup\alpha) = \Delta(\alpha)$ by 10.1. □

10.3. We shall now consider "computation" of the cap product by means of resolutions and shall obtain in the process some additional facts about the cap product.

For the remainder of this section we let $\mathscr{L}^*$ be a c-soft, torsion-free resolution of L on X. Let $\mathscr{N}^*$ be a c-soft resolution of L on a space Y. Then $\mathscr{L}^*\widehat{\otimes}\mathscr{N}^*$ is a c-soft resolution of L on $X\times Y$ by Exercise 14(b) of Chapter II. Moreover, if $X = Y$ then $\mathscr{L}^*\otimes\mathscr{N}^*$ is a c-soft resolution of L on X, by II-9.2 applied to the diagonal, or, more directly, by II-16.31.

By II-15.5, $\Gamma_c(\mathscr{L}^*\hat{\otimes}\mathscr{N}^*) \approx \Gamma_c(\mathscr{L}^*) \otimes \Gamma_c(\mathscr{N}^*)$ and so, using the identity $\mathrm{Hom}(A \otimes B, C) \approx \mathrm{Hom}(A, \mathrm{Hom}(B, C))$, we obtain the canonical isomorphism

$$\mathrm{Hom}(\Gamma_c(\mathscr{L}^*\hat{\otimes}\mathscr{N}^*), L^*) \approx \mathrm{Hom}(\Gamma_c(\mathscr{L}^*), \mathrm{Hom}(\Gamma_c(\mathscr{N}^*), L^*)).$$

The homology of the left-hand side of this equation is $H_*(X \times Y; L)$. If $\mathscr{N}^*$ is torsion-free, then $\mathrm{Hom}(\Gamma_c(\mathscr{N}^*), L^*)$ is divisible, whence injective, so that the algebraic Künneth formula (see [54] or [75]) yields the exact sequence

$$\boxed{\bigoplus_{p+q=n+1} \mathrm{Ext}(H_c^p(X;L), H_q(Y;L)) \rightarrowtail H_n(X \times Y; L) \twoheadrightarrow \bigoplus_{p+q=n} \mathrm{Hom}(H_c^p(X;L), H_q(Y;L)),} \tag{58}$$

which splits.

We now restrict our attention to the case of one space $X = Y$, and we let $\mathscr{N}^* = \mathscr{I}^*(X; L)$ (or any injective resolution of L) for the remainder of this section. Consider the homomorphism

$$\cap : \mathrm{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{N}^*|U), L^*) \otimes \mathscr{L}^*(U) \to \mathrm{Hom}(\Gamma_c(\mathscr{N}^*|U), L^*)$$

of presheaves on X defined by $(f \cap s)(t) = f(s \cup t)$, where $\cup : \mathscr{L}^*(U) \otimes \Gamma_c(\mathscr{N}^*|U) \to \Gamma_c(\mathscr{L}^* \otimes \mathscr{N}^*|U)$. This defines a homomorphism of sheaves

$$\cap : \mathscr{D}(\mathscr{L}^* \otimes \mathscr{N}^*) \otimes \mathscr{L}^* \to \mathscr{D}(\mathscr{N}^*).$$

Let $\mathscr{M}^* = \mathscr{I}^*(X; L)$ (or any injective resolution of L) and let $\mathscr{L}^* \otimes \mathscr{N}^* \to \mathscr{M}^*$ be a homomorphism of resolutions. We have the induced homomorphism $\mathscr{D}(\mathscr{M}^*) \to \mathscr{D}(\mathscr{L}^* \otimes \mathscr{N}^*)$ and hence a homomorphism

$$\cap : \Gamma_\Phi(\mathscr{D}_m(\mathscr{M}^*)) \otimes \Gamma_\Psi(\mathscr{L}^p) \to \Gamma_{\Phi\cap\Psi}(\mathscr{D}_{m-p}(\mathscr{N}^*)). \tag{59}$$

We shall show shortly that this product induces the cap product defined earlier.

We claim that for $f \in \Gamma_\Phi(\mathscr{D}_m(\mathscr{M}^*))$ and $s \in \Gamma_\Psi(\mathscr{L}^*)$ we have

$$d(f \cap s) = df \cap s + (-1)^m f \cap ds. \tag{60}$$

In fact, for $t \in \Gamma_c(\mathscr{N}^*)$ we compute [recall the definition of the differential in $\mathscr{D}^*(\mathscr{N}^*)$]

$$\begin{aligned}(d(f \cap s))(t) &= d((f \cap s)(t)) - (-1)^{m-p}(f \cap s)(dt)\\ &= d(f(s \cup t)) - (-1)^{m-p} f(s \cup dt)\\ &= (df)(s \cup t) + (-1)^m f(d(s \cup t)) - (-1)^{m-p} f(s \cup dt)\\ &= (df)(s \cup t) + (-1)^m f(ds \cup t)\\ &= [df \cap s + (-1)^m f \cap ds](t).\end{aligned}$$

Now, if $\mathscr{A}$ and $\mathscr{B}$ are sheaves on X, we have the homomorphism

$$(\mathscr{D}(\mathscr{M}^*) \otimes \mathscr{A}) \otimes (\mathscr{L}^* \otimes \mathscr{B}) \to \mathscr{D}(\mathscr{N}^*) \otimes (\mathscr{A} \otimes \mathscr{B}) \tag{61}$$

and hence

$$\Gamma_\Phi(\mathscr{D}(\mathscr{M}^*) \otimes \mathscr{A}) \otimes \Gamma_\Psi(\mathscr{L}^* \otimes \mathscr{B}) \to \Gamma_{\Phi\cap\Psi}(\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{A} \otimes \mathscr{B}), \tag{62}$$

which, of course, also satisfies (60). Since $\mathscr{L}^*$ is torsion free, $\mathscr{L}^* \otimes \mathscr{B}$ is c-soft and hence Ψ-soft for Ψ paracompactifying, by II-16.31 and II-16.5. Also, if $\mathscr{L}^*$ is flabby and $\mathscr{B}$ is elementary, then $\mathscr{L}^* \otimes \mathscr{B}$ is flabby by 3.8. Thus we see that (62) induces a product

$$\cap : H_m^\Phi(X;\mathscr{A}) \otimes H_\Psi^p(X;\mathscr{B}) \to H_{m-p}^{\Phi\cap\Psi}(X;\mathscr{A}\otimes\mathscr{B}) \tag{63}$$

when either Ψ is paracompactifying or $\mathscr{B}$ is elementary.

When Ψ and $\Phi \cap \Psi$ are both paracompactifying, it is easy to see that (63) coincides with our previous definition (52). This can be seen directly for $p = 0$ and follows in general from II-6.2 and the fact from (60) that (63) commutes with connecting homomorphisms in the variable $\mathscr{B}$.

10.4. We shall digress a moment to consider a slight generalization of the exact sequence (9) on page 292. Let $\mathscr{T}$ be a *locally constant* sheaf on X with *stalks isomorphic to* L and let $\mathscr{R}^*$ be a replete resolution of L on X. (Note that repletion is easily seen to be a local property, so that $\mathscr{R}^* \otimes \mathscr{T}$ is a replete resolution of $\mathscr{T}$.) The homomorphism

$$\theta : \operatorname{Hom}(\Gamma_c(\mathscr{R}^* \otimes \mathscr{T}|U), L^*) \otimes \mathscr{T}(U) \to \operatorname{Hom}(\Gamma_c(\mathscr{R}^*|U), L^*)$$

of presheaves defined as in the development of (59) by $\theta(f\otimes t)(s) = f(s\cup t)$ induces a homomorphism

$$\mathscr{H}om(\Gamma_c\{\mathscr{R}^* \otimes \mathscr{T}\}, L^*) \otimes \mathscr{T} \to \mathscr{H}om(\Gamma_c\mathscr{R}^*, L^*)$$

of sheaves. This is an isomorphism since $\mathscr{T}$ is locally isomorphic to L.

Now, $\mathscr{T}$ has an "inverse" sheaf $\mathscr{T}^{-1}$ (i.e., such that $\mathscr{T} \otimes \mathscr{T}^{-1} \approx L$). Thus we obtain the natural isomorphism

$$\mathscr{H}om(\Gamma_c\{\mathscr{R}^* \otimes \mathscr{T}\}, L^*) \approx \mathscr{H}om(\Gamma_c\mathscr{R}^*, L^*) \otimes \mathscr{T}^{-1},$$

and in particular, we have that

$$\operatorname{Hom}(\Gamma_c(\mathscr{R}^* \otimes \mathscr{T}), L^*) \approx \Gamma(\mathscr{H}om(\Gamma_c\mathscr{R}^*, L^*) \otimes \mathscr{T}^{-1}). \tag{64}$$

If we take $\mathscr{R}^* = \mathscr{I}^*(X;L)$, then (64) and 2.3 provide the split exact sequence

$$\boxed{\operatorname{Ext}(H_c^{p+1}(X;\mathscr{T}), L) \rightarrowtail H_p(X;\mathscr{T}^{-1}) \twoheadrightarrow \operatorname{Hom}(H_c^p(X;\mathscr{T}), L).} \tag{65}$$

We claim that when L is a field, the isomorphism provided by (65) is given by the cap product pairing. (Recall that we may take $L^* = L$ when L is a field, by 2.7.)

10.5. Theorem. *If L is a field and if $\mathscr{T}$ is a locally constant sheaf with stalks L, then for each degree p the cap product pairing*

$$\cap : H_p(X;\mathscr{T}^{-1}) \otimes H_c^p(X;\mathscr{T}) \to H_0^c(X;L) \xrightarrow{\varepsilon} L$$

is nonsingular, and in fact, the induced map

$$H_p(X;\mathscr{T}^{-1}) \to \operatorname{Hom}(H_c^p(X;\mathscr{T}),L)$$

is an isomorphism coinciding with that of (65).

Proof. Note that over a field every c-soft sheaf is replete. Let $\mathscr{L}^*$, $\mathscr{N}^*$ and $\mathscr{M}^*$ all be $\mathscr{I}^*(X;L)$, and as above, let $\cup : \mathscr{L}^* \otimes \mathscr{N}^* \to \mathscr{M}^*$ be any given homomorphism of resolutions. Use $\cup$ to also denote the obvious map $\mathscr{L}^* \otimes \mathscr{T} \otimes \mathscr{N}^* \to \mathscr{M}^* \otimes \mathscr{T}$, and consider the homomorphism

$$\zeta : \operatorname{Hom}(\Gamma_c(\mathscr{M}^* \otimes \mathscr{T}),L) \to \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}),\Gamma_c(\mathscr{D}(\mathscr{N}^*))) \tag{66}$$

defined by

$$(\zeta(f)(s))\,(t) = f(s \cup t).$$

Clearly, $|\zeta(f)(s)| \subset |f| \cap |t| \in c$; compare (62).

Since L is a field, the homology in degree m of the right-hand side of (66) can be identified with

$$\bigoplus_p \operatorname{Hom}(H_c^p(X;\mathscr{T}),H_{m-p}^c(X;L)).$$

Thus, using (64), (66) induces a map

$$H_m(X;\mathscr{T}^{-1}) \to \bigoplus_p \operatorname{Hom}(H_c^p(X;\mathscr{T}),H_{m-p}^c(X;L)),$$

which is $\alpha \mapsto \bigoplus \alpha \cap (\bullet)$ by the definition of (66) and of (63).

Now, by (19) on page 300, there is a natural map

$$\eta : \Gamma_c(\mathscr{H}om(\Gamma_c\mathscr{N}^*,L)) \to \operatorname{Hom}(\Gamma(\mathscr{N}^*),L)$$

defined as follows: If $f \in \operatorname{Hom}(\Gamma_c(\mathscr{N}^*),L)$ with $|f| \in c$ and if $s \in \Gamma(\mathscr{N}^*)$, select $s' \in \Gamma_c(\mathscr{N}^*)$ such that $s = s'$ on some neighborhood of $|f|$ and put

$$\eta(f)(s) = f(s').$$

The augmentation $L \rightarrowtail \mathscr{N}^*$ provides a canonical section $1 \in \Gamma(\mathscr{N}^*)$ and a homomorphism

$$\operatorname{Hom}(\Gamma(\mathscr{N}^*),L) \to \operatorname{Hom}(L,L) \to L$$

taking g into $g(1) \in L$. Together with η, we obtain, finally, a homomorphism

$$\nu : \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}),\Gamma_c(\mathscr{D}(\mathscr{N}^*))) \to \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}),L)$$

defined by $\nu(f)(s) = \eta(f(s))(1)$. Note that the map $\Gamma_c(\mathscr{D}(\mathscr{N}^*)) \to L$ induces the canonical surjection $H_0^c(X;L) \xrightarrow{\varepsilon} L$ defined by mapping X into a point.

The homomorphism $\mathscr{L}^* \to \mathscr{L}^* \otimes \mathscr{N}^*$ defined by tensoring with $1 \in \Gamma(\mathscr{N}^*)$ induces a map

$$\mu : \operatorname{Hom}(\Gamma_c(\mathscr{M}^* \otimes \mathscr{T}), L) \to \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}), L)$$

defined by $\mu(g)(s) = g(s \cup 1) \in L$. Thus we have the diagram

$$\begin{array}{ccc} \operatorname{Hom}(\Gamma_c(\mathscr{M}^* \otimes \mathscr{T}), L) & \xrightarrow{\zeta} & \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}), \Gamma_c(\mathscr{D}(\mathscr{N}^*))) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} \\ \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}), L) & \xrightarrow{1} & \operatorname{Hom}(\Gamma_c(\mathscr{L}^* \otimes \mathscr{T}), L). \end{array} \tag{67}$$

We claim that this commutes. In fact, for $f \in \operatorname{Hom}(\Gamma_c(\mathscr{M}^* \otimes \mathscr{T}), L)$ we have $\mu(f)(s) = f(s \cup 1)$, while

$$\nu(\zeta(f))(s) = \eta(\zeta(f)(s))(1) = (\zeta(f)(s))(1') = f(s \cup 1') = f(s \cup 1),$$

where $1' = 1$ on a neighborhood of $|s| \cup |\zeta(f)(s)| \in c$ and the support of $1'$ is compact.

Now using (64), (67) induces the commutative diagram

$$\begin{array}{ccc} H_m(X; \mathscr{T}^{-1}) & \xrightarrow{\zeta^*} & \bigoplus_p \operatorname{Hom}(H_c^p(X; \mathscr{T}), H_{m-p}^c(X; L)) \\ \downarrow{\scriptstyle \mu^*=1} & & \downarrow{\scriptstyle \nu^*} \\ H_m(X; \mathscr{T}^{-1}) & \xrightarrow{\approx} & \operatorname{Hom}(H_c^m(X; \mathscr{T}), L), \end{array}$$

where $\zeta^*(\alpha)(\beta) = \alpha \cap \beta$ and ν^* is the projection onto the factor with $p = m$ followed by the canonical epimorphism $\varepsilon : H_0^c(X;L) \twoheadrightarrow L$. The bottom horizontal map coincides, by definition, with that given by (65). The result follows. □

10.6. Corollary. *Let X be a paracompact n-whm$_L$, where L is a field. Let $\mathscr{T}$ be a locally constant sheaf on X with stalks L. Then the cup product pairing $\alpha \otimes \beta \mapsto \varepsilon\Delta(\alpha \cup \beta) \in L$ of*

$$H^p(X; \mathscr{O} \otimes \mathscr{T}^{-1}) \otimes H_c^{n-p}(X; \mathscr{T}) \to H_c^n(X; \mathscr{O}) \xrightarrow[\approx]{\Delta} H_0^c(X; L) \xrightarrow{\varepsilon} L$$

is nonsingular, and in fact, the induced homomorphism

$$H^p(X; \mathscr{O} \otimes \mathscr{T}^{-1}) \to \operatorname{Hom}(H_c^{n-p}(X; \mathscr{T}), L)$$

is an isomorphism.

Proof. We have $\Delta(\alpha \cup \beta) = (\Delta\alpha) \cap \beta$ by 10.1, and the result follows immediately from this and 10.5. □

10.7. Corollary. *Let X be a compact n-hm$_L$, where L is a field. Let $\mathscr{T}$ be a locally constant sheaf on X with stalks L. Then $H^p(X; \mathscr{T})$ has finite dimension over L for all p.*

Proof. We have

$$H^p(X;\mathcal{T}) \approx \operatorname{Hom}(H^{n-p}(X;\mathcal{O}\otimes\mathcal{T}^{-1}),L)$$

and

$$H^{n-p}(X;\mathcal{O}\otimes\mathcal{T}^{-1}) \approx \operatorname{Hom}(H^p(X;\mathcal{T}),L)$$

since $\mathcal{O}\otimes(\mathcal{O}\otimes\mathcal{T}^{-1})^{-1} \approx \mathcal{O}\otimes\mathcal{O}^{-1}\otimes\mathcal{T}\approx\mathcal{T}$. Therefore

$$H^p(X;\mathcal{T}) \approx \operatorname{Hom}(\operatorname{Hom}(H^p(X;\mathcal{T}),L),L),$$

which can happen only for a finite-dimensional vector space. □

Let us briefly indicate another definition of the cap product. By an inductive process we can define a natural homomorphism

$$\mathcal{A}\otimes\mathcal{C}^*(X;\mathcal{B}) \to \mathcal{C}^*(X;\mathcal{A}\otimes\mathcal{B})$$

of differential sheaves. Thus we have the map

$$\mathcal{C}_*(X;\mathcal{A})\otimes\mathcal{C}^*(X;\mathcal{B}) \to \mathcal{C}_*(X;\mathcal{C}^*(X;\mathcal{A}\otimes\mathcal{B}))$$

[recalling that $\mathcal{C}_*(X;\mathcal{A}) = \mathcal{C}_*(X;L)\otimes\mathcal{A}$] and hence also

$$C_m^{\Phi}(X;\mathcal{A})\otimes C_{\Psi}^p(X;\mathcal{B}) \to C_m^{\Phi\cap\Psi}(X;\mathcal{C}^p(X;\mathcal{A}\otimes\mathcal{B})). \tag{68}$$

If $\Phi\cap\Psi$ is paracompactifying, so that $C_*^{\Phi\cap\Psi}(X;\bullet)$ is exact, a spectral sequence argument on the double complex on the right shows that its total homology is naturally isomorphic to $H_*^{\Phi\cap\Psi}(X;\mathcal{A}\otimes\mathcal{B})$. Thus (68) induces a product that, as the reader is invited to prove, coincides with the cap product (52).

10.8. Example. We shall illustrate the basic spectral sequence of 8.4 for 3-manifolds with isolated singularities. Let K be a finite simplicial complex that is an orientable 3-manifold with boundary. (The main case of interest is that for which K is a convex solid polyhedron in $\mathbb{R}^3$.) Let M arise from K by identification, in pairs, of the 2-faces in ∂K. Then it is easy to see that M is locally euclidean except at its vertices. It will be orientable if the identifications all reverse orientation. In this case let $L=\mathbb{Z}$ or $L=\mathbb{Z}_2$. Otherwise, take $L=\mathbb{Z}_2$. The boundary of a star of a vertex (possibly after subdivision) is then a closed 2-manifold that is orientable if $L=\mathbb{Z}$. Thus M is a manifold $\Leftrightarrow$ each of these are 2-spheres. The sheaf $\mathcal{H}_3(M;L)$ is constant with stalks L since it is represented everywhere by the 3-cycle that is the sum of all 3-simplices (coherently oriented if $L=\mathbb{Z}$). The sheaf $\mathcal{H}_2(M;L)$ is concentrated at the vertices, and the stalk at a vertex is $H_2(U;L)\approx H_2(\overline{U},B;L)\approx H_1(B;L)$; see 5.10, where U is the star at the vertex and B is its boundary, a 2-manifold. Hence $\mathcal{H}_2(M;L)=0 \Leftrightarrow M$ is

a 3-manifold. Therefore $\Gamma(\mathscr{H}_2(M;L)) \approx L^k$ for some k, where $k = 0 \Leftrightarrow M$ is a 3-manifold. In the spectral sequence of 8.4 we have

$$E_2^{p,q} = H^p(M; \mathscr{H}_{-q}(M;L)) \approx \begin{cases} H^p(M;L), & \text{for } q = -3, \\ L^k, & \text{for } q = -2 \text{ and } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the spectral sequence degenerates into the exact sequence

$$0 \to E_2^{1,-3} \to H_2(M;L) \to E_2^{0,-2} \xrightarrow{d_2} E_2^{2,-3} \to H_1(M;L) \to 0,$$

which is

$$0 \to H^1(M;L) \xrightarrow{[M]\cap} H_2(M;L) \to L^k \to H^2(M;L) \xrightarrow{[M]\cap} H_1(M;L) \to 0$$

by 10.2. The Euler characteristic of this sequence gives $b_1 - b_2 + k - b_2 + b_1 = 0$, where b_i is the ith Betti number of M over L. Hence $\chi(M) = 1 - b_1 + b_2 - 1 = b_2 - b_1 = k/2$. We conclude that M is a 3-manifold $\Leftrightarrow$ $\chi(M) = 0$. Of course, this is easy to derive by the elementary methods of simplicial homology and, in fact, is given in [73, p. 216]. ◇

11 Intersection theory

This section is a continuation of the previous one. In it we show how to define a very nice intersection product in an n-hm_L. For the sake of simplicity we restrict attention, for the most part, to the case of compact supports. Again for simplicity we shall assume that the n-hm_L we deal with is *locally hereditarily paracompact*. This is so that we can conclude that $\dim_{c|A} X < \infty$ for any subspace $A \subset M$.

Let M be a locally hereditarily paracompact n-hm_L with orientation sheaf $\mathscr{O} = \mathscr{H}_n(M;L)$. Let $\mathscr{M}$ and $\mathscr{N}$ be elementary sheaves on M. By 5.7 we have that

$$H_p^c(F; \mathscr{O}^{-1} \otimes \mathscr{M}) \approx H_p^{c|F}(M; \mathscr{O}^{-1} \otimes \mathscr{M})$$

for any closed subset $F \subset M$. By 9.6 we also have the isomorphism

$$\boxed{D = \Delta^{-1} : H_p^{c|F}(M; \mathscr{O}^{-1} \otimes \mathscr{M}) \xrightarrow{\approx} H_{c|F}^{n-p}(M; \mathscr{M}).}$$

For A and B both closed in M there is also the cup product

$$H_{c|A}^{n-p}(M;\mathscr{M}) \otimes H_{c|B}^{n-q}(M;\mathscr{N}) \xrightarrow{\cup} H_{c|A\cap B}^{2n-p-q}(M; \mathscr{M} \otimes \mathscr{N}).$$

Combining these we define the *intersection product*

$$\bullet : H_p^c(A; \mathscr{O}^{-1} \otimes \mathscr{M}) \otimes H_q^c(B; \mathscr{O}^{-1} \otimes \mathscr{N}) \to H_{p+q-n}^c(A \cap B; \mathscr{O}^{-1} \otimes \mathscr{M} \otimes \mathscr{N})$$

by

$$\boxed{a \bullet b = \Delta(Db \cup Da).}$$

(The reason for the reversal of order is indicated in [19].) [The reader can verify that this generalizes to arbitrarily locally closed A and B with arbitrary support families Φ on A, Ψ on B, and $\Phi \cap \Psi$ on $A \cap B$.]

For the case $A = M = B$ we have, using 10.2,

$$a \bullet b = \Delta(Db \cup Da) = [M] \cap (Db \cup Da) = ([M] \cap Db) \cap Da,$$

so that

$$\boxed{a \bullet b = b \cap Da.}$$

In the general case, using the notation $i^{M,A} : A \hookrightarrow M$, etc., we conclude from naturality that

$$\boxed{i_*^{M,A\cap B}(a \bullet b) = i_*^{M,A}(a) \bullet i_*^{M,B}(b) = i_*^{M,B}(b) \cap D(i_*^{M,A}a)} \tag{69}$$

for $a \in H_p^c(A; \mathscr{O}^{-1} \otimes \mathscr{M})$ and $b \in H_q^c(B; \mathscr{O}^{-1} \otimes \mathscr{N})$. Also, note the formulas

$$\boxed{a \bullet b = (-1)^{(n-\deg(a))(n-\deg(b))} b \bullet a}$$

and

$$\boxed{a \bullet (b \bullet c) = (a \bullet b) \bullet c.}$$

An obvious consequence of (69) is that for closed sets $A, B \subset M$, if $a \in H_p^c(A; \mathscr{O}^{-1} \otimes \mathscr{M})$, $b \in H_q^c(B; \mathscr{O}^{-1} \otimes \mathscr{N})$, and $i_*^{M,A}(a) \bullet i_*^{M,B}(b) \neq 0$, then $A \cap B \neq \varnothing$.

If $\mathscr{O}^{-1} \otimes \mathscr{M} \otimes \mathscr{N} \approx L$, then there is the augmentation $\varepsilon : H_0^c(A \cap B; L) \twoheadrightarrow L$, and so we can define the *intersection number*

$$\boxed{a \cdot b = \varepsilon(a \bullet b) \in L}$$

for $a \in H_p^c(A; \mathscr{O}^{-1} \otimes \mathscr{M})$, $b \in H_q^c(B; \mathscr{O}^{-1} \otimes \mathscr{N})$, and $\mathscr{O}^{-1} \otimes \mathscr{M} \otimes \mathscr{N} \approx L$, with a given such isomorphism understood. Note then that

$$\boxed{a \cdot b = \varepsilon(a \bullet b) = \varepsilon(b \cap Da)}$$

for $a \in H_p^c(A; \mathscr{O}^{-1} \otimes \mathscr{M})$, $b \in H_q^c(B; \mathscr{O}^{-1} \otimes \mathscr{N})$, and $\mathscr{M} \otimes \mathscr{N} \approx \mathscr{O}$. In particular, $a \cdot b$ is defined when $a \in H_p^c(A; L)$ and $b \in H_q^c(B; \mathscr{O}^{-1} \otimes L)$ or when $a \in H_p^c(A; \mathscr{O}^{-1} \otimes L)$ and $b \in H_q^c(B; L)$. Note that the intersection number is a duality pairing.

In [19, p. 373] an example is given of two tori A and B embedded in a 3-torus M such that $0 \neq [A] \bullet [B] \in H_1(M)$ but with ${}_SH_1(A \cap B) = 0$. In Borel-Moore homology, of course, we must have that $H_1(A \cap B) \neq 0$. It is clear that Borel-Moore homology is the proper domain of intersection theory.

11.1. For the remainder of this section we shall restrict our attention to the case in which L is a *field*. Let $\mathcal{T}$ be a locally constant sheaf on X with stalks L. Then we have the isomorphism

$$\eta : H_p(X;\mathcal{T}^{-1}) \xrightarrow{\approx} \operatorname{Hom}(H_c^p(X;\mathcal{T}), L)$$

of 10.5 given by $\eta(a)(\beta) = \varepsilon(a \cap \beta)$. We introduce the notation

$$\boxed{\langle a, \beta\rangle = \varepsilon(a \cap \beta)}$$

for this pairing. Here $a \in H_p(X;\mathcal{T}^{-1})$ and $\beta \in H_c^p(X;\mathcal{T})$. Note the formulas

$$\boxed{\begin{aligned} \langle a \cap \alpha, \beta\rangle &= \langle a, \alpha \cup \beta\rangle,\\ \langle f_*(a), \beta\rangle &= \langle a, f^*(\beta)\rangle,\\ a \cdot b &= \langle a, Db\rangle,\\ \varepsilon(a) &= \langle a, 1\rangle, \end{aligned}}$$

all of which are immediate. Also note that since this "Kronecker pairing" is nonsingular, to define an element $a \in H_p(X;\mathcal{T}^{-1})$ it suffices to define $\langle a, \beta\rangle$ for β in a basis of $H_c^p(X;\mathcal{T})$.

We digress for a moment to discuss the homology cross product. Let $\mathcal{T}$ and $\mathcal{S}$ be locally constant sheaves on X and Y respectively with stalks L. By the Künneth theorem, $H_c^n(X \times Y;\mathcal{T}\widehat{\otimes}\mathcal{S})$ has a basis consisting of elements of the form $\alpha \times \beta$, where $\alpha \in H_c^p(X;\mathcal{T})$ and $\beta \in H_c^q(Y;\mathcal{S})$. Therefore we can define a homology cross product

$$\boxed{\times : H_p(X;\mathcal{T}^{-1}) \otimes H_q(Y;\mathcal{S}^{-1}) \to H_{p+q}(X \times Y;\mathcal{T}^{-1}\widehat{\otimes}\mathcal{S}^{-1})}$$

by

$$\boxed{\langle a \times b, \alpha \times \beta\rangle = (-1)^{\deg(\alpha)\deg(b)}\langle a, \alpha\rangle\langle b, \beta\rangle.}$$

(Note that $\deg(\alpha) = \deg(a)$ and $\deg(\beta) = \deg(b)$ when this expression is nonzero.) If $T : X \times Y \to Y \times X$ is $T(x, y) = (y, x)$, then

$$\boxed{T_*(a \times b) = (-1)^{(\deg a)(\deg b)} b \times a} \tag{70}$$

since

$$\begin{aligned} \langle a \times b, \alpha \times \beta\rangle &= (-1)^{\deg(a)\deg(b)}\langle a, \alpha\rangle\langle b, \beta\rangle\\ &= (-1)^{\deg(a)\deg(b)}\langle b, \beta\rangle\langle a, \alpha\rangle\\ &= \langle b \times a, \beta \times \alpha\rangle\\ &= (-1)^{\deg(a)\deg(b)}\langle b \times a, T^*(\alpha \times \beta)\rangle\\ &= (-1)^{\deg(a)\deg(b)}\langle T_*(b \times a), \alpha \times \beta\rangle. \end{aligned}$$

In dimension zero we have

$$\boxed{\varepsilon(a \times b) = \langle a \times b, 1\rangle = \langle a \times b, 1 \times 1\rangle = \langle a, 1\rangle\langle b, 1\rangle = \varepsilon(a)\varepsilon(b).}$$

In general, we have

$$\boxed{(a \times b) \cap (\alpha \times \beta) = (-1)^{\deg(\alpha)\deg(b)}(a \cap \alpha) \times (b \cap \beta),}$$

as can be verified easily by pairing both sides with cohomology classes of the form $\alpha' \times \beta'$ and comparing results. It is also easy to verify that in an n-hm_L we have

$$\boxed{(a \bullet b) \times (c \bullet d) = (-1)^{(n-\deg(a))(n-\deg(d))}(a \times c) \bullet (b \times d).}$$

Remark: It is natural to ask how much more generally the homology cross product can be defined. In the present situation where L is a field, it is not hard to generalize to the case of arbitrary paracompactifying support families and arbitrary coefficient sheaves. If L is not a field, then one can still define the cross product for closed supports and coefficients in L by using the fact that $H_p(X;L) \approx H_p(\mathrm{Hom}(F^*\Gamma_c(\mathscr{I}^*(X;L)),L))$, where F^* denotes the passage to a projective resolution. However, I see no way to pass from this to general coefficient sheaves.

11.2. Now we confine our attention to the case in which M^n is a compact, connected, metrizable n-hm_L and L is a field. Recall from 10.7 that $H_*(M;L)$ is finitely generated. Let $[M] \in H_n(M;\mathscr{O}^{-1}) \approx H^0(M;L) \approx L$ be the fundamental class or *any* generator and let $\gamma \in H^n(M;\mathscr{O})$ be its Kronecker dual class; i.e., $\langle [M], \gamma\rangle = 1$. Note then that $[M] \times [M] \neq 0$ since

$$\langle [M] \times [M], \gamma \times \gamma \rangle = \pm \langle [M], \gamma\rangle\langle [M], \gamma\rangle = \pm 1.$$

Thus we may take

$$[M \times M] = [M] \times [M].$$

Note that $([M] \times [M]) \cap (Da \times Db) = (-1)^{n\deg(b)}([M] \cap Da) \times ([M] \cap Db) = (-1)^{n\deg(b)} a \times b$, so that

$$\boxed{D(a \times b) = (-1)^{n\deg(b)} D(a) \times D(b).}$$

Let $d : M \to M \times M$ be the diagonal map. Then there are two "diagonal" classes,

$$[\Delta_1] = d_*[M] \in H_n(M \times M; L \widehat{\otimes} \mathscr{O}^{-1})$$

and

$$[\Delta_2] = d_*[M] \in H_n(M \times M; \mathscr{O}^{-1} \widehat{\otimes} L).$$

11.3. Theorem. *We have*

$$\boxed{[\Delta_1] = \textstyle\sum_{a \in B} a \times a^{\circ},}$$

where B is a basis of $H_*(M;L)$ and $a^{\circ} \in H_*(M;\mathscr{O}^{-1})$ is the element corresponding to a in the intersection product dual basis (i.e., $a \cdot b^{\circ} = \delta_{a,b}$ for $a, b \in B$). Also,

$$\boxed{[\Delta_2] = \textstyle\sum_{a \in B} (-1)^{\deg(a)(n-\deg(a))} a^{\circ} \times a.}$$

Proof. We can write $[\Delta_1] = \sum_{a,b} \lambda_{a,b} a \times b^\circ$ for some coefficients $\lambda_{a,b} \in L$. Note that $\deg(a) = \deg(b)$ in this sum. Then, using $|a| = \deg(a)$, we compute

$$\begin{aligned}
\lambda_{a,b} &= \varepsilon(\lambda_{a,b}(a \bullet a^\circ) \times (b \bullet b^\circ)) \\
&= (-1)^{|a|(n-|a|)}\varepsilon(\lambda_{a,b}(a^\circ \bullet a) \times (b \bullet b^\circ)) \\
&= (-1)^{|a|(n-|a|)+|a||b|}\varepsilon(\lambda_{a,b}(a^\circ \times b) \bullet (a \times b^\circ)) \\
&= (-1)^{n|a|}\varepsilon((a^\circ \times b) \bullet \lambda_{a,b}(a \times b^\circ)) \\
&= (-1)^{n|a|}\varepsilon((a^\circ \times b) \bullet [\Delta_1]) \\
&= (-1)^{n|a|}\langle [\Delta_1], D(a^\circ \times b)\rangle \\
&= (-1)^{n|a|}\langle d_*[M], D(a^\circ \times b)\rangle \\
&= (-1)^{n|a|}\langle [M], d^* D(a^\circ \times b)\rangle \\
&= \langle [M], d^*(D(a^\circ) \times D(b))\rangle \\
&= \langle [M], D(a^\circ) \cup D(b)\rangle \\
&= \langle [M] \cap D(a^\circ), D(b)\rangle \\
&= \langle a^\circ, D(b)\rangle = b \cdot a^\circ = \delta_{a,b},
\end{aligned}$$

which gives the first formula. The second formula follows from the first by the rule (70) for changing order in a cross product. □

Now suppose that $f : M \to M$ is a self map, and put

$$\boxed{[\Gamma_f] = (1 \times f)_*[\Delta_2] \in H_n(M \times M; \mathscr{O}^{-1}\hat{\otimes}L),}$$

the fundamental class of the graph of f. Now, $f_* : H_n(M; L) \to H_n(M; L)$ can be written in terms of the basis B as $f_*(b) = \sum_{a\in B} \lambda_{a,b} a$ for some coefficients $\lambda_{a,b} \in L$. Then the Lefschetz fixed-point number is defined to be $L(f) = \sum_{a\in B}(-1)^{\deg(a)}\lambda_{a,a}$.

11.4. Theorem. (Lefschetz fixed-point theorem.) *Let M be a compact metrizable n-hm_L where L is a field, and let $f : M \to M$. Then*

$$\boxed{L(f) = [\Gamma_f] \cdot [\Delta_1].}$$

In particular, if $L(f) \neq 0$, then f has a fixed point.

Proof. We compute

$$\begin{aligned}
[\Gamma_f] \cdot [\Delta_1] =& (1 \times f)_*[\Delta_2] \cdot [\Delta_1] \\
=& \varepsilon\left((1_* \times f_*)\left(\textstyle\sum_{b\in B}(-1)^{|b|(n-|b|)}(b^\circ \times b)\right) \bullet \textstyle\sum_{a\in B}(a \times a^\circ)\right) \\
=& \varepsilon\left(\textstyle\sum_{b\in B}(-1)^{|b|(n-|b|)}(b^\circ \times f_*(b)) \bullet \textstyle\sum_{a\in B}(a \times a^\circ)\right) \\
=& \varepsilon\left(\textstyle\sum_{a,b\in B}(-1)^{|b|(n-|b|)}b^\circ \times \textstyle\sum_{c\in B}\lambda_{c,b}c \bullet (a \times a^\circ)\right) \\
=& \varepsilon\left(\textstyle\sum_{a,b,c\in B}\lambda_{c,b}(-1)^{|b|(n-|b|)}(b^\circ \times c) \bullet (a \times a^\circ)\right) \\
=& \varepsilon\left(\textstyle\sum_{a,b,c\in B}\lambda_{c,b}(-1)^{|b|(n-|b|)+|a||b|}(b^\circ \bullet a) \times (c \bullet a^\circ)\right) \\
=& \varepsilon\left(\textstyle\sum_{a,b,c\in B}\lambda_{c,b}(-1)^{|b|(n-|b|)+|a||b|+|b|(n-|a|)}(a \bullet b^\circ) \times (c \bullet b^\circ)\right) \\
=& \textstyle\sum_{b\in B}(-1)^{|b|}\lambda_{b,b} = L(f).
\end{aligned}$$

Of course, the last statement holds because $[\Gamma_f]\cdot[\Delta_1]\neq 0$ implies that the graph $\Gamma_f = \{\langle x, f(x)\rangle\}$ intersects the diagonal $\Delta = \{\langle x, x\rangle\}$ nontrivially, since $[\Gamma_f]\bullet[\Delta_1]$ can be regarded as a class in $H_0(\Gamma_f\cap\Delta; L)$. □

This can easily be generalized to the case of homology manifolds with boundary (see Section 16) either directly or by the technique of doubling; see [19, p. 402]. The present version of the Lefschetz fixed-point theorem is not quite subsumed by the classical one, as given in [19, IV-23.5], since there are homology manifolds that are not locally contractible.

12 Uniqueness theorems

In this section we show that on the category of clc_L^∞ spaces the definition of cohomology by means of flabby resolutions can be dualized to obtain the homology theory in an essentially unique manner.

12.1. Definition. *A precosheaf $\mathfrak{A}$ on X is said to be "locally zero" if for any open set $U\subset X$ and $y\in U$ there is a neighborhood $V\subset U$ of y with $i_{U,V}:\mathfrak{A}(V)\to\mathfrak{A}(U)$ trivial.*

Note that a *cosheaf* is locally zero $\Leftrightarrow$ it is zero.

12.2. Definition. *A homomorphism $h:\mathfrak{A}\to\mathfrak{B}$ of precosheaves is said to be "local isomorphism" if the precosheaves* $\operatorname{Ker} h$ *and* $\mathfrak{B}/\operatorname{Im} h$ *are both locally zero.*

If $\mathfrak{A}$ and $\mathfrak{B}$ are *cosheaves* and $h:\mathfrak{A}\to\mathfrak{B}$ is a local isomorphism, then a simple diagram chase shows that $\mathfrak{B}/\operatorname{Im} h$ is a cosheaf. Another diagram chase then shows that h is actually an isomorphism of precosheaves. Thus a local isomorphism of *cosheaves* is an isomorphism. More generally, see Exercise 1 and Chapter VI. Note that for locally connected X the constant cosheaf $\mathfrak{L}$, as defined in Section 1, exists and that the canonical homomorphism $\mathfrak{L}\to L$ to the constant precosheaf is a local isomorphism.

For any precosheaf $\mathfrak{A}$ on X and any L-module M we put $\mathcal{H}om(\mathfrak{A}, M) = \mathcal{S}heaf(U\mapsto \operatorname{Hom}(\mathfrak{A}(U), M))$ and $\mathcal{E}xt(\mathfrak{A}, M) = \mathcal{S}heaf(U\mapsto \operatorname{Ext}(\mathfrak{A}(U), M))$.

12.3. Proposition. *If $h:\mathfrak{A}\to\mathfrak{B}$ is a local isomorphism of precosheaves, then the induced sheaf homomorphisms $\mathcal{H}om(\mathfrak{B}, M)\to\mathcal{H}om(\mathfrak{A}, M)$ and $\mathcal{E}xt(\mathfrak{B}, M)\to\mathcal{E}xt(\mathfrak{A}, M)$ are isomorphisms for any L-module M.*

Proof. Let $\mathfrak{K}(U) = \operatorname{Ker} h(U)$, $\mathfrak{I}(U) = \operatorname{Im} h(U)$, and $\mathfrak{C}(U) = \mathfrak{B}(U)/\mathfrak{I}(U)$. We have the exact sequences of precosheaves

$$0\to\mathfrak{K}\to\mathfrak{A}\to\mathfrak{I}\to 0$$

and

$$0\to\mathfrak{I}\to\mathfrak{B}\to\mathfrak{C}\to 0.$$

For $x \in X$, the definition of local isomorphism implies that for any neighborhood U of x there is a neighborhood $V \subset U$ of x with $\mathfrak{K}(V) \to \mathfrak{K}(U)$ and $\mathfrak{C}(V) \to \mathfrak{C}(U)$ both trivial. We have the following exact sequences, natural in U:

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathfrak{I}(U), M) &\to \operatorname{Hom}(\mathfrak{A}(U), M) \to \operatorname{Hom}(\mathfrak{K}(U), M) \\ &\to \operatorname{Ext}(\mathfrak{I}(U), M) \to \operatorname{Ext}(\mathfrak{A}(U), M) \to \operatorname{Ext}(\mathfrak{K}(U), M) \to 0, \end{aligned}$$

and

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathfrak{C}(U), M) &\to \operatorname{Hom}(\mathfrak{B}(U), M) \to \operatorname{Hom}(\mathfrak{I}(U), M) \\ &\to \operatorname{Ext}(\mathfrak{C}(U), M) \to \operatorname{Ext}(\mathfrak{B}(U), M) \to \operatorname{Ext}(\mathfrak{I}(U), M) \to 0. \end{aligned}$$

These sequences remain exact upon passage to the direct limit over neighborhoods U of x. However,

$$\varinjlim \operatorname{Hom}(\mathfrak{K}(U), M) = 0 = \varinjlim \operatorname{Ext}(\mathfrak{K}(U), M)$$

and

$$\varinjlim \operatorname{Hom}(\mathfrak{C}(U), M) = 0 = \varinjlim \operatorname{Ext}(\mathfrak{C}(U), M).$$

This implies that the homomorphisms $\mathscr{H}om(\mathfrak{B}, M)_x \to \mathscr{H}om(\mathfrak{A}, M)_x$ and $\mathscr{E}xt(\mathfrak{B}, M)_x \to \mathscr{E}xt(\mathfrak{A}, M)_x$ are isomorphisms. □

12.4. Definition. *A locally compact space X is said to be k-hlc$_L$ if for each $x \in X$ and each neighborhood U of x there is a neighborhood $V \subset U$ of x such that the homomorphism*

$$i_*^{U,V} : \tilde{H}_k^c(V; L) \to \tilde{H}_k^c(U; L)$$

is trivial. The space X is hlc$_L^n$ if it is k-hlc$_L$ for all $k \le n$.

For convenience we make the following definition. Note that it is not entirely symmetrical.

12.5. Definition. *Let M be an L-module. Let $\mathscr{L}^*$ be a differential sheaf and let $\varepsilon : M \to H^0(\mathscr{L}^*)$ be a homomorphism of presheaves [i.e., $M \to H^0(\mathscr{L}^*(U))$ for all open U]. Then $\mathscr{L}^*$, together with ε, is called a "quasi-n-resolution" of M provided that*

(a) *$\mathscr{L}^*$ is bounded below,*

(b) *$\mathscr{H}^p(\mathscr{L}^*) = 0$ for all $0 \ne p \le n$,*

(c) *ε induces an isomorphism $M \to \mathscr{H}^0(\mathscr{L}^*)$ of sheaves.*

Similarly, let $\mathfrak{L}_*$ be a differential cosheaf and $\eta : H_0(\mathfrak{L}_*) \to L$ a homomorphism of precosheaves. Then $\mathfrak{L}_*$, together with η, is called a "quasi-n-coresolution" of L provided that

(a′) *$\mathfrak{L}_*$ is bounded below,*

(b′) *$H_p(\mathfrak{L}_*) = 0$ for $p < 0$ and $H_p(\mathfrak{L}_*)$ is locally zero for $0 \ne p \le n$,*

(c′) *η is a local isomorphism of precosheaves.*

We call $\mathfrak{L}_*$ an "n-coresolution" if also $\mathfrak{L}_p = 0$ for $p < 0$.

In the above situation we define $\tilde{H}^0(\mathscr{L}^*) = \operatorname{Coker}\varepsilon$ and $\tilde{H}_0(\mathfrak{L}_*) = \operatorname{Ker}\eta$, the "reduced" groups in degree zero. Note, then, that the precosheaf $\tilde{H}_0(\mathfrak{L}_*)$ is locally zero, and the presheaf $\tilde{H}^0(\mathscr{L}^*)$ generates the zero sheaf.

Immediately from the definition of Borel-Moore homology we have:

12.6. Theorem. *X is hlc^n_L $\Leftrightarrow$ $C^c_*(\bullet\,; L)$ is a quasi-n-coresolution of L.* □

12.7. Theorem. *If there exists a quasi-n-coresolution $\mathfrak{L}_*$ of L by flabby cosheaves, then X is clc^n_L and $\mathscr{D}^*(\mathfrak{L}_*; M)$ is a quasi-n-resolution of M by flabby sheaves for any L-module M.*

Proof. For U open in X, we have, by 2.3, the natural exact sequence

$$\begin{aligned}0 \to \operatorname{Ext}(H_{p-1}(\mathfrak{L}_*(U)), M) &\to H^p(\mathscr{D}^*(\mathfrak{L}_*; M)(U)) \\ &\to \operatorname{Hom}(H_p(\mathfrak{L}_*(U)), M) \to 0.\end{aligned} \tag{71}$$

Since $\mathscr{E}xt(H_{p-1}(\mathfrak{L}_*), M) = 0$ for $1 \neq p \leq n+1$ and $\mathscr{E}xt(H_0(\mathfrak{L}_*), M) \approx \mathscr{E}xt(L, M) = 0$ by 12.3, we obtain, also by 12.3,

$$\mathscr{H}^p(\mathscr{D}^*(\mathfrak{L}_*; M)) \approx \mathscr{H}om(H_p(\mathfrak{L}_*), M) \approx \begin{cases}\mathscr{H}om(L, M) \approx M, & \text{for } p = 0,\\ 0, & \text{for } 0 \neq p \leq n.\end{cases}$$

Also, by 12.5(b′) we have $H^0(\mathscr{D}^*(\mathfrak{L}_*; M)(U)) \approx \operatorname{Hom}(H_0(\mathfrak{L}_*(U)), M)$ and $H^p(\mathscr{D}^*(\mathfrak{L}_*; M)) = 0$ for $0 < p \leq n$. The hypothesized map $H_0(\mathfrak{L}_*) \to L$ induces a map

$$M \approx \operatorname{Hom}(L, M) \to \operatorname{Hom}(H_0(\mathfrak{L}_*(U)); M) \approx H^0(\mathscr{D}^*(\mathfrak{L}_*; M)(U)),$$

which induces the isomorphism $M \approx \mathscr{H}^0(\mathscr{D}^*(\mathfrak{L}_*; M))$. This shows that $\mathscr{D}^*(\mathfrak{L}_*; M)$ is a quasi-n-resolution of M. Since $\operatorname{Ext}(L, M) = 0$, it is easy to see that (71) also holds for the *reduced* groups in all three terms.

Now, $\mathscr{D}^*(\mathfrak{L}_*; M)$ is flabby by 1.10, so that there is a natural isomorphism (in the reduced case also)

$$H^p(\mathscr{D}^*(\mathfrak{L}_*; M)(U)) \approx H^p(U; M) \text{ for } p \leq n \tag{72}$$

by IV-2.5. Now take $p \leq n$, $M = L$, and let $W \subset V \subset U$ be neighborhoods of any point $x \in X$ for which the maps $\tilde{H}_r(\mathfrak{L}_*(W)) \to \tilde{H}_r(\mathfrak{L}_*(V)) \to \tilde{H}_r(\mathfrak{L}_*(U))$ are both zero for $r = p$ or $p-1$. We have the commutative diagram

$$\begin{array}{ccccc}
 & & \tilde{H}^p(U; L) & \to & \operatorname{Hom}(\tilde{H}_p(\mathfrak{L}_*(U)), L) \\
 & & \downarrow & & \downarrow{\scriptstyle 0} \\
\operatorname{Ext}(\tilde{H}_{p-1}(\mathfrak{L}_*(V)), L) & \to & \tilde{H}^p(V; L) & \to & \operatorname{Hom}(\tilde{H}_p(\mathfrak{L}_*(V)), L) \\
\downarrow{\scriptstyle 0} & & \downarrow & & \\
\operatorname{Ext}(\tilde{H}_{p-1}(\mathfrak{L}_*(W)), L) & \to & \tilde{H}^p(W; L) & &
\end{array}$$

with exact middle row. By II-17.3, the map $\tilde{H}^p(U; L) \to \tilde{H}^p(W; L)$ is trivial, and it follows from II-17.5 that X is clc^n_L. □

12.8. Suppose that X is hlc_L^n. Then combining 12.7 with (72) and the sequence (71) we obtain the split exact *universal coefficient sequence*

$$\boxed{0 \to \operatorname{Ext}(H_{p-1}^c(X;L), M) \to H^p(X;M) \to \operatorname{Hom}(H_p^c(X;L), M) \to 0}$$

for $p \le n$, where M is any L-module. In 13.7 we will extend this to the relative case.

12.9. Theorem. *Suppose that X is clc_L^{n+1}. Then $\mathfrak{D}(\mathscr{L}^*)$ is a flabby quasi-n-coresolution of L for any c-soft quasi-$(n+1)$-resolution $\mathscr{L}^*$ of L. Also, X is hlc_L^n and there is a natural isomorphism $H_p(\mathfrak{D}(\mathscr{L}^*)(U)) \approx H_p^c(U;L)$ for $p \le n$.*

Proof. Recall that $\mathfrak{D}(\mathscr{L}^*)$ is the cosheaf $U \mapsto \Gamma_c(\mathscr{D}(\mathscr{L}^*)|U)$. Note that for K compact, $\mathscr{L}^*|K$ is also a c-soft quasi-$(n+1)$-resolution of L on K. Also recall that

$$\mathfrak{D}(\mathscr{L}^*)(U) = \Gamma_c(\mathscr{D}(\mathscr{L}^*)|U) \approx \varinjlim \Gamma_{c|K}(\mathscr{D}(\mathscr{L}^*)) \approx \varinjlim \Gamma(\mathscr{D}(\mathscr{L}^*)|K))$$

by 5.5, where K ranges over the compact subsets of U.

By IV-2.4 there are natural isomorphisms

$$H^p(\Gamma(\mathscr{L}^*|K)) \approx H^p(K;L) \quad \text{and} \quad H^p(\Gamma_c(\mathscr{L}^*|U)) \approx H_c^p(U;L)$$

for $p \le n+1$. By 2.4 we have the natural exact sequence

$$\operatorname{Ext}(H^{p+1}(K;L), L) \rightarrowtail H_p(\Gamma(\mathscr{D}(\mathscr{L}^*|K))) \twoheadrightarrow \operatorname{Hom}(H^p(K;L), L) \tag{73}$$

for $p \le n$, and passing to the limit over $K \subset U$ (K compact, U open) we obtain the exact sequence ($p \le n$)

$$\varinjlim \operatorname{Ext}(H^{p+1}(K), L) \rightarrowtail H_p(\mathfrak{D}(\mathscr{L}^*)(U)) \twoheadrightarrow \varinjlim \operatorname{Hom}(H^p(K), L). \tag{74}$$

In particular, we conclude that $H_p(\mathfrak{D}(\mathscr{L}^*)(U)) = 0$ for $p < 0$ because $H^0(K)$ is free, whence $\operatorname{Ext}(H^0(K), L) = 0$. We also have the surjection

$$H_0(\mathfrak{D}(\mathscr{L}^*)(U)) \twoheadrightarrow \varinjlim \operatorname{Hom}(H^0(K), L) \twoheadrightarrow \operatorname{Hom}(L, L) \approx L.$$

Thus we can form the reduced group $\tilde{H}_0(\mathfrak{D}(\mathscr{L}^*)(U))$, and it is clear that (74) is defined and remains exact for the reduced groups in all three terms. It is also clear that it now suffices to show that for $p \le n$ the precosheaf $\tilde{H}_p(\mathfrak{D}(\mathscr{L}^*)(\bullet))$ is locally zero for the first part of the theorem.

Let U be a given neighborhood of $x \in X$. According to the definition of the property clc_L^{n+1}, we can find a compact neighborhood N of x with $N \subset U$ and an open neighborhood $V \subset N$ of x such that the restriction $\tilde{H}^r(N) \to \tilde{H}^r(V)$ is trivial for $r \le n+1$. Thus, if K' and K are compact with $K' \subset V \subset N \subset K \subset U$, then $\tilde{H}^r(K) \to \tilde{H}^r(K')$ is trivial, as are

$$\operatorname{Hom}(\tilde{H}^p(K'), L) \to \operatorname{Hom}(\tilde{H}^p(K), L)$$

and

$$\operatorname{Ext}(\tilde{H}^{p+1}(K'), L) \to \operatorname{Ext}(\tilde{H}^{p+1}(K), L),$$

for $p \le n$. Thus the latter two maps induce zero homomorphisms upon passage to the direct limits over $K' \subset V$ and $K \subset U$. If $W \subset V$ bears the same relationship to V as does V to U, an argument using II-17.3, similar to the one employed in the proof of 12.7, shows that the map

$$\tilde{H}_p(\mathfrak{D}(\mathscr{L}^*)(W)) \to \tilde{H}_p(\mathfrak{D}(\mathscr{L}^*)(U))$$

is trivial for $p \le n$, and this finishes the proof of the first part of the theorem.

For the second part of the theorem, note that there exists a homomorphism $\mathscr{L}^* \to \mathscr{I}^*(X; L)$ of differential sheaves, unique up to homotopy and inducing an isomorphism of derived sheaves through dimension $n+1$ (which generalizes the same statement for resolutions), and in the same way as in the proof of (10) on page 293, we obtain an isomorphism $H_p(X; L) \approx H_p(\mathscr{D}(\mathscr{L}^*)(X))$ for $p \le n$. Thus, we have (K ranging over the compact subsets of U)

$$H_p(\mathfrak{D}(\mathscr{L}^*)(U)) \approx \varinjlim H_p(\mathscr{D}(\mathscr{L}^*|K)(K)) \approx \varinjlim H_p(K; L) \approx H_p^c(U; L)$$

by 5.3 or 5.6 for $p \le n$. That X is hlc_L^n is implicit in the foregoing proof; it is also immediate from (9) on page 292. □

From 12.6, 12.7, and 12.9 we have:

12.10. Corollary. $hlc_L^n \Rightarrow clc_L^n \Rightarrow hlc_L^{n-1}$. □

12.11. Theorem. *If $\mathfrak{L}_*$ is a quasi-$(n+1)$-coresolution of L by flabby cosheaves, then there is a natural isomorphism $H_p^c(X; L) \approx H_p(\mathfrak{L}_*(X))$ for $p \le n$. If $\mathfrak{N}_*$ is also a flabby quasi-$(n+1)$-coresolution of L and $f : \mathfrak{L}_* \to \mathfrak{N}_*$ is a chain map preserving the augmentations, then the induced map $H_p(\mathfrak{L}_*(X)) \to H_p(\mathfrak{N}_*(X))$ is an isomorphism for $p \le n$.*[22]

Proof. Consider the differential cosheaf $\mathfrak{A}_* = \mathfrak{D}(\mathfrak{D}(\mathfrak{L}_*))$. The natural map $\mathfrak{L}_* \to \mathfrak{A}_*$ of differential cosheaves from (6) on page 290 induces a homomorphism

$$\mathscr{D}(\mathfrak{A}_*; M) \to \mathscr{D}(\mathfrak{L}_*; M) \tag{75}$$

of differential sheaves. Both sides of (75) are flabby quasi-n-resolutions of M, by 12.7 and 12.9. It is easy to check that (75) preserves the augmentations and hence that the induced homomorphism of derived sheaves is an isomorphism through degree n. By IV-2.2 it follows that the induced map

$$H^p(\mathscr{D}(\mathfrak{A}_*; M)(X)) \to H^p(\mathscr{D}(\mathfrak{L}_*; M)(X))$$

[22] If we also assume that X is hlc_L^{n+1} in this theorem, then the conclusion can be improved to $p \le n+1$ by using results from Chapter VI.

is an isomorphism for $p \le n$.

If M is injective, Hom($\bullet$, M) is exact, and since we may take $M^* = M$, we obtain the commutative diagram

$$\begin{array}{ccc} H^p(\mathscr{D}(\mathfrak{A}_*; M)(X)) & \xrightarrow{\approx} & \operatorname{Hom}(H_p(\mathfrak{A}_*(X)), M) \\ \downarrow\approx & & \downarrow \\ H^p(\mathscr{D}(\mathfrak{L}_*; M)(X)) & \xrightarrow{\approx} & \operatorname{Hom}(H_p(\mathfrak{L}_*(X)), M) \end{array}$$

for M injective and for $p \le n$ [or use (71)]. Thus, for $p \le n$,

$$\operatorname{Hom}(H_p(\mathfrak{A}_*(X)), M) \to \operatorname{Hom}(H_p(\mathfrak{L}_*(X)), M)$$

is an isomorphism for all injective M. We claim that it follows that $H_p(\mathfrak{L}_*(X)) \to H_p(\mathfrak{A}_*(X))$ must be an isomorphism for $p \le n$. Indeed, suppose $h : A \to B$ to be a homomorphism of L-modules with $\operatorname{Hom}(B, M) \xrightarrow{\approx} \operatorname{Hom}(A, M)$ for all injective M. Letting $K = \operatorname{Ker} h$, $I = \operatorname{Im} h$, and $C = B/I$, we have the exact sequences

$$0 \to \operatorname{Hom}(I, M) \to \operatorname{Hom}(A, M) \to \operatorname{Hom}(K, M) \to 0$$

and

$$0 \to \operatorname{Hom}(C, M) \to \operatorname{Hom}(B, M) \to \operatorname{Hom}(I, M) \to 0.$$

If follows that $\operatorname{Hom}(C, M) = 0 = \operatorname{Hom}(K, M)$ for all injective M. But any module admits a *monomorphism* into some injective module, whence $C = 0 = K$, and hence $h : A \xrightarrow{\approx} B$ as claimed.

Moreover, by 12.7 and 12.9, $H_p(\mathfrak{A}_*(X)) \approx H_p^c(X; L)$ for $p \le n$, and this completes the proof of the first part of the theorem. The last part of the theorem follows easily from the proof of the first part. □

12.12. Example. Take $L = \mathbb{R}$ and let Ω^p be the sheaf of germs of differentiable p-forms on the oriented smooth manifold M^n. Put $\mathfrak{L}_p = \Gamma_c \Omega^{n-p}$ with the induced differential. If $V \approx \mathbb{R}^n$ is an open set in M^n, then $H_p(\mathfrak{L}_*(V)) = H_c^{n-p}(V; \mathbb{R}) = \mathbb{R}$ for $p = 0$ and is zero otherwise. There is the augmentation

$$H_0(\mathfrak{L}_*(U)) = H_c^n(U; \mathbb{R}) \xrightarrow{\int_M} \mathbb{R}.$$

It follows that $\mathfrak{L}_*$ is a flabby coresolution of $\mathbb{R}$. Thus, by 12.11,

$$H_p^c(M^n; \mathbb{R}) \approx H_p(\mathfrak{L}_*(M^n)) = H_c^{n-p}(M^n; \mathbb{R}),$$

which is Poincaré duality in this situation. ◇

12.13. Example. Let $X = \mathbb{R}^n$ and suppose that $F \subset X$ is any closed subspace. Let $0 \to L \to \mathscr{A}^0 \to \cdots \to \mathscr{A}^n \to 0$ be any flabby resolution of L on X. For example, by Exercise 28, we can take $\mathscr{A}^p = \mathscr{C}^p(X; L)$ for

$p < n$ and $\mathscr{A}^n = \mathscr{J}^n(X; L)$. For an open set $U \subset X$, define the precosheaf $\mathfrak{L}_p$ on F by

$$\mathfrak{L}_p(U \cap F) = \Gamma_{c|U\cap F}(\mathscr{A}^{n-p}),$$

with the induced differential. (Note that this depends only on $U \cap F$ rather than on U.) Then $\mathfrak{L}_p$ is a flabby cosheaf on F by Exercise 3 and the remarks below (27) on page 94. Now,

$$\begin{aligned} H_p(\mathfrak{L}_*(U \cap F)) &= H^{n-p}_{c|U\cap F}(U; L) && \text{since } \mathscr{A}^* \text{ is flabby} \\ &\approx H^{n-p}_c(U, U - F; L) && \text{by II-12.1} \\ &\approx H^c_p(U \cap F; L) && \text{by 9.3.} \end{aligned}$$

There is the augmentation

$$H_0(\mathfrak{L}_*(U \cap F)) \approx H^c_0(U \cap F; L) \xrightarrow{\varepsilon} L.$$

This is a local isomorphism of precosheaves on F if F is clc^1_L by 5.14. Also, for $U \subset W$, the map $H_k(\mathfrak{L}_*(U \cap F)) \to H_k(\mathfrak{L}_*(W \cap F))$ is equivalent to the map $H^c_k(U \cap F; L) \to H^c_k(W \cap F; L)$ induced by inclusion. For $k \neq 0$ this is zero when U is a sufficiently small neighborhood of a given point $x \in W$ provided that F is k-hlc_L. Consequently, $H_k(\mathfrak{L}_*)$ is locally zero $\Leftrightarrow$ F is k-hlc_L. Thus $\mathfrak{L}_*$ is a flabby n-coresolution of L on F if F is hlc^n_L. $\diamond$

12.14. Theorem. *If X is HLC^n_L, then the singular cosheaf $\mathfrak{S}_*$ of 1.3 is a flabby quasi-n-coresolution of L.*

Proof. Since $H_*(\mathfrak{S}_*(U))$ is naturally isomorphic to the singular homology of U, by 1.3, the result follows immediately from the definition of the property HLC^n_L. □

12.15. Corollary. *If X is HLC^{n+1}_L then the Borel-Moore homology groups $H^c_p(X; L)$ are naturally isomorphic to the classical singular homology groups of X with coefficients in L for $p \leq n$.* □

We shall now consider general sheaves of coefficients. Recall the notation introduced in 1.14.

12.16. Theorem. *Suppose that $\mathfrak{L}_*$ is a flabby, torsion-free quasi-$(n+2)$-coresolution of L on X. For sheaves $\mathscr{B}$ on X, there is a natural isomorphism*

$$\boxed{H^c_p(X; \mathscr{B}) \approx H_p((\mathfrak{L}_* \otimes \mathscr{B})(X))}$$

for $p \leq n$.

Proof. Note that X is clc_L^{n+2} by 12.7 and hence hlc_L^{n+1} by 12.10. Thus $\mathfrak{D}(\mathscr{I}^*(X;L))$ is a flabby, torsion-free quasi-$(n+1)$-coresolution of L but perhaps not a quasi-$(n+2)$-coresolution. Let $\mathfrak{L}_* = \Gamma_c\mathscr{L}_*$. There is the natural homomorphism

$$\mathscr{L}_* \to \mathscr{D}(\mathscr{D}(\mathscr{L}_*))$$

resulting from (6) on page 290. Note that $\mathscr{D}(\mathscr{D}(\mathscr{L}_*))$ is also torsion-free by 3.4. We have the induced homomorphism

$$\mathscr{L}_* \otimes \mathscr{B} \to \mathscr{D}(\mathscr{D}(\mathscr{L}_*)) \otimes \mathscr{B}$$

and hence, applying Γ_c,

$$\mathfrak{L}_* \otimes \mathscr{B} \to \mathfrak{D}(\mathfrak{D}(\mathfrak{L}_*)) \otimes \mathscr{B} = \mathfrak{D}(\mathscr{N}^*) \otimes \mathscr{B},$$

where $\mathscr{N}^* = \mathscr{D}(\mathscr{L}_*)$, which is a flabby quasi-$(n+2)$-resolution of L by 12.7. Note that $\mathfrak{D}(\mathscr{N}^*)$ is a quasi-$(n+1)$-coresolution of L by 12.9.

Letting $\mathscr{I}^* = \mathscr{I}^*(X;L)$, there is a homomorphism of quasi-$(n+2)$-resolutions $\mathscr{N}^* \to \mathscr{I}^*$. We have the induced maps

$$(\mathfrak{L}_* \otimes \mathscr{B})(X) \to (\mathfrak{D}(\mathscr{N}^*) \otimes \mathscr{B})(X) \leftarrow (\mathfrak{D}(\mathscr{I}^*) \otimes \mathscr{B})(X),$$

that is,

$$\Gamma_c(\mathscr{L}_* \otimes \mathscr{B}) \to \Gamma_c(\mathscr{D}(\mathscr{N}^*) \otimes \mathscr{B}) \leftarrow \Gamma_c(\mathscr{D}(\mathscr{I}^*) \otimes \mathscr{B}). \tag{76}$$

The terms in (76) are all exact functors of $\mathscr{B}$ by II-16.31. Thus we have the induced maps of connected sequences of functors

$$H_p((\mathfrak{L}_* \otimes \mathscr{B})(X)) \to H_p((\mathfrak{D}(\mathscr{N}^*) \otimes \mathscr{B})(X)) \leftarrow H_p^c(X;\mathscr{B}). \tag{77}$$

By II-14.5 the terms of (76), and hence of (77), commute with direct limits in $\mathscr{B}$. By 12.11 the homomorphisms of (77) are isomorphisms when $\mathscr{B} = L$ and $p \le n$. The same fact also follows easily for $\mathscr{B} = L_U$, U open, since, for example, $(\mathfrak{L}_* \otimes L_U)(X) = \mathfrak{L}_*(U)$. By the 5-lemma and by II-16.12 it follows that the class of sheaves $\mathscr{B}$ for which the maps (77) are isomorphisms for $p \le n$ consists of all sheaves.[23] □

12.17. Corollary. *If X is HLC_L^{n+2}, then for any sheaf $\mathscr{B}$ on X there is the natural isomorphism*

$${}_S H_p^c(X;\mathscr{B}) \approx H_p^c(X;\mathscr{B})$$

for $p \le n$; see 1.18. □

Note that if $\mathscr{B}$ is locally constant, then ${}_S H_p^c(X;\mathscr{B})$ is the classical singular homology group of X with local coefficients in $\mathscr{B}$.

[23] For $\mathscr{B}$ constant, an alternative proof is suggested in Exercises 12 and 13.

12.18. Let $\mathfrak{L}_*$ be a flabby, torsion-free quasi-$(n+2)$-coresolution of L on X and let $U \subset X$ be open. Then

$$H_p((\mathfrak{L}_* \otimes \mathscr{B}_{X-U})(X)) \approx H_p^c(X; \mathscr{B}_{X-U}) \approx H_p^c(X, U; \mathscr{B})$$

for $p \leq n$, by 12.16 and (35) on page 306. If $\mathfrak{L}_* = \Gamma_c \mathscr{L}_*$, then

$$(\mathfrak{L}_* \otimes \mathscr{B}_U)(X) = \Gamma_c(\mathscr{L}_* \otimes \mathscr{B}_U) = \Gamma_{c|U}((\mathscr{L}_* \otimes \mathscr{B})|U) = (\mathfrak{L}_* \otimes \mathscr{B})(U).$$

Since $(\mathfrak{L}_* \otimes (\bullet))(X)$ is an exact functor of sheaves, it follows that

$$(\mathfrak{L}_* \otimes \mathscr{B}_{X-U})(X) \approx (\mathfrak{L}_* \otimes \mathscr{B})(X)/(\mathfrak{L}_* \otimes \mathscr{B})(U)$$

and hence that

$$\boxed{H_p^c(X, U; \mathscr{B}) \approx H_p((\mathfrak{L}_* \otimes \mathscr{B})(X)/(\mathfrak{L}_* \otimes \mathscr{B})(U))} \tag{78}$$

for $p \leq n$. In Section 13 this fact will be generalized to arbitrary locally closed subspaces.

12.19. We shall now consider arbitrary paracompactifying families of supports. We need some preliminary remarks. Suppose that $h : \mathfrak{A}_* \to \mathfrak{B}_*$ is a homomorphism of flabby differential cosheaves, where $\mathfrak{A}_* = \Gamma_c \mathscr{A}_*$ and $\mathfrak{B}_* = \Gamma_c \mathscr{B}_*$ with $\mathscr{A}_*$ and $\mathscr{B}_*$ c-soft. By 1.9, h induces (and is induced by) a homomorphism $\mathscr{A}_* \to \mathscr{B}_*$, which we shall also denote by h. On stalks at $x \in X$ the latter is the induced map

$$\mathfrak{A}_*(X)/\mathfrak{A}_*(X - \{x\}) \to \mathfrak{B}_*(X)/\mathfrak{B}_*(X - \{x\}) \tag{79}$$

of quotient groups. Suppose that

$$h_* : H_p(\mathfrak{A}_*(U)) \to H_p(\mathfrak{B}_*(U)) \tag{80}$$

is an *isomorphism* for all $p \leq n$ and all open $U \subset X$. Then the 5-lemma implies that (79) induces an isomorphism in homology, and thus the map

$$\mathscr{H}_p(\mathscr{A}_*) \to \mathscr{H}_p(\mathscr{B}_*)$$

is an isomorphism for all $p \leq n$. If Φ is a paracompactifying family of supports on X and if $\dim_\Phi X < \infty$, then it follows from IV-2.2 and II-16.5 that

$$h_* : H_p(\Gamma_\Phi(\mathscr{A}_*)) \to H_p(\Gamma_\Phi(\mathscr{B}_*))$$

is also an isomorphism for $p \leq n$.

12.20. Theorem. *Let $\mathfrak{L}_* = \Gamma_c \mathscr{L}_*$ be a flabby quasi-$(n+2)$-coresolution of L on X, where $\dim_L X < \infty$. Let $\mathscr{B}$ be a sheaf on X, and assume that either $\mathscr{B} = L$ or that $\mathfrak{L}_*$ is torsion-free. Then for any paracompactifying family Φ of supports on X, there is a natural isomorphism*

$$\boxed{H_p^\Phi(X; \mathscr{B}) \approx H_p(\Gamma_\Phi(\mathscr{L}_* \otimes \mathscr{B})) \text{ for } p \leq n.}$$

Proof. Consider the maps in (76) that are maps of differential *cosheaves*, as follows either from their definition or by replacing $\mathscr{B}$ by $\mathscr{B}_U$ and using their functorial nature. Both of these maps satisfy the hypotheses of 12.19 that (80) is an isomorphism, as shown in the proof of 12.16, and the present result follows from 12.19. □

12.21. Corollary. *If X is HLC_L^{n+2} and has $\dim_L X < \infty$, then for Φ paracompactifying and $\mathscr{B}$ arbitrary there is a natural isomorphism*

$$\boxed{{}_S H_p^{\Phi}(X;\mathscr{B}) \approx H_p^{\Phi}(X;\mathscr{B})}$$

for $p \le n$; see 1.18. □

12.22. Let X be hlc_L^n; $n \ge 0$. Consider the canonical flabby quasi-n-coresolution $\mathfrak{L}_* = \Gamma_c \mathscr{L}_*$, where $\mathscr{L}_* = \mathscr{C}_*(X;L)$. By the proof of 5.13 we see that $d : \mathfrak{L}_0 \to \mathfrak{L}_{-1}$ is an epimorphism and that its kernel $\mathfrak{K}$ is a *flabby, torsion-free cosheaf.* It follows that the differential cosheaf

$$\cdots \to \mathfrak{L}_2 \to \mathfrak{L}_1 \to \mathfrak{K} \to 0$$

is a flabby, torsion-free n-coresolution of L; i.e., it *vanishes in negative degrees.*

13 Uniqueness theorems for maps and relative homology

In this section we shall apply the methods of the last section to maps of spaces and to relative homology theory.

Suppose that $f : X \to Y$ is a map between locally compact spaces. Let $\mathscr{N}^*$ and $\mathscr{L}^*$ be replete differential sheaves on X and Y respectively and let $k' : \mathscr{L}^* \rightsquigarrow \mathscr{N}^*$ be an f-cohomomorphism. By (24) on page 301 we have the induced homomorphism

$$\Gamma_{\Phi(c)}(\mathscr{D}(\mathscr{N}^*) \otimes f^*\mathscr{A}) \to \Gamma_{\Phi}(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}) \tag{81}$$

for any sheaf $\mathscr{A}$ on Y and family Φ of supports on Y. If $\mathscr{N}^*$ and $\mathscr{L}^*$ are quasi-resolutions of L, then as in (27) on page 302 it can be seen that in homology, (81) induces the canonical homomorphism

$$f_* : H_*^{\Phi(c)}(X; f^*\mathscr{A}) \to H_*^{\Phi}(Y;\mathscr{A}) \tag{82}$$

either when $\Phi = c$ or when Φ is paracompactifying and $\dim_L X$ and $\dim_L Y$ are both finite, as well as in other cases (see 5.6). This was proved for resolutions, but the proofs apply to quasi-resolutions.

Let $\mathfrak{N}_*$ and $\mathfrak{L}_*$ be flabby, torsion-free quasi-$(n+2)$-coresolutions of L on X and Y respectively, with $\mathfrak{N}_* = \Gamma_c \mathscr{N}_*$ and $\mathfrak{L}_* = \Gamma_c \mathscr{L}_*$. Suppose we are given an f-homomorphism

$$k : \mathfrak{N}_* \rightsquigarrow \mathfrak{L}_* \tag{83}$$

(i.e., a homomorphism $f\mathfrak{N}_* \to \mathfrak{L}_*$). Since $f\mathfrak{N}_* = \Gamma_c\{f_c\mathscr{N}_*\}$, k corresponds to a homomorphism $f_c\mathscr{N}_* \to \mathscr{L}_*$. The homomorphisms $k_U : \mathfrak{N}_*(f^{-1}U) \to \mathfrak{L}_*(U)$ induce homomorphisms

$$\mathscr{D}(\mathfrak{L}_*)(U) \to \mathscr{D}(\mathfrak{N}_*)(f^{-1}U),$$

that is, an f-cohomomorphism

$$k' : \mathscr{D}(\mathfrak{L}_*) \rightsquigarrow \mathscr{D}(\mathfrak{N}_*). \tag{84}$$

Letting $\mathscr{L}^* = \mathscr{D}(\mathfrak{L}_*)$ and $\mathscr{N}^* = \mathscr{D}(\mathfrak{N}_*)$, we obtain the diagram

$$\begin{array}{ccc} f_c\mathscr{N}_* & \longrightarrow & \mathscr{L}_* \\ \downarrow & & \downarrow \\ f_c\mathscr{D}(\mathscr{N}^*) & \longrightarrow & \mathscr{D}(\mathscr{L}^*), \end{array} \tag{85}$$

where the lower map is from (81) and the vertical maps are from 1.13. It is easy to check that this diagram commutes. Tensoring with $\mathscr{A}$, using 4.3, and applying Γ_Φ, we obtain the commutative diagram

$$\begin{array}{ccc} \Gamma_{\Phi(c)}(\mathscr{N}_* \otimes f^*\mathscr{A}) & \longrightarrow & \Gamma_\Phi(\mathscr{L}_* \otimes \mathscr{A}) \\ \downarrow & & \downarrow \\ \Gamma_{\Phi(c)}(\mathscr{D}(\mathscr{N}^*) \otimes f^*\mathscr{A}) & \longrightarrow & \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{A}). \end{array} \tag{86}$$

If $\Phi = c$ or if Φ is paracompactifying and $\dim_L X$ and $\dim_L Y$ are both finite, then the vertical maps in (86) induce isomorphisms in homology in degrees at most $n+1$ by 12.11 and 12.19. Thus, in either of these cases, the induced homomorphism

$$H_p(\Gamma_{\Phi(c)}(\mathscr{N}_* \otimes f^*\mathscr{A})) \to H_p(\Gamma_\Phi(\mathscr{L}_* \otimes \mathscr{A}))$$

may be identified, via 12.16 or 12.20, with f_* for $p \le n$.

In particular, if X and Y are both HLC_L^{n+2}, the classical homomorphism ${}_SH_p^c(X) \to {}_SH_p^c(Y)$ coincides with $f_* : H_p^c(X) \to H_p^c(Y)$ via 12.15 for $p \le n$.

We shall now consider the relative homology of a pair (X, A) where A is locally closed in X. For A open this already has been discussed in 12.18. The discussion will be divided into several subsections.

13.1. Let $\mathscr{N}^*$ be a c-soft differential sheaf on A. Then there is the canonical homomorphism

$$\begin{aligned}(i_c\mathscr{D}(\mathscr{N}^*))(X) = \Gamma_{cld|A}(\mathscr{H}\!om(\Gamma_c\mathscr{N}^*, L^*)) &\xrightarrow{\eta} \operatorname{Hom}(\Gamma_{c\cap A}(\mathscr{N}^*), L^*)\\ &= \operatorname{Hom}(\Gamma_c(i\mathscr{N}^*), L^*) = \mathscr{D}(i\mathscr{N}^*)(X)\end{aligned}$$

(see Section 4), where η is the *monomorphism* (19) on page 300. We have a similar map when X is replaced by an open subset $U \subset X$, and therefore we obtain the canonical *monomorphism*

$$\mathscr{D}(\mathscr{N}^*)^X = i_c\mathscr{D}(\mathscr{N}^*) \rightarrowtail \mathscr{D}(i\mathscr{N}^*). \tag{87}$$

Now let $\mathscr{L}^*$ and $\mathscr{N}^*$ be replete quasi-resolutions of the L-module M on X and A respectively, and let

$$\mathscr{L}^* \twoheadrightarrow i\mathscr{N}^*$$

be an epimorphism with c-soft kernel $\mathscr{K}^*$. Then the composition

$$\mathscr{D}(\mathscr{N}^*)^X \to \mathscr{D}(i\mathscr{N}^*) \to \mathscr{D}(\mathscr{L}^*)$$

is a *monomorphism*. Denote its cokernel by $\mathscr{M}_*$, so that the sequence

$$0 \to \mathscr{D}_*(\mathscr{N}^*)^X \to \mathscr{D}_*(\mathscr{L}^*) \to \mathscr{M}_* \to 0 \tag{88}$$

is exact.

Note that if A were *closed*, then (87) would be an isomorphism by 2.6, and it would follow that $\mathscr{M}_* = \mathscr{D}(\mathscr{K}^*)$.

As in Section 4 (also see II-8), we can construct a diagram

$$\begin{array}{ccc} \mathscr{L}^* & \longrightarrow & i\mathscr{N}^* \\ \downarrow & & \downarrow \\ \mathscr{I}^*(X;M) & \longrightarrow & i\mathscr{I}^*(A;M), \end{array}$$

which commutes up to chain homotopy. Letting $M = L$, we then obtain the "dual" diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{C}_*(A;L)^X & \to & \mathscr{C}_*(X;L) & \to & \mathscr{C}_*(X,A;L) & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathscr{D}(\mathscr{N}^*)^X & \to & \mathscr{D}(\mathscr{L}^*) & \to & \mathscr{M}_* & \to & 0, \end{array} \tag{89}$$

in which the square commutes up to chain homotopy. According to II-Exercise 47 this diagram induces, in a natural way, a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & \mathscr{H}_*(A;L)^X & \to & \mathscr{H}_*(X;L) & \to & \mathscr{H}_*(X,A;L) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \mathscr{H}_*(\mathscr{D}(\mathscr{N}^*))^X & \to & \mathscr{H}_*(\mathscr{D}(\mathscr{L}^*)) & \to & \mathscr{H}_*(\mathscr{M}_*) & \to \cdots \end{array} \tag{90}$$

and it follows from the 5-lemma and (16) on page 294 that the vertical maps are all isomorphisms.

The diagram (89) can be tensored with any sheaf $\mathscr{A}$, and the rows will remain exact provided that $\mathscr{M}_* * \mathscr{A} = 0$ [e.g., when A is *closed* and $\mathscr{K}^*$ is replete, since $\mathscr{M}_* = \mathscr{D}(\mathscr{K}^*)$ is then torsion free]. Thus, by the same argument we have that

$$\mathscr{H}_*(X,A;\mathscr{A}) \approx \mathscr{H}_*(\mathscr{M}_* \otimes \mathscr{A}) \tag{91}$$

when $\mathscr{M}_* * \mathscr{A} = 0$.

Taking sections with supports in Φ, the rows of (89) (having previously been tensored with $\mathscr{A}$) will remain exact provided that Φ is paracompactifying (or that A is closed and $\mathscr{A}$ is elementary). Thus, using 3.5, 5.6,

and the above argument [on the analogue of (90) for global homology], we obtain the isomorphism

$$\boxed{H_*^\Phi(X, A; \mathscr{A}) \approx H_*(\Gamma_\Phi(\mathscr{M}_* \otimes \mathscr{A}))} \tag{92}$$

when $\mathscr{M}_* * \mathscr{A} = 0$, provided either that $\Phi = c$ or that Φ is paracompactifying and $\dim_L X < \infty$ [as well as other cases resulting from 5.6].

13.2. In this subsection we shall specialize the discussion to the case of a closed subset $F = A$ and shall consider the situation in which $\mathscr{N}^* = \mathscr{L}^*|F$. Then the sequence $0 \to \mathscr{K}^* \to \mathscr{L}^* \to i\mathscr{N}^* \to 0$ becomes

$$0 \to \mathscr{L}^*_{X-F} \to \mathscr{L}^* \to \mathscr{L}^*_F \to 0 \tag{93}$$

(all of which are replete by 3.3, since $\mathscr{L}^*$ is replete). Thus, in this case, we obtain the natural isomorphism

$$\mathscr{H}_*(X, F; \mathscr{A}) \approx \mathscr{H}_*(\mathscr{D}(\mathscr{L}^*_{X-F}) \otimes \mathscr{A}). \tag{94}$$

When $\Phi = c$ or when Φ is paracompactifying and $\dim_L X < \infty$, then

$$\boxed{H_*^\Phi(X, F; \mathscr{A}) \approx H_*(\Gamma_\Phi(\mathscr{D}(\mathscr{L}^*_{X-F}) \otimes \mathscr{A})).} \tag{95}$$

Moreover, the homology sequence of the pair (X, F) may be identified with the sequence induced by (93). Also, see the remark below 5.10.

13.3. Let $\mathfrak{N}_*$ and $\mathfrak{L}_*$ be flabby quasi-$(n+1)$-coresolutions of L on A and on X respectively, and recall that by 2.6 we have

$$\mathscr{D}(\mathfrak{N}_*^X) \approx i\mathscr{D}(\mathfrak{N}_*). \tag{96}$$

Recall also that by definition $\mathfrak{N}_*^X = i\mathfrak{N}_*$ and that, if $\mathfrak{N}_* = \Gamma_c\mathscr{N}_*$ then $\mathfrak{N}_*^X = \Gamma_c\{\mathscr{N}_*^X\}$ by 1.16. Let

$$k : \mathfrak{N}_*^X \rightarrowtail \mathfrak{L}_*$$

be a *monomorphism* (of differential precosheaves) with *flabby cokernel* $\mathfrak{K}_*$ (which is automatically a cosheaf, as is seen by an easy diagram chase). Then for any L-module M we have the induced exact sequence

$$0 \to \mathscr{D}(\mathfrak{K}_*; M) \to \mathscr{D}(\mathfrak{L}_*; M) \to i\mathscr{D}(\mathfrak{N}_*; M) \to 0 \tag{97}$$

[by (96)] of differential sheaves. Note that by 12.6 this situation exists if X and A are both hlc_L^{n+1}.

13.4. Theorem. *Suppose we are given flabby quasi-$(n+1)$-coresolutions $\mathfrak{N}_*$ and $\mathfrak{L}_*$ of L on A and X respectively and a monomorphism $\mathfrak{N}_*^X \rightarrowtail \mathfrak{L}_*$*

with flabby cokernel $\mathfrak{K}_* = \Gamma_c\mathscr{K}_*$ *(where* $\mathscr{K}_*$ *is c-soft). Then there is a natural isomorphism*

$$\boxed{H_p(\mathfrak{K}_*(X)) \approx H_p^c(X, A; L)}$$

for $p \le n$. *If* $\dim_L X < \infty$, *then for any paracompactifying family* Φ *we have the natural isomorphism*

$$\boxed{H_p(\Gamma_\Phi(\mathscr{K}_*)) \approx H_p^\Phi(X, A; L)}$$

for $p < n$.

Proof. Let $\mathfrak{L}_* = \Gamma_c\mathscr{L}_*$, $\mathfrak{N}_* = \Gamma_c\mathscr{N}_*$, and $\mathfrak{K}_* = \Gamma_c\mathscr{K}_*$. Also, put $\mathscr{L}^* = \mathscr{D}(\mathfrak{L}_*)$, $\mathscr{N}^* = \mathscr{D}(\mathfrak{N}_*)$, and $\mathscr{K}^* = \mathscr{D}(\mathfrak{K}_*)$. The sequence

$$0 \to \mathfrak{N}_*^X \to \mathfrak{L}_* \to \mathfrak{K}_* \to 0$$

is equivalent to an exact sequence

$$0 \to \mathscr{N}_*^X \to \mathscr{L}_* \to \mathscr{K}_* \to 0$$

[see 1.9] and induces an exact sequence

$$0 \to \mathscr{K}^* \to \mathscr{L}^* \to i\mathscr{N}^* \to 0$$

[see (97)]. In turn, the latter sequence induces an exact sequence

$$0 \to \mathscr{D}(\mathscr{N}^*)^X \to \mathscr{D}(\mathscr{L}^*) \to \mathscr{M}_* \to 0$$

[see (88)], that *defines* $\mathscr{M}_*$.

The natural maps $\mathscr{N}_* \to \mathscr{D}(\mathscr{D}(\mathscr{N}_*)) = \mathscr{D}(\mathscr{N}^*)$ and $\mathscr{L}_* \to \mathscr{D}(\mathscr{D}(\mathscr{L}_*)) = \mathscr{D}(\mathscr{L}^*)$, from (6) on page 290, induce a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{N}_*^X & \to & \mathscr{L}_* & \to & \mathscr{K}_* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{D}(\mathscr{N}^*)^X & \to & \mathscr{D}(\mathscr{L}^*) & \to & \mathscr{M}_* & \to & 0. \end{array} \tag{98}$$

[This is a special case of (85).] Applying Γ_Φ to this diagram, we see that the first two vertical maps induce isomorphisms in homology through degree $n-1$, by 12.11 and 12.19 [i.e., the proof of 12.20]. The theorem now follows from the 5-lemma and formula (92). For $\Phi = c$ we have the same conclusion through degree n by the proof of 12.11. □

Note that if all the sheaves in (98) are torsion-free, then we can tensor (98) with any sheaf $\mathscr{A}$, retaining the exact rows, so that the conclusion of the theorem would remain true for arbitrary coefficient sheaves. In particular, this is the case when L is a field. If A is *closed* and if $\mathfrak{K}_*$ is torsion-free (i.e., $\mathscr{K}_*$ is torsion-free), then $\mathscr{K}^* = \mathscr{D}(\mathfrak{K}_*)$ is replete and $\mathscr{M}_* = \mathscr{D}(\mathscr{K}^*)$ (since A is closed) is torsion-free; see 3.4. Thus we have proved:

13.5. Theorem. *If in* 13.4, *A is closed or if L is a field, and if $\mathfrak{L}_*$, $\mathfrak{N}_*$, and $\mathfrak{K}_*$ are torsion-free and* $\dim_L X < \infty$, *then*

$$\boxed{H_p(\Gamma_\Phi(\mathscr{K}_* \otimes \mathscr{A})) \approx H_p^\Phi(X, A; \mathscr{A})}$$

for $p < n$, for any sheaf $\mathscr{A}$ and any paracompactifying family Φ. If $\Phi = c$, then $\dim_L X$ *need not be finite.* □

By 1.18 the preceding two theorems apply to the singular homology of HLC spaces. That is, we have the following consequence:

13.6. Corollary. *Let X and A be HLC_L^{n+1}. If either $\Phi = c$ or* $\dim_L X < \infty$ *with Φ paracompactifying, then there is a natural isomorphism*

$$\boxed{{}_sH_p^\Phi(X, A; L) \approx H_p^\Phi(X, A; L)}$$

for $p < n$ ($p \leq n$ if $\Phi = c$). If A is closed, or if L is a field, we also have the isomorphism

$$\boxed{{}_sH_p^\Phi(X, A; \mathscr{A}) \approx H_p^\Phi(X, A; \mathscr{A})}$$

for any sheaf $\mathscr{A}$ on X and $p < n$. □

13.7. Theorem. *Suppose that $A \subset X$ are both hlc_L^n and that M is an L-module. Then there is a natural exact sequence*

$$\boxed{\operatorname{Ext}(H_{p-1}^c(X, A; L), M) \rightarrowtail H^p(X, A; M) \twoheadrightarrow \operatorname{Hom}(H_p^c(X, A; L), M),}$$

for $p \leq n$, which splits. (*Also see* 12.8.)

Proof. First consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{K}^* & \to & \mathscr{L}^* & \to & i\mathscr{N}^* & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & '\mathscr{K}^* & \to & \mathscr{I}^*(X; M) & \to & i\mathscr{I}^*(A; M) & \to & 0 \end{array} \tag{99}$$

from 13.1, which has exact rows (by definition of $\mathscr{K}^*$ and $'\mathscr{K}^*$) and homotopy commutative square. The sheaf $'\mathscr{K}^*$ is easily seen to be flabby; see II-12, which is analogous.[24] We shall *assume* that $\mathscr{K}^*$, $\mathscr{L}^*$, and $\mathscr{N}^*$ are all flabby and that $\mathscr{L}^*$ and $\mathscr{N}^*$ are quasi-n-resolutions of the L-module M.

Applying Γ_Φ to (99) and using II-Exercise 47, we obtain the isomorphism

$$H^p(\Gamma_\Phi(\mathscr{K}^*)) \approx H^p(\Gamma_\Phi('\mathscr{K}^*)) \text{ for } p \leq n.$$

[24] Using the constructions at the beginning of Section 5, it can actually be seen that $'\mathscr{K}^*$ is injective, but this is not needed.

We could take $\mathscr{K}^* = \mathscr{C}^*(X, A; M)$, and it follows that for *general* $\mathscr{K}^*$

$$H^p(\Gamma_\Phi(\mathscr{K}^*)) \approx H^p_\Phi(X, A; M)$$

for $p \leq n$. In particular, for $p \leq n$ we have that

$$H^p(\Gamma(\mathscr{K}^*)) \approx H(X, A; M). \tag{100}$$

Now take $\mathscr{K}^* = \mathscr{D}(\mathfrak{K}_*; M)$, where $\mathfrak{K}_* = \Gamma_c \mathscr{C}_*(X, A; L)$, and similarly for $\mathscr{L}^*$ and $\mathscr{N}^*$. We have the exact sequence

$$0 \to \mathscr{K}^* \to \mathscr{L}^* \to i\mathscr{N}^* \to 0,$$

so that (100) holds for this choice of $\mathscr{K}^*$ by 12.7. The desired universal coefficient sequence now follows from 2.3 applied to $\mathfrak{K}_*$. □

Remark: The sequence analogous to 13.7, with homology and cohomology exchanged, is not generally valid if A is not closed. For example, let A be the disjoint union of infinitely many disjoint open arcs in $X = \mathbb{S}^1$ and let L be a field. Then $\dim H^1(X, A; L)$ is uncountable, but $\dim H_1(X, A)$ is countable, which rules out the existence of such a sequence.

14 The Künneth formula

Recall that in II-15 we proved a Künneth formula for cohomology. In (58) on page 339 we also found a Künneth formula for mixed homology and cohomology. The homology Künneth formula does not always hold in Borel-Moore theory, and an example of that will be given at the end of this section. In certain situations, however, one does have a Künneth formula. We shall first show that this is the case for clc^∞_L spaces, with compact supports and arbitrary coefficients, using the results of Section 12. Then we shall also derive a Künneth formula for general spaces X, Y provided that one of them has finitely generated cohomology modules.

Assume, for the time being, that X and Y are both hlc^n_L. Let $\mathfrak{L}_* = \Gamma_c \mathscr{L}_*$ and $\mathfrak{N}_* = \Gamma_c \mathscr{N}_*$ be flabby, torsion-free quasi-n-coresolutions of L on X and Y respectively, where $\mathscr{L}_*$ and $\mathscr{N}_*$ are c-soft and torsion-free. By 12.22 we may assume that they vanish in negative degrees.

Let $\mathfrak{L}_* \widehat{\otimes} \mathfrak{N}_*$ denote the differential cosheaf $\Gamma_c\{\mathscr{L}_* \widehat{\otimes} \mathscr{N}_*\}$ on $X \times Y$.[25] By II-15.5 we have that

$$(\mathfrak{L}_* \widehat{\otimes} \mathfrak{N}_*)(U \times V) = \mathfrak{L}_*(U) \otimes \mathfrak{N}_*(V)$$

for U and V open in X and Y respectively. It follows that the precosheaf $H_n(\mathfrak{L}_* \widehat{\otimes} \mathfrak{N}_*)$, where $\mathfrak{L}_* \widehat{\otimes} \mathfrak{N}_*$ is given the usual total degree and differential, satisfies a natural Künneth formula

$$0 \to \bigoplus_{p+q=k} H_p(\mathfrak{L}_*(U)) \otimes H_q(\mathfrak{N}_*(V)) \to H_k((\mathfrak{L}_* \widehat{\otimes} \mathfrak{N}_*)(U \times V))$$
$$\to \bigoplus_{p+q=k-1} H_p(\mathfrak{L}_*(U)) * H_q(\mathfrak{N}_*(V)) \to 0,$$

[25]The sheaf $\mathscr{L}_* \widehat{\otimes} \mathscr{N}_*$ is c-soft by II-Exercise 14.

and it follows that $\mathfrak{L}_*\widehat{\otimes}\mathfrak{N}_*$ is a flabby quasi-n-coresolution of L on $X \times Y$. In particular, $X \times Y$ is clc_L^n by 12.7.

Now let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and Y, respectively, such that $\mathscr{A}\widehat{*}\mathscr{B} = 0$. By definition we have

$$(\mathfrak{L}_*\widehat{\otimes}\mathfrak{N}_*) \otimes (\mathscr{A}\widehat{\otimes}\mathscr{B}) = (\mathfrak{L}_* \otimes \mathscr{A})\widehat{\otimes}(\mathfrak{N}_* \otimes \mathscr{B}), \tag{101}$$

and the value of this on $X \times Y$ is

$$(\mathfrak{L}_* \otimes \mathscr{A})(X) \otimes (\mathfrak{N}_* \otimes \mathscr{B})(Y) = \Gamma_c(\mathscr{L}_* \otimes \mathscr{A}) \otimes \Gamma_c(\mathscr{N}_* \otimes \mathscr{B}) \tag{102}$$

because of II-16.31, II-Exercise 36, and II-15.5. By II-Exercise 36 we have that

$$\Gamma_c(\mathscr{L}_* \otimes \mathscr{A}) * \Gamma_c(\mathscr{N}_* \otimes \mathscr{B}) = 0, \tag{103}$$

so that the algebraic Künneth formula [54], [75] may be applied to (102). Thus, using (101), 12.11, and 12.16, we obtain the following result:

14.1. Theorem. *Let X and Y be hlc_L^{n+1} and let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and Y, respectively, such that $\mathscr{A}\widehat{*}\mathscr{B} = 0$. Then there is a natural exact sequence*

$$\boxed{\begin{aligned} 0 \to \bigoplus_{p+q=k} H_p^c(X;\mathscr{A}) \otimes H_q^c(Y;\mathscr{B}) \to H_k^c(X \times Y; \mathscr{A}\widehat{\otimes}\mathscr{B}) \\ \to \bigoplus_{p+q=k-1} H_p^c(X;\mathscr{A}) * H_q^c(Y;\mathscr{B}) \to 0 \end{aligned}}$$

for $k < n$ ($k \le n$ if $\mathscr{A} = L$ and $\mathscr{B} = L$) which splits. □

Taking Y to be a point, we deduce the universal coefficient formula:

14.2. Corollary. *Let X be hlc_L^{n+1} and let $\mathscr{A}$ be a sheaf on X. Let M be an L-module such that $\mathscr{A} * M = 0$. Then there is a split exact sequence*

$$\boxed{0 \to H_k^c(X;\mathscr{A}) \otimes M \to H_k^c(X;\mathscr{A} \otimes M) \to H_{k-1}^c(X;\mathscr{A}) * M \to 0}$$

for $k < n$. □

We wish to obtain, from 14.1, a Künneth sequence for the sheaves of local homology groups. Note that, for example,

$$\mathscr{H}_*(X;L) = \mathscr{S}heaf(U \mapsto H_*^c(X, X - \overline{U}; L)), \tag{104}$$

since this is clearly the same as the sheaf generated by the presheaf

$$U \mapsto H_*^c(X, X - U; L)$$

and since

$$H_*^c(X, X - U; L) \approx H_*(U; L)$$

when $\overline{U}$ is compact, by 5.10. Also, we have that

$$H_*^c(X, X-\overline{U}; L) \approx H_*^c(X; L_{\overline{U}}) \tag{105}$$

by (35) on page 306. Taking $\mathscr{A} = L_{\overline{U}}$ and $\mathscr{B} = L_{\overline{V}}$ in 14.1 and passing to generated sheaves yields the desired result:

14.3. Corollary. *If X and Y are hlc_L^{n+1}, then there is an exact sequence*

$$\boxed{\begin{aligned} 0 \to \bigoplus_{p+q=k} \mathscr{H}_p(X;L)\widehat{\otimes}\mathscr{H}_q(Y;L) \to \mathscr{H}_k(X\times Y;L) \\ \to \bigoplus_{p+q=k-1} \mathscr{H}_p(X;L)\widehat{*}\mathscr{H}_q(Y;L) \to 0 \end{aligned}}$$

of sheaves on $X \times Y$ for $k < n$. □

We now turn to the case of general (locally compact) spaces X and Y. We shall indicate the proof of the following result:

14.4. Theorem. *Suppose that $H_c^p(X;L)$ is finitely generated for each p. Let Φ be any family of supports on Y. Then there is a natural exact sequence*

$$\boxed{\begin{aligned} 0 \to \bigoplus_{p+q=k} H_p(X;L)\otimes H_q^{\Phi}(Y;L) \to H_k^{X\times\Phi}(X\times Y;L) \\ \to \bigoplus_{p+q=k-1} H_p(X;L) * H_q^{\Phi}(Y;L) \to 0. \end{aligned}}$$

This sequence splits when $\Phi = cld$.

Proof. By 5.6(i) and naturality it suffices to treat the case in which $\Phi = cld$. For the proof we introduce the notation

$$C_X^p = C_c^p(X;L) \quad \text{and} \quad C_Y^q = C_c^q(Y;L).$$

Note that $C_X^* \otimes C_Y^* = \Gamma_c(\mathscr{C}^*(X;L)\widehat{\otimes}\mathscr{C}^*(Y;L))$, so that

$$H_p(\operatorname{Hom}(C_X^*\otimes C_Y^*, L^*)) \approx H_p(X\times Y;L) \tag{106}$$

by (10) on page 293.

Let F_X^* and F_Y^* be free chain complexes such that there exist chain maps

$$F_X^* \to C_X^* \quad \text{and} \quad F_Y^* \to C_Y^*$$

inducing isomorphisms in homology.[26] Since $H_c^p(X;L)$ is finitely generated for each p, it can be shown that F_X^* can be chosen to be finitely generated in each degree.[27] We assume that this is done.

[26]For this one could take projective resolutions of the C^*.

[27]The proof, which is not difficult, can be found in [75].

The induced homomorphisms

$$\mathrm{Hom}(C^*, L^*) \to \mathrm{Hom}(F^*, L^*) \leftarrow \mathrm{Hom}(F^*, L)$$

induce isomorphisms in homology (the first, as in the proof of (10) on page 293; the second, since F^* is free). Also, the map

$$F_X^* \otimes F_Y^* \to C_X^* \otimes C_Y^*$$

induces a homology isomorphism since the C^* are torsion-free. It follows that

$$H_p(X \times Y; L) \approx H_p(\mathrm{Hom}(F_X^* \otimes F_Y^*, L)). \tag{107}$$

However,

$$\mathrm{Hom}(F_X^* \otimes F_Y^*, L) \approx \mathrm{Hom}(F_X^*, L) \otimes \mathrm{Hom}(F_Y^*, L) \tag{108}$$

(naturally) since F_X^* is free of finite type. Thus the algebraic Künneth formula applied to the right-hand side of (108) yields the desired result. The reader may make the straightforward verification of naturality. □

14.5. We shall now provide the promised counterexample to the general Künneth formula (for compact spaces). Let $X = Y$ be a solenoid, that is, the inverse limit of a sequence of circles C_n, where $C_n \to C_{n-1}$ has degree n. By continuity II-14.6 we see that $H^1(X; \mathbb{Z}) \approx \mathbb{Q}$, the groups in higher degrees vanishing. From II-15.2 and (9) on page 292 we calculate

$$H_1(X \times Y; \mathbb{Z}) \approx \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \neq 0$$

[see 14.8 below]. However, $H_0(X) \otimes H_1(Y) = 0$ since $H_1(Y) = \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$, and $H_0(X) * H_0(Y) = 0$ since $H_0(Y) = \mathbb{Z} \oplus \mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ and $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$, being a vector space over $\mathbb{Q}$, is torsion free. Thus the Künneth formula is not valid in this case.

14.6. If $L = \mathbb{Z}$ or if L is a field, then the condition in 14.4 that $H_c^p(X; L)$ be finitely generated for all p is equivalent to the condition that $H_p(X; L)$ be finitely generated for all p. This is clear when L is a field. When $L = \mathbb{Z}$, it follows from (9) on page 292 and the following algebraic fact.

14.7. Proposition. *For an abelian group A, if $\mathrm{Hom}(A, \mathbb{Z})$ and $\mathrm{Ext}(A, \mathbb{Z})$ are both finitely generated (respectively zero) then A is also finitely generated (respectively zero).*

Proof. (The parenthetical case can be found in [64].) It is easy to reduce the proposition to the cases in which A is either a torsion group or is torsion-free. If A is all torsion, then the exact sequence

$$0 = \mathrm{Hom}(A, \mathbb{R}) \to \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{R}) = 0$$

(induced by $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$) shows that

$$\operatorname{Ext}(A, \mathbb{Z}) \approx \hat{A} \quad \text{if } A \text{ is all torsion}, \tag{109}$$

where $\hat{A}$ denotes the Pontryagin dual of the discrete group A. The proposition follows immediately for torsion groups.

If A is torsion-free, consider the canonical map

$$A \to \operatorname{Hom}(\operatorname{Hom}(A, \mathbb{Z}), \mathbb{Z}). \tag{110}$$

Let the kernel of this map be denoted by A_0. Since the right-hand side of (110) is a finitely generated free abelian group, we have an exact sequence

$$0 \to A_0 \to A \to F \to 0, \tag{111}$$

where F is free and finitely generated. Note that (111) must split. Since by definition, the restriction $\operatorname{Hom}(A, \mathbb{Z}) \to \operatorname{Hom}(A_0, \mathbb{Z})$ is trivial, it follows from (111) that $\operatorname{Hom}(A_0, \mathbb{Z}) = 0$ and that $\operatorname{Ext}(A_0, \mathbb{Z})$ is finitely generated. Since A_0 is torsion-free, the exact sequence

$$0 \to A_0 \xrightarrow{n} A_0 \to A_0/nA_0 \to 0$$

implies that $\operatorname{Ext}(A_0, \mathbb{Z})$ is divisible. Hence $\operatorname{Ext}(A_0, \mathbb{Z}) = 0$ since it is finitely generated and divisible. The latter sequence also implies that $\operatorname{Ext}(A_0/nA_0, \mathbb{Z}) = 0$ and hence that $A_0/nA_0 = 0$ by (109). Thus A_0, being divisible and torsion-free, is a vector space over the field $\mathbb{Q}$ of rational numbers. Thus it suffices to show that $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \neq 0$. This follows from the next lemma. □

14.8. Lemma. *$\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ is a $\mathbb{Q}$-vector space of uncountable dimension.*

Proof. Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. By (109), $\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = \widehat{\mathbb{Q}/\mathbb{Z}}$. This is a *compact* group. It is not finite since its dual $\mathbb{Q}/\mathbb{Z}$ is not finite. Thus it must be uncountable by the Baire category theorem on locally compact spaces [19, p. 57]. We have the exact sequence

$$\mathbb{Z} = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \to \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \to \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0,$$

and the lemma follows. □

15 Change of rings

Suppose that K is a principal ideal domain that is also an L-module, and let $\mathscr{B}$ be a sheaf of K-modules on X. The homology of X with coefficients in $\mathscr{B}$ then has two interpretations depending on whether L or K is used as the base ring. We shall indicate the base ring, in this section only, by affixing it as a left superscript. Thus

$${}^L H_n^{\Phi}(X; \mathscr{B}) = H_n(\Gamma_{\Phi}({}^L\mathscr{D}(\mathscr{I}^*(X; L); L) \otimes_L \mathscr{B})), \tag{112}$$

while

$$ {}^{K}H_n^{\Phi}(X;\mathscr{B}) = H_n(\Gamma_\Phi({}^{K}\mathscr{D}(\mathscr{I}^*(X;K);K)\otimes_K \mathscr{B})). \tag{113}$$

Both of these are K-modules, but it is not clear whether or not there exists any relationship between them in general.

The K-modules (112) and (113) need not be isomorphic, as is shown by the following example: Let X be a compact space with $H^2(X;\mathbb{Z}) \approx \mathbb{Q}/\mathbb{Z}$ and $H^1(X;\mathbb{Z}) = 0$ (e.g., an inverse limit of lens spaces). By (9) on page 292, 3.13, and (109) we have ${}^{\mathbb{Z}}H_1(X;\mathbb{Q}) \approx (\widehat{\mathbb{Q}/\mathbb{Z}})\otimes\mathbb{Q} \neq 0$, since $\widehat{\mathbb{Q}/\mathbb{Z}}$ is torsion-free, $\mathbb{Q}/\mathbb{Z}$ being divisible. However, by II-15.3 and (9), we have that ${}^{\mathbb{Q}}H_1(X;\mathbb{Q}) = 0$. Another example is provided by a solenoid; see 14.5.

We shall say that "change of rings is valid," for X, Φ, K, and L, if (112) and (113) are naturally isomorphic. In the present section we shall see that change of rings is valid in two general situations. First this is shown for clc_L^∞ spaces, with suitable restrictions on Φ, using the results of Section 12. Then we show that change of rings is valid when $K = L_p = L/pL$, where p is a prime in L, for general spaces X and support families Φ.

15.1. Theorem. *Let X be hlc_L^{n+2}. Then there is a natural isomorphism*

$$ {}^{L}\mathscr{H}_k(X;\mathscr{B}) \approx {}^{K}\mathscr{H}_k(X;\mathscr{B}) $$

of sheaves of K-modules for $k \le n$. If $\Phi = c$ or if $\dim_{\Phi,L} X < \infty$ with Φ paracompactifying, then there is a natural isomorphism

$$ \boxed{{}^{L}H_k^{\Phi}(X;\mathscr{B}) \approx {}^{K}H_k^{\Phi}(X;\mathscr{B})} $$

of K-modules for $k \le n$.[28]

Proof. By definition, $\dim_{\Phi,L} X \ge \dim_{\Phi,K} X$. Let $\mathfrak{L}_*$ be a flabby, torsion-free quasi-$(n+2)$-coresolution of L on X [such as $\mathfrak{D}(\mathscr{I}^*(X;L))$] and let $\mathfrak{L}_* = \Gamma_c\mathscr{L}_*$ with $\mathscr{L}_*$ c-soft. The algebraic universal coefficient theorem implies, quite easily, that $\mathfrak{L}_*\otimes_L K = \Gamma_c\{\mathscr{L}_*\otimes_L K\}$ is a flabby, torsion-free quasi-$(n+2)$-coresolution of K on X. Since $(\mathscr{L}_*\otimes_L K)\otimes_K\mathscr{B} \approx \mathscr{L}_*\otimes_K\mathscr{B}$, the last statement of the theorem follows directly from 12.16 and 12.20. The first statement of the theorem follows from the fact that, for example,

$$ {}^{L}\mathscr{H}_k(X;\mathscr{B}) \approx \mathscr{H}_k(\mathscr{L}_*\otimes_L\mathscr{B}) $$

[see 12.19 and the proof of 12.20]. Alternatively, the first statement of the theorem follows from the last and the fact that

$$ {}^{L}\mathscr{H}_k(X;\mathscr{B}) = \mathscr{S}\!heaf\left(U \mapsto {}^{L}H_k^c(X, X-\overline{U};\mathscr{B}) = {}^{L}H_k^c(X;\mathscr{B}_U)\right). \quad \square $$

15.2. Theorem. *Let $\mathscr{B}$ be a sheaf of L_p-modules, p prime in L, and let Φ be any family of supports on the arbitrary locally compact Hausdorff space X. Then there is a natural isomorphism of L_p-modules*

$$ \boxed{{}^{L}H_n^{\Phi}(X;\mathscr{B}) \approx {}^{L_p}H_n^{\Phi}(X;\mathscr{B}).} $$

[28] Also see VI-11.5.

Proof. We note first that since L can be embedded in a *torsion-free* injective module (e.g., its field of quotients), $\mathscr{I}^*(X;L)$ may be replaced by an injective resolution $\mathscr{I}^*$ such that $\mathscr{I}^0$ is torsion-free [see the construction of II-3.2].

Since an injective L-module is divisible, there is, for any L-injective sheaf $\mathscr{I}$, an exact sequence

$$0 \to {}_p\mathscr{I} \to \mathscr{I} \xrightarrow{p} \mathscr{I} \to 0,$$

where p stands for multiplication by $p \in L$ and ${}_p\mathscr{I}$ is its kernel. Clearly, ${}_p\mathscr{I}$ is a sheaf of L_p-modules. We need the following lemma:

15.3. Lemma. *${}_p\mathscr{I}$ is an L_p-injective sheaf.*

Proof. Consider the commutative diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{A} & \xrightarrow{f} & \mathscr{B} \\ & & {\scriptstyle g}\downarrow & & \vdots{\scriptstyle h} \\ & & {}_p\mathscr{I} & \longrightarrow & \mathscr{I} \end{array}$$

where f and g are given homomorphisms of sheaves of L_p-modules. Since $\mathscr{I}$ is L-injective, there exists the L-homomorphism h, as indicated, making the diagram commute. The diagram

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{p=0} & \mathscr{B} \\ {\scriptstyle h}\downarrow & & \downarrow \\ \mathscr{I} & \xrightarrow{p} & \mathscr{I} \end{array}$$

shows that $\operatorname{Im} h \subset {}_p\mathscr{I}$. As a homomorphism into ${}_p\mathscr{I}$, h is clearly L_p-linear, and this completes the proof of the Lemma. □

Since ${}_p\mathscr{I}$ is L_p-injective, it is also flabby, and it follows that the sequence

$$0 \to \Gamma_c({}_p\mathscr{I}|U) \to \Gamma_c(\mathscr{I}|U) \xrightarrow{p} \Gamma_c(\mathscr{I}|U) \to 0 \tag{114}$$

is exact for all open sets U in X.

Note that the homology sequence (of sheaves) of

$$0 \to {}_p\mathscr{I}^* \to \mathscr{I}^* \xrightarrow{p} \mathscr{I}^* \to 0$$

shows that the differential sheaf $\mathscr{J}^*$, where $\mathscr{J}^n = {}_p\mathscr{I}^{n+1}$, is a resolution of L_p (since ${}_p\mathscr{I}^0 = 0$).

Let us fix the open set $U \subset X$ for the moment and use the abbreviation $I^n = \Gamma_c(\mathscr{I}^n|U)$ and $J^n = {}_pI^{n+1} = \Gamma_c({}_p\mathscr{I}^{n+1}|U)$; see (114). Then the sequence

$$0 \to J^{n-1} \to I^n \xrightarrow{p} I^n \to 0 \tag{115}$$

is exact for each n.

Let Q denote the field of quotients of L and note that we may take $L^0 = Q$ and $L^1 = Q/L$ [see 2.7]. Similarly, we take $(L_p)^0 = L_p$ and $(L_p)^1 = 0$. Then there is an exact sequence

$$0 \to L_p \to Q \oplus (Q/L) \xrightarrow{p} Q \oplus (Q/L) \to 0.$$

Applying this to the functor $\operatorname{Hom}(I^*, \bullet)$ and using the fact that $A \otimes L_p \approx \operatorname{Coker}(p : A \to A)$ for an L-module A, we obtain the isomorphism

$$\operatorname{Hom}(I^*, L^*)_n \otimes L_p \approx \operatorname{Ext}(I^*, L_p)_{n+1}. \tag{116}$$

(The subscripts $n, n+1$ indicate total degree.) The sequences (115) also show that

$$\operatorname{Ext}(I^*, L_p)_{n+1} \approx \operatorname{Hom}(J^*, L_p)_n \tag{117}$$

since $p : \operatorname{Ext}(I^*, L_p) \to \operatorname{Ext}(I^*, L_p)$ is trivial and $\operatorname{Hom}(I^*, L_p) = 0$ (since I^* is divisible). (Also note that $\operatorname{Hom}_L = \operatorname{Hom}_{L_p}$ and $\otimes_L = \otimes_{L_p}$ for L_p-modules so that we need not affix these subscripts.)

The isomorphisms (116) and (117) are clearly natural in U. Thus they induce an isomorphism of sheaves of L_p-modules:

$${}^L\mathscr{D}(\Gamma_c\mathscr{I}^*; L) \otimes L_p \approx {}^{L_p}\mathscr{D}(\Gamma_c\mathscr{J}^*; L_p) \tag{118}$$

[where we are using L^* and $(L_p)^*$ as defined above; see 2.7]. The isomorphism (118) may be tensored over L_p with any sheaf $\mathscr{B}$ of L_p-modules, and we may take sections with supports in any family Φ. Since $\mathscr{I}^*$ is an L-injective resolution of L and $\mathscr{J}^*$ is an L_p-injective resolution of L_p, the theorem follows upon passage to homology. □

15.4. It is possible to define an alternative homology theory that does satisfy change of rings provided one is willing to sacrifice other desirable properties. For simplicity we shall confine the discussion to *compact* spaces and supports in this discussion. For an L-module M, define

$$\boxed{\overline{H}_p(X; M) = H_p(\Gamma(\mathscr{D}(\mathscr{I}^*; M))).}$$

This does satisfy change of rings because[29]

$$\begin{aligned}
\overline{H}_p(X; M) &\approx H_p(\Gamma(\mathscr{D}(\mathscr{C}^*(X;\mathbb{Z}) \otimes L; M))) && \text{2.4 and (9)}\\
&\approx H_p(\operatorname{Hom}^L(C^*(X;\mathbb{Z}) \otimes L; {}^LM^*)) && \text{II-15.5(b)}\\
&\approx H_p(\operatorname{Hom}^L(F^*C^*(X;\mathbb{Z}) \otimes L; M)) && \text{hyperhomology}\\
&\approx H_p(\operatorname{Hom}^{\mathbb{Z}}(F^*C^*(X;\mathbb{Z}); M)) && \text{algebra}\\
&\approx H_p(\operatorname{Hom}^{\mathbb{Z}}(C^*(X;\mathbb{Z}); {}^{\mathbb{Z}}M^*)) && \text{hyperhomology,}
\end{aligned}$$

where F^* denotes taking a free resolution over $\mathbb{Z}$. However, this theory has two weaknesses: it does not extend[30] to general coefficient sheaves, and it does not satisfy a universal coefficient formula such as 3.13. Indeed, we have (roughly stated):

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[29] The references are to *similar* items.

[30] As far as I know.

15.5. Theorem. *There exists no "homology theory" ${}^LH_*(X;M)$ defined for compact X and L-modules M that satisfies all three of the following conditions:*

(a) *There is an exact sequence (with sheaf cohomology)*

$$0 \to \operatorname{Ext}^L(H^{p+1}(X;L),L) \to {}^LH_p(X;L) \to \operatorname{Hom}^L(H^p(X;L),L) \to 0.$$

(b) *There is an exact sequence*

$$0 \to {}^LH_p(X;L)\otimes_L M \to {}^LH_p(X;M) \to {}^LH_{p-1}(X;L) *_L M \to 0.$$

(c) *Change of rings: ${}^LH_p(X;M) \approx {}^MH_p(X;M)$ when M is an L-module that is also a principal ideal domain.*

Also, there is no theory satisfying (a′) *and* (b) *where* (a′) *is the stronger condition:*

(a′) *There is an exact sequence*

$$\operatorname{Ext}^L(H^{p+1}(X;L),M) \rightarrowtail {}^LH_p(X;M) \twoheadrightarrow \operatorname{Hom}^L(H^p(X;L),M).$$

Proof. Assume that $H_*(\bullet\,;M)$ satisfies all three conditions. (The superscript L can be omitted because of (c).) Let X be the solenoid of 14.5. Then $H^0(X;\mathbb{Z}) \approx \mathbb{Z}$, $H^1(X;\mathbb{Z}) \approx \mathbb{Q}$ and $H^1(X;\mathbb{Q}) \approx \mathbb{Q}$ by II-15.3. By (a) we have $H_1(X;\mathbb{Z}) \approx \operatorname{Hom}^{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$. From (b) we have $H_1(X;\mathbb{Q}) \approx H_1(X;\mathbb{Z})\otimes\mathbb{Q} = 0\otimes\mathbb{Q} = 0$. However, from (a) we have $H_1(X;\mathbb{Q}) \approx \operatorname{Hom}^{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) \approx \mathbb{Q}$, a contradiction.

For the second statement let $L = \mathbb{Z}$. Then by (b) we have

$${}^{\mathbb{Z}}H_0(X;\mathbb{Q}) \approx {}^{\mathbb{Z}}H_0(X;\mathbb{Z})\otimes\mathbb{Q} \overset{(a)}{\approx} \operatorname{Ext}(\mathbb{Q},\mathbb{Z})\otimes\mathbb{Q}\oplus\mathbb{Q},$$

which is a rational vector space of uncountable dimension. But from (a′) we have

$${}^{\mathbb{Z}}H_0(X;\mathbb{Q}) \approx \operatorname{Ext}^{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})\oplus\operatorname{Hom}^{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}) \approx 0\oplus\mathbb{Q}\approx\mathbb{Q},$$

a contradiction. □

Note that Borel-Moore theory satisfies (a) and (b); singular theory satisfies (b) and (c); and the $\overline{H}_*(X;M)$ theory satisfies (a′) and (c).

In a similar vein we have the following result showing that on compact spaces, sheaf-theoretic cohomology is not a "co" theory to any homology theory with integer coefficients.

15.6. Theorem. *There do not exist functors $H_*(X)$ of compact spaces X that provide an exact universal coefficient sequence of the form*

$$0 \to \operatorname{Ext}(H_{k-1}(X),M) \to H^k(X;M) \to \operatorname{Hom}(H_k(X),M) \to 0$$

for abelian groups M (with sheaf-theoretic cohomology).

Proof. Consider again the solenoid X, and note that $H^1(X;\mathbb{Z}_p) = 0$ for all primes p. Then the sequence implies that $\operatorname{Hom}(H_1(X),\mathbb{Z}_p) = 0$, whence also $\operatorname{Hom}(H_1(X),\mathbb{Z}) = 0$. Similarly, since $H^2(X;\mathbb{Z}) = 0$, the sequence implies that $\operatorname{Ext}(H_1(X),\mathbb{Z}) = 0$. By the result 14.7 of Nunke we have that $H_1(X) = 0$. However, the sequence for $M = \mathbb{Q}$ shows that $\mathbb{Q} \approx H^1(X;\mathbb{Q}) \approx \operatorname{Hom}(H_1(X),\mathbb{Q}) = 0$, a contradiction. □

Even easier arguments of dimensionality give a similar result over the rationals as base ring.

It is of interest to ask under what conditions the $\overline{H}_*$ theory coincides with H_*. In VI-11 it is shown that $\overline{H}_*^c(X;M) \approx H_*^c(X;M)$ when X is clc_L^∞, but we do not know whether or not this extends to the case of general paracompactifying supports. See VI-11 for further remarks on this.

16 Generalized manifolds

In this section we shall study various conditions on a locally compact space X that are equivalent to X being an n-hm_L. We also prove a number of results about these spaces.

First suppose that the homology sheaves $\mathscr{H}_p(X;L)$ are all locally constant. We shall show that under suitable conditions this implies that X is an n-hm_L for some n. We first consider the case in which L is a field:

16.1. Theorem. *If L is a field and X is a connected and locally connected space for which $\dim_L X = n < \infty$ and such that $\mathscr{H}_p(X;L)$ is locally constant for all p, then X is an n-hm_L.*

Proof. Let A_p denote the stalk of $\mathscr{H}_p(X;L)$. We must show that $A_n \approx L$ and that $A_p = 0$ for all $p \neq n$. Let $r = \max\{p \mid A_p \neq 0\}$ and $s = \min\{p \mid A_p \neq 0\}$. Consider the spectral sequence of 8.4:

$$E_2^{p,q} = H_c^p(U;\mathscr{H}_{-q}(X;L)) \Longrightarrow H^c_{-p-q}(U;L)$$

(for U open in X). By passing to a small open subset we may as well assume that $\mathscr{H}_*(X;L)$ is constant. Then $E_2^{p,q} = 0$ for $p > n$ or $q > -s$, and hence $E_2^{n,-s} = H_c^n(U;A_s) \approx H_c^n(U;L) \otimes A_s$ is isomorphic to $H^c_{s-n}(U;L)$. This must vanish for $n > s$. By II-16.14, $H_c^n(U;A_s) \neq 0$ for some U. Hence $n \leq s$. Also, (11) on page 293 shows that $r \leq n$. Thus $n = r = s$. We now have that

$$H_0^c(U;L) \approx H_c^n(U;L) \otimes A_n.$$

Taking U to be connected, we have that $H_0^c(U;L) \approx L$ by 5.14, and hence $A_n \approx L$. □

We now take up the general case.

16.2. Theorem. *Let X be a connected and locally connected space for which $\dim_L X = n < \infty$ and such that $\mathscr{H}_p(X;L)$ is locally constant and has finitely generated stalks for all p. Then X is an n-hm_L.*

Proof. For the field $K = L/pL$ where p is a prime of L we have the exact sequence

$$0 \to \mathscr{H}_k(X;L) \otimes K \to \mathscr{H}_k(X;K) \to \mathscr{H}_{k-1}(X;L) * K \to 0$$

of (13) on page 294. By I-Exercise 9, $\mathscr{H}_k(X;K)$ is also locally constant, and by 15.2 the hypotheses of 16.1 are satisfied by K in place of L. If A_k is the stalk of $\mathscr{H}_k(X;L)$, which is finitely generated by assumption, then A_k can have no p-torsion because otherwise $\mathscr{H}_k(X;K)$ and $\mathscr{H}_{k+1}(X;K)$ would both be nonzero. Since this is true for all primes p of L, A_k is free over L, and it follows that $A_m \approx L$ for some m and $A_k = 0$ for $k \neq m$. Thus X is an m-hm_L. From 9.6 it follows that $m = n$. □

16.3. Theorem. *Let X be a connected clc_L^∞ space for which $\dim_L X = n < \infty$ and such that $\mathscr{H}_p(X;L)$ is locally constant for each p. Then X is an n-hm_L.*

Proof. Let m be the largest integer such that $\mathscr{H}_m(X;L) \neq 0$. By 8.7, $H_m(U;L)$ can be identified with $\Gamma(\mathscr{H}_m(X;L)|U)$ for U open and paracompact. For $V \subset U$, the canonical homomorphism $H_m(U) \to H_m(V)$ corresponds to section restriction. Since $\mathscr{H}_m(X)$ is locally constant, there is an open set U over which it is constant. We may take U to be paracompact and connected, and thus $\Gamma(\mathscr{H}_m(X)|U)$ is isomorphic to the stalk. Let V be another connected paracompact open set with compact closure in U. Then $\Gamma(\mathscr{H}_m(X)|U) \to \Gamma(\mathscr{H}_m(X)|V)$ is an isomorphism. But $H_m(U) \to H_m(V)$ has finitely generated image by II-17.5 and II-17.3, since X is clc_L^∞. Thus $\mathscr{H}_m(X;L)$ has finitely generated stalks. For any prime p of L, X is an hm_{L_p}, and this implies that $\mathscr{H}_m(X;L)$ has no p-torsion; see the proof of 16.2. Similarly, the stalks of $\mathscr{H}_m(X;L)$ must be free of rank 1. If $M = \mathscr{H}_s(X;L)$, $s \neq m$, then we must have, for the same reason, $M \otimes K = 0 = M * K$ for any L-module K that is a field; see 15.1. This implies that $M = 0$, whence X is an m-hm_L and $m = n$. □

16.4. Definition. *Precosheaves $\mathfrak{A}$ and $\mathfrak{B}$ on X are said to be "equivalent" if $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent under the smallest equivalence relation containing the relation of local isomorphism of* 12.2 *(see Exercise* 5*).*

16.5. Definition. *A precosheaf $\mathfrak{A}$ will be said to be "locally constant" if each point $x \in X$ has a neighborhood U such that the precosheaf $\mathfrak{A}|U$ on U [i.e., $V \mapsto \mathfrak{A}(V)$ for $V \subset U$] is equivalent to a constant precosheaf. If this is the constant precosheaf M, where M is an L-module, then $\mathfrak{A}$ is said to be "locally equivalent to M."*

16.6. Definition. *The space X will be said to possess "locally constant cohomology modules over L locally equivalent to M^*," where M^* is a graded L-module, if the precosheaf $\mathfrak{H}_c^p(X;L) : U \mapsto H_c^p(U;L)$ is locally equivalent to M^p for all p.*

16.7. Definition. *The space X is called an "n-dimensional cohomology manifold over L" (denoted n-cm_L) if X has locally constant cohomology modules, locally equivalent to L in degree n, and to zero in degrees other than n, and if* $\dim_L X < \infty$.

16.8. Theorem. *If X is connected, then the following four conditions are equivalent:*

(a) *X is an n-cm_L.*

(b) [27] $\dim_L X = n < \infty$ *and X has locally constant cohomology modules locally equivalent to M^*, where M^* is finitely generated.*

(c) *X is clc_L^∞ and is an n-hm_L.*

(d) [20], [14] *X is clc_L^∞, $\dim_L X < \infty$, and the stalks of $\mathscr{H}_i(X;L)$ are zero for $i \neq n$ and are isomorphic to L for $i = n$.*[31]

Proof. Using II-16.17, (a) $\Rightarrow$ (b) is clear. Assume that (b) holds. Then it follows from II-17.5 that X is clc_L^∞. By 12.3 we see that the sheaves $\mathscr{H}om(\mathfrak{H}_c^p(X;L),L)$ and $\mathscr{E}xt(\mathfrak{H}_c^p(X;L),L)$ are locally constant with stalks $\operatorname{Hom}(M^p,L)$ and $\operatorname{Ext}(M^p,L)$, respectively (which are finitely generated). By (9) on page 292 we have the exact sequence of sheaves:

$$0 \to \mathscr{E}xt(\mathfrak{H}_c^{p+1}(X;L),L) \to \mathscr{H}_p(X;L) \to \mathscr{H}om(\mathfrak{H}_c^p(X;L),L) \to 0,$$

and it follows from I-Exercise 9 that $\mathscr{H}_p(X;L)$ is locally constant with finitely generated stalks. Thus X is an n-hm_L by 16.2.

Now suppose that condition (c) holds. Using the definition of clc_L^∞, II-17.3, and the exact sequences

$$0 \to \operatorname{Ext}(H^{p+1}(K;L),L) \to H_p(K;L) \to \operatorname{Hom}(H^p(K;L),L) \to 0$$

for K compact in X, we see that for any compact neighborhood K of a point $x \in X$ there is a compact neighborhood $K' \subset K$ of x with $H_p(K') \to H_p(K)$ trivial for $p \neq 0$. It follows that for every neighborhood U of x there is a neighborhood $V \subset U$ of x with $H_p^c(V) \to H_p^c(U)$ trivial for $p \neq 0$. By Poincaré duality (assuming, as we may, that $\mathscr{O}$ is *constant*), $H_c^q(V) \to H_c^q(U)$ is trivial for $q \neq n$. Also, for any open set U, $H_c^n(U;L) \approx H_0^c(U;L)$ is the free L-module on the components of U by 5.14. Thus $\mathfrak{H}_c^q(X;L)$ is locally equivalent to L for $q = n$, and to zero for $q \neq n$, which is condition (a).

Trivially, (c) implies (d). Assuming (d), to prove (c) we must show that X is "locally orientable," meaning that $\mathscr{H}_n(X;L)$ is locally constant. Since this is a local matter and every point in X has an arbitrarily small

[31] The implication (d) $\Rightarrow$ X is an n-hm_L requires only the assumption that X is locally connected rather than the full clc_L condition. This implication is the "Wilder local orientability (former) conjecture." Also see 16.15.

open paracompact neighborhood (the union of an increasing sequence of compact neighborhoods), we may as well assume that X is connected and paracompact. With $\mathscr{O} = \mathscr{H}_n(X;L)$, suppose that L is a *field* and that $\sigma \in \Gamma(\mathscr{O})$ is nonzero. We claim that σ must be everywhere nonzero. Let $A = |\sigma|$ and suppose that $A \neq X$. Since L is a field, the section σ induces a homomorphism $L \to \mathscr{O}$ that is an isomorphism over A. Therefore

$$H_c^n(A;L) \approx H_c^n(A;\mathscr{O}|A) \approx H_0^c(X,X-A;L) = 0$$

by 9.2 and the exact sequence

$$H_0^c(X-A;L) \to H_0^c(X;L) \to H_0^c(X,X-A;L) \to 0,$$

the first map of which is onto. The exact sequence

$$H_c^n(X-A;L) \xrightarrow{j^*} H_c^n(X;L) \to H_c^n(A;L) = 0$$

shows that j^* is onto. Now $H_c^n(X-A;L) \approx \varinjlim H_c^n(U;L)$, where U ranges over the open paracompact sets with compact closure in $X-A$, by II-14.5 applied to $L \approx \varinjlim L_U$.[32] Since $H_n(U;L) \approx \operatorname{Hom}(H_c^n(U;L),L)$, naturally in U, by (9) on page 292, we deduce that

$$j_* : H_n(X;L) \to \varprojlim H_n(U;L)$$

is a *monomorphism*. But by Poincaré duality 9.2 or 8.7, j_* can be identified with section restriction

$$\Gamma(\mathscr{O}) = H^0(X;\mathscr{O}) \to \varprojlim H^0(U;\mathscr{O}|U) = \varprojlim \Gamma(\mathscr{O}|U) = \Gamma(\mathscr{O}|X-A).$$

Since $\sigma \in \Gamma(\mathscr{O})$ restricts to zero on $X-A$, this is a contradiction. Thus σ must be everywhere nonzero as claimed.

Now, in the general case where L is a principal ideal domain, which may not be a field, note that if p is a prime in L then X satisfies (d) over the field $K = L/pL$ with orientation sheaf $\mathscr{O} \otimes K$ by 15.2. Let $x \in X$ and let σ be a local section of $\mathscr{O}$ near x that gives a generator of the stalk at x. Since x has a neighborhood basis at x consisting of open paracompact sets, we may assume that σ is defined on a connected paracompact open neighborhood U of x. Then σ defines a homomorphism of the constant sheaf L on U to $\mathscr{O}|U$ that is an isomorphism on the stalks at x. By the case of a field applied to $K = L/pL$, the map $L/pL \to (\mathscr{O} \otimes L/pL)|U$ is an *isomorphism* for each prime p of L. But a homomorphism $L \to L$ of L-modules that induces an isomorphism $L/pL \to L/pL$ for all primes p of L is necessarily an isomorphism. Therefore the homomorphism $L \to \mathscr{O}|U$ of sheaves over U is an isomorphism on each stalk and hence is an isomorphism of sheaves. Thus $\mathscr{O}$ is constant over U, proving (c). □

For the case in which $L = \mathbb{Z}$ or is a field, we have the following simple cohomological criterion:

[32] This is to avoid a paracompactness assumption on $X - A$.

16.9. Corollary. [20] *If L is the integers or a field, then a locally compact space X is an n-cm_L $\Leftrightarrow$ X is clc_L^∞, $\dim_L X < \infty$, and*

$$H^p(X, X - \{x\}; L) \approx \begin{cases} L, & \textit{for } p = n, \\ 0, & \textit{for } p \neq n \end{cases}$$

for all $x \in X$.

Proof. By 7.2 applied to the identity map we have

$$\mathscr{H}_i(X; L)_x \approx H_i^c(X, X - \{x\}; L).$$

Thus from 13.7 we have the exact sequence

$$\operatorname{Ext}(\mathscr{H}_{i-1}(X; L)_x, L) \rightarrowtail H^i(X, X - \{x\}; L) \twoheadrightarrow \operatorname{Hom}(\mathscr{H}_i(X; L)_x, L).$$

It follows from this and 14.7 that $H^*(X, X - \{x\}; L)$ is of finite type $\Leftrightarrow$ $\mathscr{H}_*(X; L)_x$ is of finite type. Hence the condition in the corollary is equivalent to

$$\mathscr{H}_p(X; L)_x \approx \begin{cases} L, & \text{for } p = n, \\ 0, & \text{for } p \neq n \end{cases}$$

for all $x \in X$. But this is precisely condition (d) of 16.8 (given the other conditions). □

16.10. Example. Let C be the open cone on $\mathbb{RP}^2$, let E be the open ball of radius 1 in $\mathbb{R}^3$, and let X be the one point union $C \vee E$ at the vertex x_0 of the cone. Then if $\overline{C}$ denotes the closed cone on $\mathbb{RP}^2$ and $\overline{E} = \mathbb{D}^3$, we have

$$\mathscr{H}_n(X; \mathbb{Z})_{x_0} \approx H_n(\overline{C} \vee \overline{E}, \mathbb{RP}^2 + \mathbb{S}^2) \approx \tilde{H}_{n-1}(\mathbb{RP}^2 + \mathbb{S}^2) \approx \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}_2, & n = 2, \\ \mathbb{Z}, & n = 3 \end{cases}$$

and is otherwise zero. Outside x_0, of course, X is a 3-manifold. The sheaf $\mathscr{H}_3(X; \mathbb{Z})$ is isomorphic to $\mathscr{O}^X \oplus \mathbb{Z}_E$, where $\mathscr{O}$ is the orientation sheaf of $C - \{x_0\}$. Thus $\mathscr{H}_3(X; \mathbb{Z})$ has stalks $\mathbb{Z}$ everywhere but is not locally constant. The sheaves $\mathscr{H}_1$ and $\mathscr{H}_2$ are, of course, concentrated at x_0. Note that with coefficients in a field K of characteristic other than two, we have

$$\mathscr{H}_n(X; K)_{x_0} \approx \begin{cases} K, & \text{for } n = 1, 3, \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$\mathscr{H}_3(X; K) \approx (\mathscr{O}^X \otimes K) \oplus K_E$$

has stalks K everywhere but is not locally constant, and the homology sheaves in the adjacent dimensions 2 and 4 are zero.

This shows that the local orientability theorem, (d) $\Rightarrow$ (a) of 16.8, does not generalize in any obvious way to spaces in which the other homology sheaves are not all zero. There are many variations on this example. ◇

The following result is responsible for much of the interest in generalized manifolds, because there are spaces X that are not manifolds but for which $X \times \mathbb{R}$ is a manifold. Factorizations of manifolds of this form are important in the theory of topological transformation groups on manifolds.

16.11. Theorem. [6] *The locally compact space $X \times Y$ is an n-cm_L $\Leftrightarrow$ X is a p-cm_L and Y is a q-cm_L for some p, q with $n = p + q$. Moreover, the orientation sheaves satisfy the equation $\mathscr{O}_{X\times Y} \approx \mathscr{O}_X \widehat{\otimes} \mathscr{O}_Y$ (with the obvious notation).*

Proof. It is not hard to prove that $X \times Y$ is clc_L^∞ $\Leftrightarrow$ both X and Y are clc_L^∞. The proof of this follows from the Künneth formula II-15.2 applied to compact product neighborhoods of a point in $X \times Y$, and will be left to the reader. The $\Leftarrow$ part of the theorem is also easy, so we shall prove only the $\Rightarrow$ part.

Let $x \in X$ and $y \in Y$ and let $M_p = \mathscr{H}_p(X; L)_x$ and $N_q = \mathscr{H}_q(Y; L)_y$. By 14.3 we have the exact sequence

$$0 \to \bigoplus_p M_p \otimes N_{n-p} \to L \to \bigoplus_p M_p * N_{n-p-1} \to 0,$$

and we also have that

$$M_p \otimes N_q = 0 \quad \text{for} \quad p + q \neq n$$

and

$$M_p * N_q = 0 \quad \text{for} \quad p + q \neq n - 1.$$

Clearly, we must have $M_p \otimes N_{n-p} \approx L$ for some p (which we now fix).

Let $b \in N_{n-p}$ be such that $a \otimes b \neq 0 \in M_p \otimes N_{n-p}$ for some $a \in M_p$. The map $M_p \to M_p \otimes N_{n-p} \approx L$ defined by $a \mapsto a \otimes b$ has a nontrivial ideal of L as its image. Since L is a principal ideal domain, this implies that M_p has L as a direct summand. Similarly, N_{n-p} has L as a direct summand. The fact that $M_p \otimes N_{n-p} \approx L$ implies that $M_p \approx L$ and $N_{n-p} \approx L$. Since $M_r \otimes N_s = 0$ for $r + s \neq n$, we see that $M_r = 0$ for $r \neq p$ and $N_s = 0$ for $s \neq n - p$.

Now p depends on x, but $n - p$ depends only on $y \in Y$. Thus p is, in fact, constant. By 14.3

$$\mathscr{H}_p(X) \widehat{\otimes} \mathscr{H}_{n-p}(Y) \approx \mathscr{H}_n(X \times Y).$$

Thus

$$\mathscr{H}_p(X) \approx \mathscr{H}_p(X) \widehat{\otimes} (\mathscr{H}_{n-p}(Y)|\{y\}) \approx \mathscr{H}_n(X \times Y)|X \times \{y\}$$

is locally constant (with stalks L). Similarly, $\mathscr{H}_{n-p}(Y)$ is locally constant. (Of course, local constancy also follows from 16.8(d).) □

Now we shall consider further weakenings of the condition (d) of 16.8. In particular, we shall show that the condition clc_L^∞ can be dropped if L is a field and X is first countable.[33] First we need to discuss some technical items about "local Betti numbers."

We define, for L a *field*, the "ith local Betti number of X at x" to be

$$\boxed{b_i(x) = \dim_L \mathscr{H}_i(X;L)_x = \dim_L H_i(X, X - \{x\}; L).}$$

The following lemma is an elaboration of a method due to J. H. C. Whitehead [82] and Wilder [84].

16.12. Lemma. [14] *Let $V_1 \leftarrow V_2 \leftarrow V_3 \leftarrow \cdots$ be an inverse sequence of vector spaces over the field L. Put $V_i^* = \operatorname{Hom}(V_i, L)$ and $W = \varinjlim V_i^*$. Then $\dim_L W$ is countable $\Leftrightarrow$ for each i there exists an index $j > i$ such that $\dim_L \operatorname{Im}(V_j \to V_i) < \infty$.*

Proof. Put $G_i = \operatorname{Im}(V_j \to V_1)$. It suffices to show that if each G_i has infinite dimension then W has uncountable dimension. Now $G_1 \supset G_2 \supset \cdots$ Suppose there is a j with $G_j = G_{j+1} = \cdots$ Then for $i > j$, $V_i \to G_i = G_j$ is onto, and hence $G_j^* \to V_i^*$ is injective. Therefore $G_j^* \to W$ is injective, showing that W has uncountable dimension.

If, on the contrary, the G_i are not eventually constant, we may assume that they decrease strictly. Then let $g_i \in G_i - G_{i+1} \subset V_1$. The g_i are independent and hence are a basis of a subspace H of V_1. Let $V_1 \to H$ be a projection, and consider the dual $H^* \to V_1^* \to W$. If $u \in H^*$ maps to zero in W, then $u(g_i) = 0$ except for finitely many i. Since such elements u form a countable-dimensional subspace of H^*, we deduce that the image of $H^* \to W$, and hence W itself, has uncountable dimension. The converse is clear. □

16.13. Theorem. [14] *Let L be a field. If the locally compact space X is first countable and has $\dim_L X < \infty$, then $b_i(x)$ is countable for all i and all $x \in X \Leftrightarrow X$ is clc_L.*[34]

Proof. Let $U_1 \supset U_2 \supset \cdots$ be a countable neighborhood basis at x. Then $H_p(U_i) \approx \operatorname{Hom}(H_c^p(U_i), L)$ by (9) on page 292. By 16.12, $\mathscr{H}_p(X)_x = \varinjlim H_p(U_i)$ has countable dimension $\Leftrightarrow$ for each i there exists a $j > i$ with $\dim_L\{\operatorname{Im}(H_c^p(U_j) \to H_c^p(U_i)\} < \infty$. Since $\dim_L X < \infty$, this is equivalent to the condition clc_L by II-17.5. □

An old problem of Alexandroff [1] asks whether a finite-dimensional space that has constant finite local Betti numbers must be a manifold. Since *hm*'s satisfy this, it is false; but it is reasonable to ask whether such spaces must be homology manifolds. We shall show that this is, indeed, the

[33] We do not know whether these conditions are needed.

[34] See [60] for a generalization to countable principal ideal domains L.

case under some mild restrictions. The original, slightly weaker, versions of these results can be found in [14].

16.14. Lemma. [14] *Let L be a field. Suppose that X is second countable and that $\dim_L X < \infty$. For each i assume that $b_i(x)$ is finite and independent of x. Then X is an n-cm_L for some n.*

Proof. By 16.13, X is clc_L^∞. By IV-4.12, there is an open set U such that $\mathscr{H}_i(X;L)$ is constant over U with finitely generated stalks. By 16.8(b), U is an n-cm_L for some n. Thus $b_n(x) = 1$ and $b_i(x) = 0$ for $i \neq n$ for all $x \in X$. The result now follows from 16.8(d). □

16.15. Theorem. [14] *Let X be second countable with $\dim_L X < \infty$, and assume that the stalks $\mathscr{H}_i(X;L)_x$ are finitely generated and mutually isomorphic (i.e., independent of x) for each i. Then X is an n-hm_L for some n.*

Proof. (If we assume that X is clc_L^∞, then there is a direct proof out of IV-4.12.) Let p be a prime of L. Then $L_p = L/pL$ is a field. Note that X is $clc_{L_p}^\infty$ by 16.13 and change of rings 15.2. Let A_i be an isomorph of $\mathscr{H}_i(X;L)_x$. There is the universal coefficient sequence

$$0 \to A_i \otimes L_p \to \mathscr{H}_i(X;L_p)_x \to A_{i-1} * L_p \to 0$$

of (13) on page 294. If A_i contains a factor L_{p^r}, then $\mathscr{H}_i(X;L_p)_x$ and $\mathscr{H}_{i+1}(X;L_p)_x$ are both nonzero, contrary to 16.14. Thus A_i is free. Similarly, it follows that at most one A_i, say A_n, is nonzero, and this one has rank one. Since X is $clc_{L_p}^\infty$, it is locally connected. Thus by 16.8(d) and its footnote, X is an n-hm_L. □

It has been shown in [43] and [60] that a first-countable n-$hm_{\mathbb{Z}}$ is necessarily $clc_{\mathbb{Z}}$ and hence is an n-$cm_{\mathbb{Z}}$. Thus, for $L = \mathbb{Z}$ or a field, the conclusion of 16.15 can be sharpened to say that X is an n-$cm_{\mathbb{Z}}$. [No example of an n-hm_L that is not clc_L is known to the author.]

16.16. Theorem. *Let X be a connected n-cm_L, let $F \subsetneq X$ be a proper closed subset, let $U \subset X$ be any nonempty open subset, and let $\mathscr{M}$ be an elementary sheaf on X. Then:*

(a) *If $\mathscr{M}$ has stalks L, then $H_n(F;\mathscr{M}) = 0$ and $H_{n-1}(F;\mathscr{M})$ is torsion-free;*

(b) *$H_c^n(U;\mathscr{O})$ is the free L-module on the components of U;*

(c) *$H_c^n(F;\mathscr{M}) = 0$;*

(d) *$j_{X,U}^* : H_c^n(U;\mathscr{M}) \to H_c^n(X;\mathscr{M})$ is surjective;*

(e) *$j_{X,U}^* : H_c^n(U;\mathscr{O}) \to H_c^n(X;\mathscr{O})$ is an isomorphism if U is connected;*

(f) *X is orientable* $\Leftrightarrow H_c^n(X;L)\approx L$.

Moreover, if $L=\mathbb{Z}$ *and* $\mathscr{T}$ *is a locally constant sheaf on* X *with stalks* $\mathbb{Z}$, *then:*

(g) $H_c^n(X;\mathscr{T})\approx\mathbb{Z}_2$ *unless* $\mathscr{T}\approx\mathscr{O}$;

(h) $H_c^n(U;\mathscr{T})\approx\bigoplus_\alpha\mathbb{Z}\oplus\bigoplus_\beta\mathbb{Z}_2$, *where* α *ranges over the components* V *of* U *for which* $\mathscr{T}|V\approx\mathscr{O}|V$ *and* β *ranges over the remaining components;*

(i) *The torsion subgroup of* $H_{n-1}(X;\mathscr{T}^{-1})$ *is zero if* $\mathscr{T}\approx\mathscr{O}$ *and is* $\mathbb{Z}_2$ *otherwise.*

Proof. The case $n=0$ is trivial, so assume that $n>0$. Assuming part (c) for the moment, we have that $H_n(F;\mathscr{M})\approx\operatorname{Hom}(H_c^n(F;\mathscr{M}^{-1}),L)=0$ by 10.5, giving the first part of (a). The last part of (a) follows from the exact sequence (3.12) on page 297 induced by the coefficient sequence $0\to\mathscr{M}\xrightarrow{p}\mathscr{M}\to\mathscr{M}\otimes L_p\to 0$, where p is a prime in L; from (a) in the case of the field L_p as base ring; and from change of rings 15.2.

From 9.2 we have that $H_c^n(U;\mathscr{O})\approx H_0^c(U;L)$, whence (b) follows from 5.14. Also, if $U=X-F$, then there is the exact sequence

$$H_c^n(U;\mathscr{O})\to H_c^n(X;\mathscr{O})\to H_c^n(F;\mathscr{O})\to H_c^{n+1}(U;\mathscr{O})=0,$$

since $\dim_L X=n$, whence (b) implies that $H_c^n(F;\mathscr{O})=0$. Therefore $H_c^n(F;L)=0$ provided that $\mathscr{O}|F$ is constant, hence for F sufficiently small. The cohomology sequence of the coefficient sequence $0\to L\to L\to L/pL\to 0$, for $p\in L$ a prime, shows that $H_c^n(F;L/pL)=0$ for F small. It follows that $H_c^n(F;\mathscr{M})=0$ for F small (so small that $\mathscr{M}|F$ and $\mathscr{O}|F$ are both constant). Now suppose that $0\neq a\in H_c^n(F;\mathscr{M})$. By II-10.8 there is a closed subset $C\subset F$ with $a|C\neq 0\in H_c^n(C;\mathscr{M})$ but with $a|C'=0$ for all proper subsets $C'\subset C$. Now C cannot be a singleton since $n>0$. Let $x\in C$ and let U be a connected open neighborhood of x such that $\mathscr{O}$ and $\mathscr{M}$ are both constant on U. Then the exact sequence

$$H_c^n(U\cap C;\mathscr{M})\xrightarrow{j^*}H_c^n(C;\mathscr{M})\to H_c^n(C-U;\mathscr{M})$$

shows that $a\in\operatorname{Im}j^*$. But $H_c^n(U\cap C;\mathscr{M})=0$, provided that $U\cap C\neq U$, by the part of (c) already proven, since $\mathscr{M}$ is constant on $U\cap C$ and $U\cap C$ is a proper closed subset of U. Thus we must have that $U\cap C=U$, and so x is an interior point of C. Hence $C=X$ since C is open and closed and X is connected. Thus F is not proper, contrary to assumption, proving (c).

Part (d) follows from the cohomology sequence of (X,U) and part (c) with $F=X-U$. Part (e) holds because an epimorphism $L\twoheadrightarrow L$ must be an isomorphism since L is a principal ideal domain. For (f), suppose that $H_c^n(X;L)\approx L$. Let $V\subset X$ be connected, open, and orientable. Then

by (e), $H^n_c(V;L) \to H^n_c(X;L)$ is surjective and hence is an isomorphism. From this and the natural isomorphisms

$$\mathscr{O}(U) \approx H^0(U;\mathscr{O}) \approx H_n(U;L) \approx \operatorname{Hom}(H^n_c(U;L),L)$$

we deduce that the restriction $\mathscr{O}(X) \to \mathscr{O}(V)$ is an isomorphism, whence X is orientable, giving (f).

For part (g), note that $H^n_c(X;\mathscr{T})$ is a quotient group of $\mathbb{Z}$ by (b) and (d), and so it has the form $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for some m, possibly 0. Since the union of an increasing sequence of compact subsets of X is paracompact, $H^n_c(X;\mathscr{T}) \approx \varinjlim_V H^n_c(V;\mathscr{T})$, where V ranges over the connected, paracompact, and relatively compact subspaces of X. This shows, using (b) and (d), that we may assume that X is paracompact in the proof of (g). The exact sequence

$$0 \to \operatorname{Ext}(H^n_c(X;\mathscr{T}),\mathbb{Z}) \to H_{n-1}(X;\mathscr{T}^{-1}) \to \operatorname{Hom}(H^{n-1}_c(X;\mathscr{T}),\mathbb{Z}) \to 0$$

of (65) on page 340 shows that the torsion subgroup of $H_{n-1}(X;\mathscr{T}^{-1})$ is $\mathbb{Z}_m$. This sequence also gives (i), assuming (b) and (g). From duality 9.2 we have

$$H_n(X;\mathscr{T}^{-1}) \approx H^0(X;\mathscr{O}\otimes\mathscr{T}^{-1}) = \Gamma(\mathscr{O}\otimes\mathscr{T}^{-1}) = 0,$$

and similarly, for any integer $k > 1$,

$$H_n(X;\mathscr{T}^{-1}\otimes\mathbb{Z}_k) \approx \Gamma(\mathscr{O}\otimes\mathscr{T}^{-1}\otimes\mathbb{Z}_k) \approx \begin{cases} \mathbb{Z}_2, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

because $\mathscr{O}\otimes\mathscr{T}^{-1}$ is a nonconstant bundle with fiber $\mathbb{Z}$ and structure group $\mathbb{Z}_2$. The coefficient sequence $0 \to \mathscr{T}^{-1} \xrightarrow{k} \mathscr{T}^{-1} \to \mathscr{T}^{-1}\otimes\mathbb{Z}_k \to 0$ induces the exact sequence

$$0 = H_n(X;\mathscr{T}^{-1}) \to H_n(X;\mathscr{T}^{-1}\otimes\mathbb{Z}_k) \to H_{n-1}(X;\mathscr{T}^{-1}) \xrightarrow{k} H_{n-1}(X;\mathscr{T}^{-1}),$$

which shows that $H_{n-1}(X;\mathscr{T}^{-1})$ contains 2-torsion $\mathbb{Z}_2$ and no odd torsion, whence $m = 2$, proving (g).[35] Part (h) is an immediate consequence of (b) and (g). □

16.17. Corollary. (F. Raymond [69].) *Let X be an n-cm_L, and assume that $H^*_c(X;L) \approx H^*_c(\mathbb{R}^n;L)$. Then the one-point compactification X^+ of X is an n-cm_L with $H^*(X^+;L) \approx H^*(\mathbb{S}^n;L)$.*

Proof. That X^+ has the stated global cohomology is immediate from the cohomology sequence of the pair (X^+,∞). By 16.16(f), X is orientable. Thus $H^c_q(X;L) \approx H^{n-q}_c(X;L)$, whence $\tilde{H}^c_*(X;L) = 0$. By II-17.16, X^+ is clc^∞_L. By (9) on page 292, $H_*(X^+;L) \approx H_*(\mathbb{S}^n;L)$. The exact homology sequence of (X^+,X) then shows that $\mathscr{H}_p(X^+;L)_\infty \approx H_p(X^+,X)$ is L for $p = n$ and is zero for $p \neq n$. This gives the result by 16.8(d). □

[35] Part (g) can also be proved by an argument similar to that of II-16.29.

16.18. Corollary. *If A is a locally closed subspace of an n-cm$_L$ X, then* $\dim_L A = n \Leftrightarrow \operatorname{int} A \neq \varnothing$.[36]

Proof. The $\Leftarrow$ part is trivial. Also, we may as well assume that X is connected and orientable and that A is closed. In that case, if $\operatorname{int} A = \varnothing$ then $H_c^n(A \cap U; L) = 0$ by 16.16(c) for all open U. Then it follows from II-16.14 that $\dim_L A < n$. □

16.19. Corollary. (Invariance of domain.) *If $A \subset X$ and if A and X are both n-cm$_L$'s, then A is open in X.*

Proof. By passing to a small open neighborhood of a point $x \in A$ we can assume that A and X are both orientable, that A is closed in X, and that X is connected. Since $H_c^n(A; L) \neq 0$ by 16.16(b), we must have $A = X$ (locally at x in general) by 16.16(c). □

16.20. Theorem. *Let X be a connected n-cm$_L$ and let $F \subset X$ be a closed subspace with* $\dim_L F \leq n - 2$. *Then $X - F$ is connected.*

Proof. There is the exact sequence

$$0 = H_c^{n-1}(F; \mathscr{O}) \to H_c^n(X - F; \mathscr{O}) \to H_c^n(X; \mathscr{O}) \to 0,$$

so that the result follows from part (b) of 16.16. □

16.21. Theorem. *Let X be an n-cm$_L$ and let $F \subset X$ be a closed subspace with* $\dim_L F = n - 1$. *Then there exists a point $x \in F$ such that all sufficiently small open neighborhoods of x are disconnected by F; i.e., there exists an open neighborhood U of x such that for all open neighborhoods $V \subset U$ of x, $V - F$ is disconnected. Moreover, every point of X has an arbitrarily small open connected neighborhood V such that $V - F$ is disconnected for all compact subspaces F of V with* $\dim_L F = n - 1$ *and* $H^{n-1}(F; L) \neq 0$.

Proof. This is a local matter, and so we may as well assume that X is connected and orientable. By II-16.17 there exists a point $x \in F$ and an open neighborhood W of x such that for all open neighborhoods $U \subset W$ of x, $H_c^{n-1}(U \cap F; L) \to H_c^{n-1}(W \cap F; L)$ is nonzero. However, by definition of an n-cm, there is a neighborhood U of x so small that $H_c^{n-1}(U; L) \to H_c^{n-1}(W; L)$ is zero. Let $V \subset U$ be a connected neighborhood of x. In the commutative diagram

$$\begin{array}{ccccccc} H_c^{n-1}(V; L) & \to & H_c^{n-1}(V \cap F; L) & \to & H_c^n(V - F; L) & \to & H_c^n(V; L) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \neq 0} & & & & \\ H_c^{n-1}(W; L) & \to & H_c^{n-1}(W \cap F; L) & & & & \end{array}$$

[36] In a topological manifold, this holds for arbitrary subspaces; see [49, p. 46].

the map $H_c^n(V-F;L) \to H_c^n(V;L)$ is an isomorphism by 16.16(e) if $V-F$ is connected. That contradicts the remainder of the diagram. Therefore $V-F$ is disconnected for *all* open neighborhoods $V \subset U$ of x. The last statement also follows from the displayed diagram, where now $F \subset V$. □

16.22. Corollary. *If $F \subset X$ is a closed subspace of the n-cm_L X, then* $\dim_L F = n-1 \Leftrightarrow \operatorname{int} F = \varnothing$ *and F separates X locally near some point.* □

We shall now introduce the notion of a generalized manifold *with boundary.* As always in this chapter, all spaces considered are assumed to be locally compact Hausdorff spaces.

16.23. Definition. *A space X is said to be an n-cm_L with boundary B if $B \subset X$ is closed, $X - B$ is an n-cm_L, B is an $(n-1)$-cm_L, and the homology sheaf $\mathscr{H}_n(X;L)$ vanishes on B.*

By 16.8 this is equivalent to the following five conditions:

1. $B \subset X$ is closed.
2. X and B are clc_L^∞.
3. $\dim_L X < \infty$.
4. The stalks $\mathscr{H}_p(X;L)_x$ are isomorphic to L for $x \in X - B$ and $p = n$, and are zero otherwise.
5. The stalks $\mathscr{H}_p(B;L)_y$ are isomorphic to L for $p = n-1$ and are zero for $p \neq n-1$.

Also, it follows from 9.6 and II-Exercise 11 that $\dim_L X = n$. Note that an n-cm_L with boundary is an n-whm_L.

16.24. Proposition. *If X is hereditarily paracompact and $X - B$ is orientable over L, then condition* 5 *follows from the other conditions.*

Proof. By 9.2, $H_{n-p}^c(X, X-B) \approx H_c^p(B; \mathscr{O}|B) = 0$ since $\mathscr{O}|B = 0$. By 13.7, $H^*(X, X-B; L) = 0$. By II-16.10, 9.3, and the cohomology sequence of $(X, X-B)$ and the coefficient sequence $0 \to L_{X-B} \to L \to L_B \to 0$ we have $H_p(B) \approx H^{n-p}(X, X-B; L_{X-B}) \approx H^{n-p-1}(X, X-B; L_B)$. By the exact sequence of the pair $(X, X-B)$ with coefficients in L_B, this is isomorphic to $H^{n-p-1}(X; L_B) \approx H^{n-p-1}(B)$. These isomorphisms are natural, and so we have the natural isomorphism $H_p(B \cap U) \approx H^{n-p-1}(B \cap U)$ for U open. Taking the limit over neighborhoods U of $x \in B$ gives $\mathscr{H}_p(B)_x = 0$ for $p \neq n-1$ and $\mathscr{H}_{n-1}(B)_x \approx L$. □

Remark: That orientability is needed for 16.24 is shown by the example of an open cone over $\mathbb{RP}^2$ with B the vertex and L a field of characteristic other than 2. We conjecture, however, that orientability is not needed if $L = \mathbb{Z}$.

16.25. Theorem. *Let X and Y have common intersection $B = X \cap Y$ that is closed and nowhere dense in both X and Y. Then $X \cup Y$ is an n-cm_L and B is an $(n-1)$-$cm_L \Leftrightarrow$ both X and Y are n-cm_L's with boundary B.*

Proof. From a diagram using the Mayer-Vietoris sequence one sees easily from II-17.3 that if B is clc_L^∞ then $X \cap Y$ are $clc_L^\infty \Leftrightarrow X$ and Y are clc_L^∞. From the first Mayer-Vietoris sequence of Exercise 9 one derives the Mayer-Vietoris exact sequence (coefficients in L)

$$\cdots \to \mathscr{H}_p(B) \to \mathscr{H}_p(X)|B \oplus \mathscr{H}_p(Y)|B \to \mathscr{H}_p(X \cup Y)|B \to \mathscr{H}_{p-1}(B) \to \cdots$$

of sheaves on B, and the $\Leftarrow$ implication follows immediately from this by examining stalks at points of B. For the $\Rightarrow$ portion, let W be a connected paracompact open neighborhood in $X \cup Y$ of any point in B. Then $W \cap X$ and $W \cap Y$ are nonempty proper closed subsets of W. By 16.16(a) we have that $H_n(W \cap X) = 0 = H_n(W \cap Y)$ and that $H_{n-1}(W \cap X)$ and $H_{n-1}(W \cap Y)$ are torsion-free. It follows that $\mathscr{H}_n(X)|B = 0 = \mathscr{H}_n(Y)|B$ and that $\mathscr{H}_{n-1}(X)|B$ and $\mathscr{H}_{n-1}(Y)|B$ are torsion-free. This again gives the result upon examination of the displayed sequence. □

16.26. Corollary. *Let X be a connected, orientable n-cm_L such that $H_1^c(X;L) = 0$ and let $B \subset X$ be a closed, connected $(n-1)$-cm_L. Assume either that $L = \mathbb{Z}$ or that B is orientable. Then $X - B$ has two components U and V. Moreover, $\overline{U} = U \cup B$ is an n-cm_L with orientable boundary B, and similarly with $\overline{V}$.*

Proof. Since $H_c^{n-1}(X;L) \approx H_1^c(X;L) = 0$, the exact cohomology sequence of (X, B) has the segment

$$0 \to H_c^{n-1}(B;L) \to H_c^n(X - B;L) \to H_c^n(X;L) \to 0,$$

from which we deduce from 16.16(b, g) that $X - B$ has exactly two components, say U and V (and that B is orientable in the case $L = \mathbb{Z}$). That $\overline{U} = U \cup B$ follows from 16.21, and the result then follows from 16.25. □

Remark: Even in the case $n = 3$ and $X = \mathbb{S}^3$ the Alexander horned sphere shows that $\overline{U}$ need not be an actual 3-manifold with boundary. The standard embedding of $\mathbb{RP}^2$ in $\mathbb{RP}^3$ and $L = \mathbb{Z}_3$ shows that the condition "$L = \mathbb{Z}$ or B is orientable" is required.

16.27. Theorem. *Let $B \subset X$ be closed and nowhere dense. Then X is an n-cm_L with boundary $B \Leftrightarrow$ the double $dX = X \cup_B X$ is an n-cm_L.*

Proof. The $\Rightarrow$ part follows immediately from the preceding result. For $\Leftarrow$, let Y denote a copy of X, let $T : X \cup Y \to X \cup Y$ be the map switching copies, let $i : X \hookrightarrow X \cup Y$ be the inclusion, and let $\pi : X \cup Y \to X$ be the

folding map. Then $\pi i : X \to X$ is the identity. Let U be a connected open neighborhood in $X \cup Y$ of some point in $B = X \cap Y$ that is invariant under T. Then the composition

$$H_p(U \cap X) \xrightarrow{i_*} H_p(U) \xrightarrow{\pi_*} H_p(U \cap X)$$

is the identity, and it induces maps of sheaves

$$\mathscr{H}_p(X)|B \to \mathscr{H}_p(X \cup Y)|B \to \mathscr{H}_p(X)|B$$

whose composition is the identity. It follows that $\mathscr{H}_p(X)|B = 0$ for $p \neq n$. But $H_n(U \cap X) = 0$ by 16.16(a). Thus $\mathscr{H}_p(X)|B = 0$ for all p. Then the Mayer-Vietoris sequence

$$\cdots \to \mathscr{H}_p(B) \to \mathscr{H}_p(X)|B \oplus \mathscr{H}_p(Y)|B \to \mathscr{H}_p(X \cup Y)|B \to \mathscr{H}_{p-1}(B) \to \cdots$$

shows that $\mathscr{H}_{n-1}(B) \approx \mathscr{H}_n(X \cup Y)|B$ is locally constant with stalks L and that $\mathscr{H}_p(B) = 0$ for $p \neq n-1$; that is, B is an $(n-1)$-cm_L. Hence X is an n-cm_L with boundary B by definition. □

16.28. Proposition. *Let $B \subset X$ be closed with both X and B being clc_L^∞. Then X is an n-cm_L with boundary B $\Leftrightarrow$ the sheaf $\overline{\mathscr{O}} = \mathscr{H}_p(X, B; L)$ has stalks L for $p = n$ and vanishes for $p \neq n$, and $\mathscr{H}_p(X; L)|B = 0$ for all p.*

Proof. It follows from 16.8(d) that under either hypothesis, $X - B$ is an n-cm_L. For an open set U of X we have the exact sequence

$$\cdots \to H_p(U) \to H_p(U, B \cap U) \to H_{p-1}(B \cap U) \to H_{p-1}(U) \to \cdots,$$

natural with respect to inclusions, and hence the exact sequence of sheaves

$$\cdots \to \mathscr{H}_p(X) \to \mathscr{H}_p(X, B) \to \mathscr{H}_{p-1}(B)^X \to \mathscr{H}_{p-1}(X) \to \cdots,$$

yielding the isomorphism $\mathscr{H}_p(X, B)|B \approx \mathscr{H}_{p-1}(B)$. Thus the result follows from 16.8(d). □

16.29. Proposition. *For an n-cm_L X with boundary B and its orientation sheaves $\mathscr{O} = \mathscr{H}_n(X; L)$ and $\overline{\mathscr{O}} = \mathscr{H}_n(X, B; L)$, we have $\mathscr{O} \approx \overline{\mathscr{O}}_{X-B}$. Moreover, $\overline{\mathscr{O}}$ is locally constant on X with stalks L.*

Proof. The first part is immediate. Let $X \cup Y$ be the double of X along B. Then similarly to the previous proof, there is the exact sequence

$$\cdots \to \mathscr{H}_p(Y)^{X \cup Y} \to \mathscr{H}_p(X \cup Y) \to \mathscr{H}_{p-1}(X \cup Y, Y) \to \mathscr{H}_{p-1}(Y)^{X \cup Y} \to \cdots$$

and by excision 5.10, $\mathscr{H}_p(X \cup Y, Y) \approx \mathscr{H}_p(X, B)^{X \cup Y}$. Since $\mathscr{H}_p(Y)^{X \cup Y}$ restricts to the zero sheaf on X, we have that $\mathscr{H}_p(X, B) \approx \mathscr{H}_p(X \cup Y)|X$. The latter sheaf is locally constant with stalks L by 16.8. □

16.30. Theorem. *Let X be an n-cm_L with boundary $A \cup B$ where A and B are closed and are $(n-1)$-cm_L's with common boundary $A \cap B$. Let $(\mathscr{M}, \Phi|A)$ be elementary on X, and assume that* $\dim_\Phi X < \infty$.[37] *Then*[38]

$$\boxed{H^p_\Phi(X, A; \mathscr{M} \otimes \overline{\mathscr{O}}) \approx H^\Phi_{n-p}(X, B; \mathscr{M}).}$$

Proof. To simplify notation we shall give the proof in the case for which $\mathscr{M} = L$ and X is orientable, i.e., $\overline{\mathscr{O}} = L$. There is no difficulty in generalizing it. Note that $\mathscr{O} = L_{X-A-B}$ in this case.

Let $X^+ = X \cup_{B=B\times\{0\}} (B \times \mathbb{I})$, let B^+ be the image of $B \times \{1\}$, and put $X^\circ = X^+ - B^+$. Extend Φ to X^+ by adding the sets of the form $K \times \mathbb{I}$ and their subsets. (Similarly, extend $\mathscr{M}$ in the general case.) We shall still call this extended family Φ. Put $A^+ = A \cup ((A \cap B) \times \mathbb{I})$, and similarly for A°. In the same manner as in previous proofs, one can show that X^+ is an n-cm_L with boundary $A^+ \cup B^+$ and X° is an n-cm_L with boundary A° and orientation sheaf $L_{X^\circ - A^\circ}$. Consider the following diagram (coefficients in L when omitted):

$$\begin{array}{ccccccc}
H^p_\Phi(X; L_{X-A}) & \xleftarrow{\approx} & H^p_\Phi(X^\circ; L_{X^\circ - A^\circ}) & \underset{\Delta}{\xrightarrow{\approx}} & H^{\Phi\cap X^\circ}_{n-p}(X^\circ) & \approx & H^\Phi_{n-p}(X^+, B^+) \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle j_*} & & \\
 & & H^p_{\Phi\cap(X-B)}(X - B; L_{X-B-A}) & \underset{\Delta}{\xrightarrow{\approx}} & H^{\Phi\cap(X-B)}_{n-p}(X - B) & \approx & H^\Phi_{n-p}(X, B),
\end{array}$$

which commutes. (The isomorphism on the upper left is a consequence of invariance of cohomology under proper homotopies.) Now, via the isomorphisms on the right (from 5.10), the map j_* is the composition

$$H^\Phi_{n-p}(X^+, B^+) \xrightarrow{\approx} H^\Phi_{n-p}(X^+, B \times \mathbb{I}) \xleftarrow{\approx} H^\Phi_{n-p}(X, B)$$

(these maps are isomorphisms due to invariance of homology under c-proper homotopies) and hence is an isomorphism. The required isomorphism is then just the composition from the upper left to the lower right. □

The typical case of interest of 16.30 is the product $(M, \partial M) \times (N, \partial N)$ of two orientable manifolds (or cm's) with boundary and $A = \partial M \times N$, $B = M \times \partial N$. Thus, with $m = \dim_L M$ and $n = \dim_L N$, one gets the isomorphism

$$\boxed{H^p_\Phi((M, \partial M) \times N; L) \approx H^\Phi_{m+n-p}(M \times (N, \partial N); L)}$$

when Φ is paracompactifying or when $M \times N$ is locally hereditarily paracompact.

The first term on the bottom of the diagram in the proof of 16.30 and the commutativity of the diagram were not used in the proof. However,

[37]E.g., Φ paracompactifying, or X locally hereditarily paracompact.

[38]See [19, p. 358] for a classical account of this version of duality.

the composition

$$H^p_\Phi(X;L_{X-A}) \xleftarrow{\approx} H^p_\Phi(X^\circ;L_{X^\circ-A^\circ}) \xrightarrow{i^*} H^p_{\Phi\cap(X-B)}(X-B;L_{X-B-A})$$

is just the restriction homomorphism for the inclusion $X-B \hookrightarrow X$, and so in the case $A=\varnothing$ we also conclude the following result from the commutativity of the diagram:

16.31. Proposition. *Let X be an n-cm_L with boundary B. Assume that* $\dim_\Phi X < \infty$ *and* $\dim_{\Phi\cap(X-B)} X-B < \infty$.[39] *Then for $\mathscr{M}$ an elementary sheaf on X and an arbitrary family of supports Φ on X, the restriction*

$$H^p_\Phi(X;\mathscr{M}) \to H^p_{\Phi\cap(X-B)}(X-B;\mathscr{M})$$

is an isomorphism for all p. Also see [68]. □

An example of a 3-*cm* with boundary that is not a manifold is the closure M of one of the components of the complement of an Alexander horned sphere B in 3-space. In this case there is no internal collar of the bounding 2-sphere. An example of a 3-*cm* without boundary that is not a manifold is the quotient space M/B, as follows from 16.35. An example of a 1-*cm* that is not a manifold is the double of a compactified long line. However, separable metric n-cm_L's are manifolds for $n \le 2$ and any L, as we now show.

16.32. Theorem. *If X is a second countable n-hm_L, with or without boundary, and $n \le 2$, then X is a topological n-manifold.*

Proof. Since X is completely regular and second countable, it is separable metrizable by the Urysohn metrization theorem; see [19, I-9.11]. Also, X is an n-hm_K for $K = L/pL$ for any prime p in L, and so it suffices to consider the case in which L is a *field*. Then by 16.13, X is clc_L. In particular, X is locally connected. A locally compact and locally connected metric space is locally arcwise connected in the sense that an "arc" is a *homeomorphic* image of $[0,1]$; see [47, p. 116].

In case $n=0$, X is totally disconnected by II-16.21. Since it is locally connected, it is discrete, proving that case.

In case $n=1$, first assume that $\partial X=\varnothing$. By 16.21 any point $x\in X$ has a connected neighborhood U with $U-\{x\}$ disconnected. Let $a,b\in U-\{x\}$ be points from different components. Then there is an arc A in U joining a and b. By invariance of domain 16.19, $A-\{a,b\}$ is open in X, so that X is locally euclidean near x, proving that case. The case in which X has a boundary can be verified by looking at the double of X. The details of that argument are left to the reader.

For the case $n=2$, we must use the topological characterization of 2-manifolds with boundary due to G. S. Young [87]. This says that a locally

[39] E.g., Φ paracompactifying and $X-B$ paracompact.

compact metric space X for which each point has a connected neighborhood U such that $U - C$ is disconnected for all simple closed curves C in U, is a 2-manifold, possibly with boundary. Consider first the case in which $\partial X = \varnothing$. Then the last part of 16.21 shows that X satisfies Young's criterion. Since the sheaf $\mathscr{H}_n(X; L)$ vanishes on the boundary of an n-manifold, the topological manifold notion of boundary coincides with that defined for homology manifolds. Now consider the case in which $B = \partial X \neq \varnothing$. Let $X \cup Y$ be the double of X along B. Let $x \in B$ and let U be a symmetric neighborhood of x in $X \cup Y$ such that $U \cap X$ is connected and with U not containing any component of B (which we now know to be a 1-manifold). Also, take U so small that U is disconnected by all simple closed curves $C \subset U$. Then note that

$$U - C = (U \cap X - C) \cup (U \cap (Y - B)).$$

Now, $U \cap (Y - B)$ is connected since $H^0(U \cap Y; L) \to H^0(U \cap (Y - B); L)$ is an isomorphism by 16.31. Also, the closure of $U \cap (Y - B)$ intersects $U \cap X - C$ nontrivially, since otherwise C would be a component of B, contrary to the selection of U as not containing a component of B. Thus, since $U - C$ is disconnected, $U \cap X - C$ must be disconnected. This shows that X satisfies Young's criterion.[40] □

16.33. Theorem. (Wilder's monotone mapping theorem.) *Let X be an n-hm_L. Assume either that X is orientable or that $L = \mathbb{Z}$. Let $f : X \to Y$ be a Vietoris map (i.e., proper, surjective, and with acyclic point inverses). Then Y is also an n-hm_L, which is orientable if X is orientable. Also, Y is clc_L^∞ (and hence is an n-cm_L) if X is clc_L^∞.*

Proof. The hypothesis implies that f is proper and that the Leray sheaf $\mathscr{H}^p(f; L)$ is L for $p = 0$ and vanishes otherwise. We shall prove the last statement first. For $U \subset Y$ open and with $U^\bullet = f^{-1}U$, the Leray spectral sequence degenerates to the natural isomorphism $H_c^p(U; L) \approx H_c^p(U^\bullet; L)$. If U and V are open with $\overline{U} \subset V$, then $\overline{U^\bullet} \subset V^\bullet$. Therefore if X is clc_L^∞, then by II-17.5, $H_c^p(U^\bullet; L) \to H_c^p(V^\bullet; L)$ has finitely generated image. Hence $H_c^p(U; L) \to H_c^p(V; L)$ has finitely generated image for all p, which means that Y is clc_L^∞ by II-17.5.

Now assume that X is orientable. Note that a point in a locally compact Hausdorff space has a fundamental system of open paracompact neighborhoods U (e.g., the union of an increasing sequence of compact neighborhoods). Since f is proper, $U^\bullet$ is also paracompact. Now $\mathscr{H}_p(f; L) = \mathscr{S}\!heaf(U \mapsto H_p(U^\bullet; L) \approx H^{n-p}(U^\bullet; L)) \approx \mathscr{H}^{n-p}(f; L)$. Thus we have

$$\begin{aligned}\mathscr{H}_p(Y; L) &\approx \mathscr{H}_p(f; L) && \text{by (48) on page 323}\\ &\approx \mathscr{H}^{n-p}(f; L) && \text{since } X \text{ is orientable}\\ &\approx \begin{cases} L, & p = n\\ 0, & p \neq n\end{cases} && \text{since } f \text{ is a Vietoris map,}\end{aligned}$$

[40] Another treatment of these matters can be found in [85].

which means that Y is an n-hm_L as claimed.

Now assume X to be nonorientable and that $L = \mathbb{Z}$. Let $y \in Y$, and put $F = f^{-1}(y)$. We claim that $\mathscr{O}$ is constant on F. If not, then the units in the stalks give a nontrivial double covering space of F, so that $H^1(F; \mathbb{Z}_2) \neq 0$ by II-19.6. This contradicts the universal coefficient theorem II-15.3. Now, $\mathscr{O}$ must be constant on some open neighborhood of $F = f^{-1}(y)$ by II-9.5, and this neighborhood can be taken to be of the form $U^\bullet = f^{-1}U$, U open. Then $U^\bullet$ is orientable, and so U is an n-$hm_{\mathbb{Z}}$ by the part of the theorem previously proved. □

This result would be false for L a field of characteristic other than 2 and X nonorientable, as shown by the map of $\mathbb{RP}^2$ to a point. Of course, it is only the dimension of Y that goes wrong here. A more convincing example is given by $X = \mathbb{RP}^2 \times \mathbb{R}$ and $Y = (-\infty, 0] \cup \mathbb{RP}^2 \times (0, \infty)$, where $f : X \to Y$ is the obvious map and Y is given the quotient topology. Here f is a Vietoris map over fields L of characteristic other than 2, but Y fails to be an hm_L for any L and even fails to have a uniform dimension over L.

As an example of the use of 16.33, the identification space $\mathbb{R}^3/A$ is a 3-$cm_{\mathbb{Z}}$, where A is a wild arc. Note that if x is the identification point, then $U - \{x\}$ is not simply connected for any neighborhood U of x in this 3-$cm_{\mathbb{Z}}$.

In Section 18 we shall present a considerable generalization of 16.33.

16.34. Corollary. *Let X be an n-cm_L, and assume either that X is orientable or that $L = \mathbb{Z}$. Let $f : X \to Y$ be a Vietoris map. Then the mapping cylinder M_f is an $(n+1)$-cm_L with boundary $X + Y$.*

Proof. The obvious map $X \times \mathbb{S}^1 \to dM_f$ of the double of $X \times \mathbb{I}$ to the double of M_f is a Vietoris map, and so dM_f is an $(n+1)$-cm_L by 16.33. Thus M_f is an $(n+1)$-cm_L with boundary by 16.27. □

16.35. Corollary. *Let X be an n-cm_L with boundary and let B be a compact component of the boundary such that $H_*(B; L) \approx H_*(\mathbb{S}^{n-1}; L)$. Then the quotient space X/B is an n-cm_L with boundary being the image of $\partial X - B$.*

Proof. The boundary component B is orientable by 16.16(f), and it follows that M is orientable in a neighborhood of B. If M is the union of X with the cone CB on B, then M is an n-cm_L by 16.9 (or 5.10) and 16.25. Also $X/B \approx M/CB$. Since the map $M \to M/CB$ is Vietoris, the result follows from 16.33. □

16.36. We conclude this section with a short discussion of compactifications of generalized manifolds. Let U be a connected n-hm_L. By a *compactification* of U we mean a compact space X containing U as a dense

subspace and such that $F = X - U$ is totally disconnected. It is the *end-point compactification* of U if, in addition, F does not disconnect X locally near any point; see Raymond [69], on which this discussion is largely based.

16.37. Theorem. *Suppose that X is a compactification of the n-hm$_L$ U. Then $\Gamma(\mathscr{H}_r(X;L)) \approx H_r(X,U;L)$ for $r < n$. Moreover, for $r < n$, $\mathscr{H}_r(X;L) = 0 \Leftrightarrow H_r(X,U;L) = 0$.*

Proof. By II-Exercise 11, $\dim_L X = n$. Let $F = X - U$. By 8.4, there is a spectral sequence with

$$E_2^{p,q} = H^p(F; \mathscr{H}_{-q}(X;L)|F) \Longrightarrow H_{-p-q}(X,U;L).$$

Since F is totally disconnected, this reduces to the isomorphism

$$\Gamma(\mathscr{H}_r(X;L)|F) \approx H_r(X,U;L).$$

For $r < n$, $\mathscr{H}_r(X;L)$ is concentrated on F, so that

$$\Gamma(\mathscr{H}_r(X;L)) \approx \Gamma(\mathscr{H}_r(X;L)|F).$$

The last statement follows from II-Exercise 61. □

16.38. Lemma. *Let X be the end-point compactification of the connected n-cm$_L$ U, $n > 1$. If U is orientable in the neighborhood of each point of X, then $\mathscr{O} = \mathscr{H}_n(X;L)$ is locally constant with stalks L. If X is hlc$_L^1$, then $\mathscr{H}_1(X;L) = 0 = \mathscr{H}_0(X;L)$.*

Proof. Suppose that V is an open connected subset of X with $U \cap V$ orientable. If W is a connected open subset of V, then $W - F$ is connected and $H_c^n(W - F;L) \to H_c^n(W;L)$ is an isomorphism. By 16.16, $H_c^n(W - F;L) \to H_c^n(V \cap U;L) \approx L$ is an isomorphism, whence $H_c^n(W;L) \xrightarrow{\approx} H^n(V;L) \approx L$. It follows that $\mathscr{O}|V$ is constant with stalks L.

Now assume, instead, that X is *hlc*$_L^1$. Then a point $x \in X$ has a connected open neighborhood V so small that $H_1^c(V;L) \to H_1^c(X;L)$ is zero. Since $V - \{x\}$ and $X - \{x\}$ are connected and locally connected, we have that $\tilde{H}_0^c(V - \{x\};L) = 0 = \tilde{H}_0^c(X - \{x\};L)$ by 5.14. Therefore we have the commutative diagram

$$\begin{array}{ccccc} H_1^c(V;L) & \to & H_1^c(V, V - \{x\};L) & \to & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \approx} & & \\ H_1^c(X;L) & \to & H_1^c(X, X - \{x\};L) & \to & 0, \end{array}$$

where the vertical isomorphism is due to excision. It follows that

$$\mathscr{H}_1(X;L)_x \approx H_1^c(X, X - \{x\};L) = 0.$$

A similar but easier argument shows that $\mathscr{H}_0(X;L) = 0$. □

16.39. Theorem. *Let X be the end-point compactification of the connected n-cm$_L$ U, $n > 1$, that is orientable in the neighborhood of each point of X. Then the following three conditions are equivalent:*[41]

(a) *X is an n-cm$_L$.*

(b) *$H^*(X; L)$ is finitely generated and $H_r(X, U; L) = 0$ for all $1 < r < n$.*

(c) *$H^*(X; L)$ is finitely generated and*

$$[X] \cap (\bullet) : H^r(X; \mathscr{O}) \to H_{n-r}(X; L)$$

is an isomorphism for all $0 < r < n$.

Proof. By II-17.15, X is $clc_L \Leftrightarrow H^*(X; L)$ is finitely generated. This gives (a) $\Leftrightarrow$ (b) by 16.37 and 16.38. The basic spectral sequence

$$E_2^{p,q} = H^p(X; \mathscr{H}_{-q}(X; L)) \Longrightarrow H_{-p-q}(X; L)$$

has $E_2^{0,-q} = \Gamma(\mathscr{H}_q(X; L)) \approx H_q(X, U; L)$ for $q < n$ by 16.38. Also, $E_2^{p,-n} \approx H^p(X; \mathscr{O})$. All other terms are zero. By 10.2, and the proof of 10.1, the edge homomorphism

$$H^p(X; \mathscr{O}) = E_2^{p,-n} \twoheadrightarrow E_\infty^{p,-n} \rightarrowtail H_{n-p}(X)$$

is the cap product $[X] \cap (\bullet)$ with $[X] = \Delta(1) \in H_n(X; \mathscr{O}^{-1})$, up to sign. Thus the latter is an isomorphism for $0 < p < n \Leftrightarrow H_q(X, U) \approx E_2^{0,-q} = 0$ for $1 < q < n$, since $E_2^{0,-1} \approx H_1(X, U) = 0$ by 16.38. Hence (b) $\Leftrightarrow$ (c). □

Remark: As noted by Raymond [69, 4.12], the example of $U = \mathbb{RP}^2 \times \mathbb{R}$, $X = \Sigma\mathbb{RP}^2$, and L a field of characteristic other than 2 shows that the hypothesis of orientability of U near points of X is essential in 16.39.

17 Locally homogeneous spaces

A (locally compact, Hausdorff) space X is called "locally homogeneous" if for every pair $x, y \in X$ there is a homeomorphism h of some neighborhood of x onto a neighborhood of y with $h(x) = y$; see [4] and [29]. Bing and Borsuk [4] have asked whether every locally contractible, locally homogeneous, finite dimensional, separable metric space is a generalized manifold (or, indeed, a topological manifold). A partial result in this direction follows immediately from 16.15:

17.1. Theorem. [14] *Suppose that X is second countable, that* $\dim_L X < \infty$, *that X is locally homogeneous, and that $H_i(X, X - \{x\}; L)$ is finitely generated for each i. Then X is an n-hm$_L$ for some n.* □

[41] Raymond's Theorem 4.5 in [69] is equivalent to the present (a) $\Leftrightarrow$ (b) by the exact sequence of his Theorem 2.16, which is the homological analogue of the second exact sequence of our II-18.2.

The problem with this result is the condition that $H_i(X, X - \{x\}; L)$ be finitely generated. This is a very strong condition that does not follow from any reasonable set-theoretic properties that we are aware of. For instance, local contractibility does not imply it. It would be nice if this condition could be replaced by condition clc_L, for example. We shall prove such a result under a stronger type of local homogeneity:

17.2. Definition. *A space X will be called "locally isotopic" if X is locally arcwise connected and for each path $\lambda : \mathbb{I} \to X$ there is a neighborhood N of $\lambda(0)$ in X and a map $\Lambda : \mathbb{I} \times N \to X$ such that $\Lambda(t, \lambda(0)) = \lambda(t)$ and such that each Λ^t is a homeomorphism of N onto a neighborhood of $\lambda(t)$, where $\Lambda^t(x) = \Lambda(t, x)$.* [*With no loss of generality we may also assume that $\Lambda(0, x) = x$ for all $x \in N$.*]

This condition is slightly stronger than the notion of local homogeneity defined by Montgomery [61]. It is weaker than that defined by Hu [48], which coincides with the condition 1-LH of Ungar [81].

Throughout the remainder of this section we shall impose the standing assumption that X is locally compact, Hausdorff, second countable and clc_L^∞, and that L is a *countable* principal ideal domain.

17.3. Lemma. *If X is also locally homogeneous, then the homology sheaf $\mathscr{H}_p(X; L)$ is a Hausdorff space.*

Proof. Let $\{V_i\}$ be a countable basis for the topology of X. For any $x \in X$ and $\alpha \in \mathscr{H}_p(X)_x$, there are indices i, j with $x \in V_j$, $\overline{V}_j \subset V_i$, and such that α is in the image of the canonical map $r^*_{x,j,i} : \operatorname{Im}(H_p(V_i) \to H_p(V_j)) \to \mathscr{H}_p(X)_x$. Since X is clc_L^∞, $\operatorname{Im}(H_p(V_i) \to H_p(V_j))$ is finitely generated, whence it is countable because L is countable. For $s_n \in \operatorname{Im}(H_p(V_i) \to H_p(V_j))$ the definition $\sigma_n(x) = r^*_{x,j,i}(s_n)$ gives a section $\sigma_n \in \Gamma(\mathscr{H}_p(U_n))$, where $U_n = V_j$. Therefore $\mathscr{H}_p(X)$ is covered by the countable collection $\{\sigma_n\}$ of local sections. Put $A_n = |\sigma_n|$ and $C_n = (\partial U_n) \cup (\partial A_n)$. By the Baire category theorem, there is a point x not in any C_n. Each point of the stalk $\mathscr{H}_p(X)_x$ has the form $\sigma_n(x)$ for some n. If $\sigma_n(x) \neq 0$, then since $x \in A_n - \partial A_n$, σ_n is nonzero on some neighborhood of x. By the group structure this implies that any two points of $\mathscr{H}_p(X)_x$ can be separated by open sets in $\mathscr{H}_p(X)$. Since X is locally homogeneous, this is true at *all* points of X. □

17.4. Lemma. *Suppose that X is also locally isotopic. Then the projection $\pi : \mathscr{H}_p(X; L) \to X$ has the path-lifting property.*

Proof. Let $\lambda : \mathbb{I} \to X$ be any path and let $\alpha \in H^c_p(X, X - \lambda(0)) = \mathscr{H}_p(X)_{\lambda(0)}$ be given. Let N be a compact neighborhood of $\lambda(0)$ so small that $\Lambda : \mathbb{I} \times N \to X$ exists as in Definition 17.2. For $t \in \mathbb{I}$, Λ^t is a

homeomorphism of $(N, N - \lambda(0))$ onto $(N_t, N_t - \lambda(t))$, where $N_t = \Lambda^t(N)$ is a neighborhood of $\lambda(t)$. Thus

$$\Lambda_*^t : H_p^c(N, N - \lambda(0)) \xrightarrow{\approx} H_p^c(N_t, N_t - \lambda(t)),$$

and we put

$$\alpha_t = \Lambda_*^t(\alpha) \in \mathscr{H}_p(X)_{\lambda(t)}.$$

We must show that $t \mapsto \alpha_t$ is continuous. For this, let $U \subset N$ be an open neighborhood of $\lambda(0)$ so small that α is the image of an element $\beta \in H_p^c(N, N - U)$. Put $U_t = \Lambda^t(U)$ and $\beta_t = \Lambda_*^t(\beta) \in H_p^c(N_t, N_t - U_t)$. Let $s \in \mathbb{I}$ be given and let V be a neighborhood of $\lambda(s)$ so small that for some $\varepsilon > 0$, $V \subset U_t$ for all $s - \varepsilon < t < s + \varepsilon$. By homotopy invariance 6.5, β_t maps to the same element $\gamma \in H_p^c(X, X - V) = H_p(V)$ for all $t \in (s - \varepsilon, s + \varepsilon)$. Thus γ defines a section of $\mathscr{H}_p(X)$ over V whose value at $\lambda(t)$ is α_t for $s - \varepsilon < t < s + \varepsilon$, and this shows that α_t is continuous in t. Since π is a local homeomorphism, the path lifting is unique. □

Since a fibration with unique path lifting over a semilocally 1-connected space is a covering map [75, p. 78], we have:

17.5. Corollary. *If X is semilocally 1-connected and locally isotopic in addition to the standing assumptions, then each $\mathscr{H}_p(X; L)$ is locally constant.* □

From 16.1 and 16.3 we conclude:

17.6. Theorem. [14] *Let L be a countable principal ideal domain. Let X be second countable, semilocally 1-connected, and locally isotopic. Assume further that $n = \dim_L X < \infty$ and that X is connected and locally connected. If L is a field, then X is an n-cm_L. If L is arbitrary and X is clc_L^∞, then X is an n-cm_L.* □

18 Homological fibrations and p-adic transformation groups

The major unsolved question in the theory of continuous (as opposed to differentiable) transformation groups on topological manifolds is whether a compact group acting effectively on a manifold must necessarily be a Lie group. It is known that a counterexample to this would imply that some p-adic group

$$A_p = \varprojlim\{\cdots \to \mathbb{Z}_{p^3} \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p\},$$

p prime, can also act effectively on a manifold.[42] In one direction, it is even unknown whether such a group A_p can act *freely*. (It might be expected

[42]This follows from the known structure of compact groups together with Newman's Theorem [15, p. 156], stating that a finite group cannot act effectively on a manifold with uniformly small orbits.

that an effective action would imply a free one on some open subspace, but no one has been able to prove that either.) In this section we will present much of what is known about the homological implications of the existence of such an action. First we will study the notion of a homological fibration.

18.1. Definition. [11] *A "cohomology fiber space" over L (abbreviated cfs_L) is a proper map $f : X \to Y$ between locally compact Hausdorff spaces X and Y such that the Leray sheaf $\mathscr{H}^i(f; L)$ is locally constant for all i. We shall call it a cfs_L^k if $\mathscr{H}^i(f; L) = 0$ for $i > k$ and $\mathscr{H}^k(f; L) \neq 0$. In this case we denote the common stalk of $\mathscr{H}^i(f; L)$ by $H^i(F)$.*

18.2. Proposition. [11] *Suppose that $f : X \to Y$ is a cfs_L^k, and assume that* $\dim_L X = n < \infty$. *If L is a field, then* $\dim_L Y \leq n - k$. *If L is any principal ideal domain, then* $\dim_L Y \leq n + 1$.

Proof. We may as well assume that $\mathscr{H}^i(f; L)$ is the constant sheaf $H^i(F)$. Let L be a field. Let $y \in Y$. For $U \subset Y$ open, consider the Leray spectral sequence

$$E_2^{p,q}(U) = H^p(U; H^q(F)) \Longrightarrow H^{p+q}(U^\bullet)$$

(and also its analogue with compact supports) with coefficients in L, where $U^\bullet = f^{-1}(U)$. Let $0 \neq a \in H^k(F)$. Since $H^k(F) = \varinjlim H^k(V^\bullet)$, there is a neighborhood V of y and an element $a_V \in H^k(V^\bullet)$ extending a. For $U \subset V$ connected, let $a_U \in H^k(U^\bullet)$ be the restriction of a_V and let $\alpha \in E_2^{0,k} = H^0(U; H^k(F))$ be the image of a_U under the edge homomorphism $H^k(U^\bullet) \twoheadrightarrow E_\infty^{0,k} \rightarrowtail E_2^{0,k}$. Then $\alpha \neq 0$ is a permanent cocycle. Let $0 \neq \beta \in H_c^r(U; H^0(F)) = E_2^{r,0}$. Then β is trivially a permanent cocycle, and so $\alpha \cup \beta \in E_2^{r,k}$ is a permanent cocycle since $d_r(\alpha \cup \beta) = d_r\alpha \cup \beta \pm \alpha \cup d_r\beta = 0$. Also, $\alpha \cup \beta \neq 0$ by II-7.4. Therefore $H_c^{r+k}(U^\bullet) \neq 0$,and so $r + k \leq n$. Consequently, $H_c^r(U; H^0(F)) = 0$ for $r > n - k$. Since $H^0(F)$ is a free L-module, the composition $L \to H^0(F) \to L$ of coefficient groups implies that $H_c^r(U; L) = 0$ for $r > n-k$. Therefore $\dim_L U \leq n-k$ by II-16.14, and $\dim_L Y \leq n - k$ by II-16.8. The case of a general principal ideal domain L follows from an easy universal coefficient argument. □

18.3. Proposition. [11][36] *Suppose that $f : X \to Y$ is a cfs_L such that each $\mathscr{H}^i(f; L)$ has finitely generated stalks for $i \leq k$. If X is clc_L^k, then Y is also clc_L^k.*[43]

Proof. We may as well assume that each $\mathscr{H}^i(f; L)$ is constant for $i \leq k$. Let $A, B \subset Y$ be compact sets with $A \subset \operatorname{int} B$. If Y is not clc_L^k, then by II-17.5 there is a *minimal* integer $m \leq k$ such that there are such sets A, B with the image of the restriction $H^m(B; L) \to H^m(A; L)$ not being finitely

[43]The proof uses only that the sheaves $\mathscr{H}^i(f; L)$ are locally constant for $i \leq k$ rather than the full cfs_L condition.

generated. From the universal coefficient sequence and II-17.3, it follows that $E_2^{p,q}(B') = H^p(B'; H^q(F)) \to H^p(A'; H^q(F)) = E_2^{p,q}(A')$ has finitely generated image for all $p \le m-2$, $q \le k$ and all such pairs A', B'. Let K be compact and such that $A \subset \operatorname{int} K$ and $K \subset \operatorname{int} B$. Then by the diagram

$$\begin{array}{ccccc}
 & & E_2^{m,0}(B) & \to & E_3^{m,0}(B) \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\
E_2^{m-2,1}(K) & \to & E_2^{m,0}(K) & \to & E_3^{m,0}(K) \\
\downarrow & & \downarrow{\scriptstyle g} & & \\
E_2^{m-2,1}(A) & \to & E_2^{m,0}(A) & &
\end{array}$$

and II-17.3 it follows that $\operatorname{Im} h$ is not finitely generated. Continuing this way, we see that there is such a K such that the image of $E_\infty^{m,0}(B) \to E_\infty^{m,0}(K)$ is not finitely generated. It follows that the image of $H^m(B^\bullet) \to H^m(K^\bullet)$ is not finitely generated, contrary to II-17.5. □

18.4. Lemma. *Suppose that $f : X \to Y$ is a cfs_L^k with each $\mathscr{H}^i(f; L)$ constant, where L is a field. Then for U open in Y there is a spectral sequence of homological type*[44] *with*

$$\boxed{E^2_{p,q} = \operatorname{Hom}(H^q(F), H_p(U)) \Longrightarrow H_{p+q}(U^\bullet)}$$

and that is natural in U. Consequently, there is a spectral sequence of sheaves with

$$\boxed{\mathscr{E}^2_{p,q} = \mathscr{H}\!om(H^q(F), \mathscr{H}_p(Y)) \Longrightarrow \mathscr{H}_{p+q}(f).}$$

(Coefficients in L are omitted.) Also, the stalks satisfy

$$\boxed{\mathscr{H}\!om(H^q(F), \mathscr{H}_p(Y))_y \approx \operatorname{Hom}(H^q(F), \mathscr{H}_p(Y)_y).}$$

Proof. In the Leray spectral sequence

$$E_2^{p,q} = H_c^p(U; H^q(F)) \Longrightarrow H_c^{p+q}(U^\bullet)$$

we have $H_c^p(U; H^q(F)) \approx H_c^p(U) \otimes H^q(F)$ since L is a field. Also, $\operatorname{Hom}(\bullet, L)$ is exact, and so if we apply it to the spectral sequence and adjust indices, we get a homological spectral sequence with

$$\begin{aligned} E^2_{p,q} = \operatorname{Hom}(H_c^p(U) \otimes H^q(F), L) &\approx \operatorname{Hom}(H^q(F), \operatorname{Hom}(H_c^p(U), L)) \\ &\approx \operatorname{Hom}(H^q(F), H_p(U)) \end{aligned}$$

and converging to $\operatorname{Hom}(H_c^{p+q}(U^\bullet), L) \approx H_{p+q}(U^\bullet)$. The last statement is from the fact that $\operatorname{Hom}(\bullet, \bullet)$ commutes with direct limits in the second variable. □

[44] This means that $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

18.5. Theorem. [11] *Suppose that $f : X \to Y$ is a cfs_L^k, and assume that X is an orientable n-hm_L, where L is a field. Then Y is an $(n-k)$-hm_L. Also, Y is orientable if each $\mathscr{H}^i(f; L)$ is constant. Moreover, each $H^i(F; L)$ is finitely generated, and $H^i(F; L) \approx H^{k-i}(F; L)$ for all i.*

Proof. We may as well assume that the Leray sheaf of f is constant. As in the proof of 16.33, we have that $\mathscr{H}_i(f) \approx \mathscr{H}^{n-i}(f)$. By assumption, this vanishes for $n - i > k$, that is, for $i < n - k$. It also vanishes, trivially, for $n - i < 0$, that is, $i > n$. Let $m = \min\{p \mid \mathscr{H}_p(Y) \neq 0\}$ and $M = \max\{p \mid \mathscr{H}_p(Y) \neq 0\}$. Then, since L is a field and $H^k(F) \neq 0 \neq H^0(F)$, the terms $\mathscr{E}^2_{m,0}$ and $\mathscr{E}^2_{M,k}$ are nonzero and survive to $\mathscr{E}^\infty_{*,*}$.[45] Therefore, $m \geq n - k$ and $M + k \leq n$, whence $m = M = n - k$. Thus $\mathscr{H}_p(Y) = 0$ for $p \neq n - k$, and the spectral sequence degenerates to the isomorphism

$$\mathscr{H}\!om(H^q(F), \mathscr{H}_{n-k}(Y)) \approx \mathscr{H}_{n-k+q}(f) \approx \mathscr{H}^{k-q}(f) = H^{k-q}(F).$$

It follows that $\mathscr{H}_{n-k}(Y)$ is constant. Call its common stalk S. Then we deduce that

$$\operatorname{Hom}(H^q(F), S) \approx H^{k-q}(F). \tag{119}$$

In particular

$$\operatorname{Hom}(H^0(F), S) \approx H^k(F) \text{ and } \operatorname{Hom}(H^k(F), S) \approx H^0(F),$$

which implies that $\operatorname{rank} H^0(F) \cdot \operatorname{rank} S = \operatorname{rank} H^k(F)$ and $\operatorname{rank} H^k(F) \cdot \operatorname{rank} S = \operatorname{rank} H^0(F)$ and that all these ranks are finite. Thus $\operatorname{rank} S = 1$ and $H^q(F) \approx H^{k-q}(F)$ by (119).[46] □

Remark: Orientability is needed in 18.5, as is shown by the orbit map of an action of $\mathbb{S}^1$ on a Klein bottle. This is a cfs_L^1 for any field L with $\operatorname{char}(L) \neq 2$. The orbit space is an interval $[0, 1]$. Also, see the examples below 16.33.

A little work with the universal coefficient theorem yields a corresponding result over the integers:

18.6. Corollary. [11] *Suppose that $f : X \to Y$ is a $cfs_{\mathbb{Z}}^k$, and assume that X is an orientable n-$hm_{\mathbb{Z}}$ and that the Leray sheaves $\mathscr{H}^i(f; \mathbb{Z})$ have finitely generated stalks. Then Y is an $(n-k)$-$hm_{\mathbb{Z}}$ and $H^k(F; \mathbb{Z}) \approx H^0(F; \mathbb{Z})$. Moreover, Y is orientable if each $\mathscr{H}^i(f; \mathbb{Z})$ is constant.*[47] □

18.7. Let us now turn to actions of the p-adic group A_p on generalized manifolds. There is the p-adic solenoid Σ_p, which is the inverse limit of circle groups $\cdots \to \mathbb{S}^1 \to \mathbb{S}^1$, where all the maps are the p-fold covering maps. Then $A_p \subset \Sigma_p$ are compact topological groups and $\Sigma_p / A_p \approx \mathbb{S}^1$. Continuity implies that

$$H^1(\Sigma_p; \mathbb{Z}) \approx Q_p,$$

[45] Look at the spectral sequence restricted to any stalk.
[46] Another proof can be based on Exercise 38.
[47] Also see Exercise 39.

the group of rational numbers whose denominators are powers of p. Also,

$$H^1(\Sigma_p;\mathbb{Q}) \approx \mathbb{Q},$$
$$H^1(\Sigma_p;\mathbb{Z}_p) \approx 0,$$
$$H^1(\Sigma_p;\mathbb{Z}_q) \approx \mathbb{Z}_q$$

for any prime $q \neq p$. All cohomology vanishes above dimension 1.

Now suppose that A_p acts freely on an orientable n-hm_L M^n. We shall assume that A_p preserves orientation; this is no loss of generality in studying the local properties of the orbit space since for L a prime field or $\mathbb{Z}$ (so that $\operatorname{Aut}(L)$ is finite) there is a subgroup of finite index in A_p that does preserve orientation. Let

$$N^{n+1} = \Sigma_p \times_{A_p} M^n$$

with the notation of IV-9. Then N^{n+1} is a bundle over $\mathbb{S}^1 = \Sigma_p/A_p$ with fiber M^n, and so N^{n+1} is an $(n+1)$-hm_L. Moreover, Σ_p acts freely on N^{n+1} and $N^{n+1}/\Sigma_p \approx M^n/A_p$. Now, Σ_p is a compact connected group, and so it operates trivially on the cohomology of N^{n+1} by II-11.11. (This is the primary reason for making this construction here.)

The orbit map

$$f : N^{n+1} \to N^{n+1}/\Sigma_p \approx M^n/A_p$$

is a Vietoris map when $L = \mathbb{Z}_p$, and so we conclude from 16.33 that

$$\boxed{M^n/A_p \text{ is an } (n+1)\text{-}hm_L \text{ if } L = \mathbb{Z}_p.}$$

18.8. Lemma. *In the above situation the Leray sheaf $\mathscr{H}^s(f;L)$ is constant with stalks $H^s(\Sigma_p;L)$ for any L.*

Proof. Let $\pi_i : \Sigma_p \to \mathbb{S}^1$ be the ith map in the inverse system defining Σ_p. Let $K_i = \operatorname{Ker}\pi_i$ and let $f_i : N^{n+1}/K_i \to N^{n+1}/\Sigma_p$ be the orbit map of $\Sigma_p/K_i \approx \mathbb{S}^1$. For $U \subset N^{n+1}/\Sigma_p$ open with $\overline{U}$ compact, $\overline{U}^\bullet = f^{-1}(\overline{U}) = \varprojlim f_i^{-1}(\overline{U})$, and so $H^s(\overline{U};L) \approx \varinjlim H^s(f_i^{-1}(\overline{U});L)$ by continuity. Since the presheaf $U \mapsto H^s(f_i^{-1}(\overline{U});L)$ generates the Leray sheaf $\mathscr{H}^s(f_i;L)$, we conclude that $\mathscr{H}^s(f;L) \approx \varinjlim \mathscr{H}^s(f_i;L)$. But the sheaf $\mathscr{H}^1(f_i;L)$, being the Leray sheaf in the top dimension of an action of a compact Lie group, is constant, as is shown in IV-14. Hence $\mathscr{H}^1(f;L)$ is constant. The same statement in other degrees is trivial. □

By the lemma the orbit map f is a cfs^1_L when L is a field of characteristic $q \neq p$. Thus, by 18.5, we have that

$$\boxed{M^n/A_p \text{ is an } n\text{-}hm_L \text{ if } L = \mathbb{Q} \text{ or } L = \mathbb{Z}_q,\ q \neq p \text{ prime.}}$$

If M^n is an n-$cm_{\mathbb{Z}}$, then it is also an n-cm_L for all fields L (for $\mathbb{Q}$ since M^n is $clc_{\mathbb{Z}}$). Thus the foregoing statements might seem contradictory,

but in fact, they appear to be logically consistent, and spaces with similar properties have been presented in the literature.[48]

18.9. Now suppose that we are given an orientation preserving free action of A_p on an orientable n-$cm_{\mathbb{Z}}$. We retain the notation of the previous subsection. Then the orbit map $f : N^{n+1} \to N^{n+1}/\Sigma_p \approx M^n/A_p$ is a $cfs^1_{\mathbb{Z}}$ by 18.8, but it does not satisfy the finite-generation condition of 18.6, and so we cannot assert that the orbit space is a homology manifold over $\mathbb{Z}$; indeed, it cannot be one by the foregoing results and the universal coefficient theorem; also see 18.11.

There are two tools we shall use to study this situation further. Consider the diagram

$$\begin{array}{ccc} N^{n+1} & \xrightarrow{\pi} & N^{n+1}/A_p \\ f\downarrow \Sigma_p & & g\downarrow \mathbb{S}^1 \\ N^{n+1}/\Sigma_p & = & N^{n+1}/\Sigma_p \approx M^n/A_p. \end{array}$$

Note that $N^{n+1}/A_p = (\Sigma_p \times_{A_p} M^n)/A_p \approx (\Sigma_p/A_p) \times M^n/A_p \approx \mathbb{S}^1 \times M^n/A_p$. Let $U \subset M^n/A_p \approx N^{n+1}/\Sigma_p$ be an open set, and put $U^\bullet = f^{-1}U$ and $U' = g^{-1}U \approx U \times \mathbb{S}^1$. The first tool is the Gysin sequence of $f : U^\bullet \to U$ as derived from the Leray spectral sequence of f. The second tool is the Borel spectral sequence

$$E_2^{s,t} = H^s(B_{Ap}; H_c^t(U^\bullet)) \Longrightarrow H_c^{s+t}(U'),$$

whose coefficients are constant because $A_p \subset \Sigma_p$, and the latter operates trivially on $H_c^t(U^\bullet)$ by II-11.11.

In order to use the Borel spectral sequence, we must calculate the cohomology of B_{A_p}:

18.10. Lemma. *We have*

$$H^s(B_{A_p};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \textit{for } s = 0, \\ Q_p/\mathbb{Z}, & \textit{for } s = 2, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. A direct proof of this can be found in [21]. We shall give a much more efficient indirect proof. The cohomology of B_{Σ_p} follows immediately from the Leray spectral sequence of $E_{\Sigma_p} \to B_{\Sigma_p}$, which gives

$$H^s(B_{\Sigma_p};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } s = 0, \\ Q_p, & \text{for } s > 0 \text{ even}, \\ 0, & \text{otherwise} \end{cases}$$

and also shows that the map

$$Q_p \approx H^{2s}(B_{\Sigma_p}; H^1(\Sigma_p)) = E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} = H^{2s+2}(B_{\Sigma_p}; H^0(\Sigma_p))$$

[48] See the bibliography in [15], which is complete on this subject.

is an isomorphism. The diagram

$$\begin{array}{ccc} E_{\Sigma_p} & \rightarrow & B_{A_p} \\ \downarrow \eta & & \downarrow \varsigma \\ B_{\Sigma_p} & = & B_{\Sigma_p} \end{array}$$

induces a map of the spectral sequence of ζ into that of η. On the $E_2^{0,1}$ terms this is just the canonical inclusion

$$\mathbb{Z} \approx H^0(B_{\Sigma_p}; H^1(\mathbb{S}^1)) \rightarrow H^0(B_{\Sigma_p}; H^1(\Sigma_p)) \approx Q_p,$$

and on the $E_2^{2s,1}$ terms, for $s > 0$, it is the isomorphism

$$Q_p \approx H^{2s}(B_{\Sigma_p}; H^1(\mathbb{S}^1)) \rightarrow H^{2s}(B_{\Sigma_p}; H^1(\Sigma_p)) \approx Q_p.$$

It follows that

$$\mathbb{Z} \approx H^0(B_{\Sigma_p}; H^1(\mathbb{S}^1)) = E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} = H^2(B_{\Sigma_p}; H^0(\mathbb{S}^1)) \approx Q_p,$$

in the spectral sequence of ζ, is the canonical inclusion and that

$$Q_p \approx H^{2s}(B_{\Sigma_p}; H^1(\mathbb{S}^1)) = E_2^{2s,1} \xrightarrow{d_2} E_2^{2s+2,0} = H^{2s+2}(B_{\Sigma_p}; H^0(\mathbb{S}^1)) \approx Q_p$$

is an isomorphism for $s > 0$. Therefore, the only nonzero terms of the $E_3^{p,q}$ for ζ are $E_3^{0,0} \approx \mathbb{Z}$ and $E_3^{2,0} \approx Q_p/\mathbb{Z}$, whence $H^*(B_{A_p}; \mathbb{Z})$ is as claimed. □

Finally, we collect some miscellaneous facts about free actions of A_p on generalized manifolds. Again we stress that no examples of such actions are known, so the following theorem may be completely empty.

18.11. Theorem. *For an orientation-preserving free action of A_p on an orientable n-cm$_{\mathbb{Z}}$ M^n with $Y = M^n/A_p$ let $\mathfrak{H}_c^i(Y)$ denote the precosheaf $U \mapsto H_c^i(U; \mathbb{Z})$ on Y. Then we have:*

(a) $\dim_{\mathbb{Z}} Y = n + 2$,[49]

(b) *Y is an $(n+1)$-cm$_{\mathbb{Z}_p}$,*

(c) *Y is an n-cm$_{\mathbb{Z}_q}$ for $q \neq p$ prime,*

(d) *Y is an n-cm$_{\mathbb{Q}}$,*

(e) *$\mathfrak{H}_c^i(Y)$ is locally zero for $i \neq n, n+2$,*

(f) *$\mathfrak{H}_c^n(Y)$ is equivalent to the precosheaf $U \mapsto H_c^n(U^\bullet; \mathbb{Z}) \approx H_1^c(U^\bullet; \mathbb{Z})$,*

(g) *$\mathfrak{H}_c^{n+2}(Y)$ is the constant cosheaf $Q_p/\mathbb{Z}$,*

(h) *$\mathscr{H}_i(Y; \mathbb{Z}) = 0$ for $i \neq n, n+1$,*

[49]Part (a) also holds for all *effective* actions; see [21] and [86].

(i) $\mathscr{H}_n(Y;\mathbb{Z})$ *is constant with stalks* Q_p,

(j) $\mathscr{H}_{n+1}(Y;\mathbb{Z})$ *is constant with stalks* $\widehat{Q_p/\mathbb{Z}}$,

(k) Y *is* j-*clc*$_{\mathbb{Z}}$ *for all* $j \neq 2$,

(l) Y *is not* 2-*clc*$_{\mathbb{Z}}$,

(m) $f : N^{n+1} \to Y$ *is a cfs*$^1_{\mathbb{Z}}$, *where* $N^{n+1} = \Sigma_p \times_{A_p} M^n$,

(n) $n \geq 3$.

Proof. Statements (b), (c), (d), and (m) have already been shown. In the Borel spectral sequence $E_2^{s,t} = H^s(B_{A_p}; H_c^t(U^\bullet)) \Longrightarrow H_c^{s+t}(U')$, for U connected, we have that $E_2^{s,t} = 0$ for $s > 2$ or $t > n+1$. Therefore $H_c^{n+3}(U') \approx H^2(B_{A_p}; H_c^{n+1}(U^\bullet)) \approx Q_p/\mathbb{Z}$, and $H_c^i(U') = 0$ for $i > n+3$. Since $U' \approx U \times \mathbb{S}^1$, we deduce that $H_c^i(U) = 0$ for $i > n+2$ and $H_c^{n+2}(U) \approx Q_p/\mathbb{Z}$ naturally in U, proving (g). By II-16.14 this also gives (a).

To prove (e), note that for an open set V with $\overline{V} \subset U$, the image of $H^i(U^\bullet) \to H^i(V^\bullet)$ is finitely generated by II-17.5. Since the direct limit of these is the cohomology of an orbit Σ_p, it follows that V can be chosen so small that $H^i(U^\bullet) \to H^i(V^\bullet)$ is zero for $i \neq 0, 1$. By duality, $H_j(U^\bullet) \to H_j(V^\bullet)$ is zero for $j \neq n, n+1$. By the sequences (9) on page 292 we derive that

$$\mathrm{Hom}(H_c^j(U^\bullet), \mathbb{Z}) \to \mathrm{Hom}(H_c^j(V^\bullet), \mathbb{Z}) \quad \text{is zero for } j \neq n, n+1$$

and

$$\mathrm{Ext}(H_c^j(U^\bullet), \mathbb{Z}) \to \mathrm{Ext}(H_c^j(V^\bullet), \mathbb{Z}) \quad \text{is zero for } j \neq n+1, n+2. \tag{120}$$

Now the maps $H_c^j(V^\bullet) \to H_c^j(U^\bullet)$ have finitely generated images by II-17.5. It follows fairly easily that V can be taken so small that $H_c^j(V^\bullet) \to H_c^j(U^\bullet)$ is zero for $j < n$. Inductively assume that V can be taken so small that $H_c^j(V) \to H_c^j(U)$ is zero for $j < j_0$. Then by the universal coefficient theorem, we have that $H_c^j(V; Q_p) \to H_c^j(U; Q_p)$ is zero for $j < j_0$. The Gysin sequence of f has the segment

$$H_c^{j_0-2}(V; Q_p) \to H_c^{j_0}(V) \to H_c^{j_0}(V^\bullet).$$

and it follows from II-17.3 that V can be taken so that $H_c^{j_0}(V) \to H_c^{j_0}(U)$ is zero, as long as $j_0 < n$. This proves (e) except in dimension $n+1$. Now, from the Borel spectral sequence we get that

$$H_c^{n+2}(U') \approx H^2(B_{A_p}; H_c^n(U^\bullet)) \approx Q_p/\mathbb{Z} \otimes H_c^n(U^\bullet)$$

is all torsion. Thus its direct factor $H_c^{n+1}(U)$ is all torsion. The Gysin sequence has the segment

$$H_c^{n-1}(U; Q_p) \to H_c^{n+1}(U) \xrightarrow{f^*} H_c^{n+1}(U^\bullet) \approx \mathbb{Z},$$

so that $f^* = 0$ here. Since the left-hand term is locally zero (by the universal coefficient theorem), so is the middle term, giving (e).

For part (f) consider the segment

$$H_c^{n-2}(U;Q_p) \to H_c^n(U) \to H_c^n(U^\bullet) \to H_c^{n-1}(U;Q_p)$$

of the Gysin sequence. The two outer terms are locally zero, so that the middle arrow is a local isomorphism as claimed.

There is the exact sequence

$$0 \to \mathscr{E}xt(\mathfrak{H}_c^{i+1}(Y),\mathbb{Z}) \to \mathscr{H}_i(Y;\mathbb{Z}) \to \mathscr{H}om(\mathfrak{H}_c^i(Y),\mathbb{Z}) \to 0,$$

from which part (h) follows immediately.[50] It also gives $\mathscr{H}_{n+1}(Y;\mathbb{Z}) \approx \mathscr{E}xt(\mathfrak{H}_c^{n+2}(Y),\mathbb{Z}) \approx \widehat{Q_p/\mathbb{Z}}$, proving (j); see (109) on page 368. Also

$$\mathscr{H}_n(Y;\mathbb{Z}) \approx \mathscr{H}om(\mathfrak{H}_c^n(Y),\mathbb{Z})$$

since $\mathfrak{H}_c^{n+1}(Y)$ is locally zero. By (f) this is the same as the sheaf generated by the presheaf $U \mapsto \operatorname{Hom}(H_c^n(U^\bullet),\mathbb{Z})$, which is the same as the sheaf generated by the presheaf $U \mapsto H_n(U^\bullet) \approx H^1(U^\bullet)$ since $\operatorname{Ext}(H_c^{n+1}(U^\bullet),\mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Z},\mathbb{Z}) = 0$. This is just the Leray sheaf of f. Thus $\mathscr{H}_n(Y;\mathbb{Z}) \approx \mathscr{H}^1(f;\mathbb{Z})$, which is constant with stalks $H^1(\Sigma_p) \approx Q_p$ by 18.8, proving (i).

To prove (k) and (l), note that by (e), for U an open neighborhood of $y \in Y$ there are open neighborhoods $W \subset V \subset U$ of y with $H_c^{i+1}(W) \to H_c^{i+1}(V)$ and $H_c^i(V) \to H_c^i(U)$ both zero for $i < n-1$. By (9) on page 292 and II-17.3, $H_i(U) \to H_i(W)$ is zero for $i < n-1$. By duality, assuming as we may that U and W are paracompact, we have that $H^{n-i+1}(U) \to H^{n-i+1}(W)$ is zero for $i < n-1$; that is, $H^j(U) \to H^j(W)$ is zero for $j > 2$. This shows that Y is j-$clc_{\mathbb{Z}}$ for $j > 2$. Consider the Gysin sequences of $f : A^\bullet \to A$ for compact connected neighborhoods A of a given point $y \in Y$. For $B \subset \operatorname{int} A$ another such set, we have the diagram

$$\begin{array}{ccccccccc}
0 & \to & H^1(A) & \to & H^1(A^\bullet) & \to & H^0(A;Q_p) & \to & H^2(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1(B) & \to & H^1(B^\bullet) & \to & H^0(B;Q_p) & \to & H^2(B) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^1(y^\bullet) & \to & H^0(y;Q_p) & &
\end{array}$$

in which $H^i(B^\bullet) \to H^i(y^\bullet)$ has finitely generated image by II-17.5. Since y is a retract of A, $H^0(A;Q_p) \to H^0(y;Q_p) \approx Q_p$ is onto and hence does not have a finitely generated image. By II-17.3, the image of $H^2(A) \to H^2(B)$ is not finitely generated. Thus Y is not 2-$clc_{\mathbb{Z}}$. The left part of the diagram shows that $H^1(A) \to H^1(B)$ has finitely generated image, and since the direct limit of these is $H^1(y) = 0$, it follows that Y is $clc_{\mathbb{Z}}^1$.

Part (n) follows from (a), (d), and 16.32. □

[50]The case $i = n-1$ uses (120).

Remark: Since Y is not $clc_{\mathbb{Z}}$, it may not satisfy change of rings between $\mathbb{Z}$ and $\mathbb{Q}$ coefficients. Indeed, by (j) we have that $\mathscr{H}_{n+1}(Y;\mathbb{Q}) \neq 0$ with $\mathbb{Z}$ as base ring even though Y is an n-$cm_{\mathbb{Q}}$. If one is trying to find a contradiction in the homology of this situation, care must be taken on this point.

19 The transfer homomorphism in homology

In homology, the strict analogue of II-11.1 is false (except for covering maps), as seen in 6.6, so that to define the transfer homomorphism we must use a method different from that employed in II-19.

Let G be a finite group of homeomorphisms of the locally compact space X and let $\pi : X \to X/G$ be the orbit map. The following lemma is basic. (The parenthetical cases are used only for dealing with arbitrary coefficient sheaves.)

19.1. Lemma. *If $\mathscr{L}$ is a sheaf on X/G that is c-soft (respectively, c-fine or replete), then $\pi^*\mathscr{L}$ is also c-soft (respectively, c-fine or replete).*[51]

Proof. For any integer t let

$$U_t = \{y \in X/G \mid \pi^{-1}(y) \text{ contains at least } t \text{ points}\}.$$

Then U_t is open and

$$X/G = U_1 \supset U_2 \supset U_3 \supset \cdots$$

is a finite filtration of X/G. Correspondingly, $\mathscr{L}$ is filtered by

$$\mathscr{L} = \mathscr{L}_{U_1} \supset \mathscr{L}_{U_2} \supset \cdots$$

Let $U_t^\bullet = \pi^{-1}(U_t)$. Then $\pi^*\mathscr{L}$ is filtered by the subsheaves

$$(\pi^*\mathscr{L})_{U_t^\bullet} \approx \pi^*(\mathscr{L}_{U_t}).$$

Let $\mathscr{L}$ be c-soft. Since an extension of a c-soft sheaf by a c-soft sheaf is c-soft, an easy decreasing induction on t shows that in order to prove that $\pi^*\mathscr{L}$ is c-soft we need only show that

$$(\pi^*\mathscr{L})_{U_t^\bullet}/(\pi^*\mathscr{L})_{U_{t+1}^\bullet} \approx (\pi^*(\mathscr{L} \mid U_t - U_{t+1}))^X$$

is c-soft. By II-9.12 it further suffices to show that $\pi^*(\mathscr{L} \mid U_t - U_{t+1})$ is c-soft. But c-softness is a local property, by II-9.14, and $\pi|U_t - U_{t+1}$ is a local homeomorphism. Thus $\pi^*\mathscr{L}$ is c-soft when $\mathscr{L}$ is c-soft.

Repletion is also a local property, as follows easily from II-Exercise 13. This property is also easily seen to be preserved by extensions. Thus, again, an inductive argument shows that if $\mathscr{L}$ is replete, then so is $\pi^*\mathscr{L}$.

[51] Compare IV-6.7.

If $\mathscr{L}$ is c-fine, then it is a module over a c-soft sheaf $\mathscr{R}$ of rings with unit; $\mathscr{R} = \mathscr{H}om(\mathscr{L}, \mathscr{L})$, for example. Then $\pi^*\mathscr{L}$ is a module over the c-soft sheaf $\pi^*\mathscr{R}$. Thus $\pi^*\mathscr{L}$ is c-fine. □

19.2. Proposition. *Let $\mathscr{L}^*$ be a replete resolution of L on X/G, let Φ be a family of supports on X/G, and let $\mathscr{B}$ be a sheaf on X/G. Assume that at least one of the following four conditions holds:*

(a) $\Phi = c$.

(b) $\pi^{-1}\Phi$ *is paracompactifying and* $\dim_{\pi^{-1}\Phi} X < \infty$.

(c) $(\mathscr{B}, \Phi)$ *is elementary.*[52]

(d) $(\pi^*\mathscr{B}, \pi^{-1}\Phi)$ *is elementary.*

Then there is a natural isomorphism

$$\boxed{H_p(\Gamma_\Phi(\mathscr{D}(\pi\pi^*\mathscr{L}^*) \otimes \mathscr{B})) \approx H_p^{\pi^{-1}\Phi}(X; \pi^*\mathscr{B}).}$$

Proof. Since π is proper, we have, by 2.6, that

$$\mathscr{D}(\pi\pi^*\mathscr{L}^*) = \pi\mathscr{D}(\pi^*\mathscr{L}^*). \tag{121}$$

By (21) on page 301 we also have that

$$\mathscr{D}(\pi\pi^*\mathscr{L}^*) \otimes \mathscr{B} = (\pi\mathscr{D}(\pi^*\mathscr{L}^*)) \otimes \mathscr{B} \approx \pi(\mathscr{D}(\pi^*\mathscr{L}^*) \otimes \pi^*\mathscr{B}), \tag{122}$$

and by applying Γ_Φ we obtain the natural isomorphism

$$\Gamma_\Phi(\mathscr{D}(\pi\pi^*\mathscr{L}^*) \otimes \mathscr{B}) \approx \Gamma_{\pi^{-1}\Phi}(\mathscr{D}(\pi^*\mathscr{L}^*) \otimes \pi^*\mathscr{B})$$

by IV-5.2 and IV-5.4(3). The proposition, in cases (a), (b), and (d), now follows from 3.5, 5.6, and 19.1. It is easily seen that $(\pi^*\mathscr{B}, \pi^{-1}\Phi)$ is elementary when $(\mathscr{B}, \Phi)$ is elementary, so that case (c) follows from case (d). □

Now, by II-19, G acts, in a natural way, as a group of automorphisms of $\pi\pi^*\mathscr{L}^*$.

The reader may verify the fact that the action of G on $\pi\pi^*\mathscr{L}^*$ induces, via 19.2, the natural action of G on $H_*^{\pi^{-1}\Phi}(X; \pi^*\mathscr{B})$. There are the maps

$$\mathscr{L}^* \underset{\mu}{\overset{\beta}{\rightleftarrows}} \pi\pi^*\mathscr{L}^*$$

of II-19, where $\mu\beta$ is multiplication by $\operatorname{ord}(G)$ and $\beta\mu = \sigma = \sum g$. We have the induced homomorphisms

$$\mathscr{D}(\mathscr{L}^*) \otimes \mathscr{B} \underset{\beta'\otimes 1}{\overset{\mu'\otimes 1}{\rightleftarrows}} \mathscr{D}(\pi\pi^*\mathscr{L}^*) \otimes \mathscr{B}. \tag{123}$$

Applying $H_*(\Gamma_\Phi(\bullet))$ and using 19.2, we obtain the homomorphisms

$$\boxed{H_p^\Phi(X/G; \mathscr{B}) \underset{\pi_*}{\overset{\mu_*}{\rightleftarrows}} H_p^{\pi^{-1}\Phi}(X; \pi^*\mathscr{B})}$$

[52] Recall that (c) is satisfied for any Φ when $\mathscr{B} = L$.

when (a), (b), (c), or (d) of 19.2 is satisfied, since $\mathscr{L}^*$ can be taken to be $\mathscr{I}^*(X/G; L)$. (The fact that π_* is indeed induced from $\beta' \otimes 1$ follows easily from its definition in Section 4.) The map μ_* is called the *transfer* homomorphism. It is clear that

$$\boxed{\mu_*\pi_* = \sigma_* = \sum g_*}$$

and that

$$\boxed{\pi_*\mu_* \text{ is multiplication by } \operatorname{ord}(G).}$$

With the appropriate restrictions on supports and coefficient sheaves, we have the diagram

$$\begin{array}{ccc} H_m^\Phi(X/G;\mathscr{A}) \otimes H_\Psi^p(X/G;\mathscr{B}) & \xrightarrow{\cap} & H_{m-p}^{\Phi\cap\Psi}(X/G;\mathscr{A}\otimes\mathscr{B}) \\ {\scriptstyle \mu_*\otimes\pi^*}\big\downarrow\big\uparrow{\scriptstyle \pi_*\otimes\mu^*} & & {\scriptstyle \mu_*}\big\downarrow\big\uparrow{\scriptstyle \pi_*} \\ H_m^{\pi^{-1}\Phi}(X;\pi^*\mathscr{A}) \otimes H_{\pi^{-1}\Psi}^p(X;\pi^*\mathscr{B}) & \xrightarrow{\cap} & H_{m-p}^{\pi^{-1}(\Phi\cap\Psi)}(X;\pi^*(\mathscr{A}\otimes\mathscr{B})), \end{array}$$

for which we claim that

$$\boxed{\mu_*(a \cap b) = \mu_*(a) \cap \pi^*(b)}$$

for $a \in H_m^\Phi(X/G;\mathscr{A})$ and $b \in H_\Psi^p(X/G;\mathscr{B})$. To see this, note that the diagram

$$\begin{array}{ccc} \Gamma_\Phi(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) & \longrightarrow & \Gamma_{\Phi\cap\Psi}(\mathscr{D}(\mathscr{L}^*)\otimes\mathscr{A}\otimes\mathscr{B}) \\ \big\downarrow{\scriptstyle \mu'\otimes 1} & & \big\downarrow{\scriptstyle \mu'} \\ \Gamma_\Phi(\mathscr{D}(\pi\pi^*\mathscr{L}^*)\otimes\mathscr{A})\otimes\Gamma_\Psi(\mathscr{B}) & \longrightarrow & \Gamma_{\Phi\cap\Psi}(\mathscr{D}(\pi\pi^*\mathscr{L}^*)\otimes\mathscr{A}\otimes\mathscr{B}) \\ \big\downarrow{\scriptstyle \approx\otimes\pi^*} & & \big\downarrow{\scriptstyle \approx} \\ \Gamma_{\pi^{-1}\Phi}(\mathscr{D}(\pi^*\mathscr{L}^*)\otimes\pi^*\mathscr{A})\otimes\Gamma_{\pi^{-1}\Psi}(\pi^*\mathscr{B}) & \longrightarrow & \Gamma_{\pi^{-1}(\Phi\cap\Psi)}(\mathscr{D}(\pi^*\mathscr{L}^*)\otimes\pi^*(\mathscr{A}\otimes\mathscr{B})) \end{array}$$

commutes. This shows that the formula holds for $p = 0$. The general formula then follows from II-6.2. There is also the formula

$$\boxed{\pi_*(a) \cap \mu^*(b) = \pi_*(a \cap \pi^*\mu^*(b))}$$

for $a \in H_m^{\pi^{-1}\Phi}(X;\pi^*\mathscr{A})$ and $b \in H_{\pi^{-1}\Psi}^p(X;\pi^*\mathscr{B})$, by the more general formula (57) on page 337. The reader can verify the additional formulas

$$\boxed{\pi_*(\mu_*(a) \cap b) = a \cap \mu^*(b)}$$

for $a \in H_m^\Phi(X/G;\mathscr{A})$ and $b \in H_{\pi^{-1}\Psi}^p(X;\pi^*\mathscr{B})$, and, for the cup product,

$$\boxed{\mu^*(b \cup \pi^*(c)) = \mu^*(b) \cup c}$$

for $b \in H_{\pi^{-1}\Phi}^p(X;\pi^*\mathscr{A})$ and $c \in H_\Psi^q(X/G;\mathscr{B})$. The reader should also verify these formulas for the case of the transfer for proper covering maps discussed in Section 6.

Consider a subgroup $H \subset G$ and the map $\varpi : X/H \to X/G$. Then, using the fact from (48) on page 140 that $(\pi\pi^*\mathscr{B})^H \approx \varpi\varpi^*\mathscr{B}$, we have that

$$H_p(\Gamma_\Phi(\mathscr{D}((\pi\pi^*\mathscr{L}^*)^H)\otimes\mathscr{B})) \approx H_p(\Gamma_\Phi(\mathscr{D}(\varpi\varpi^*\mathscr{L}^*)\otimes\mathscr{B})) \approx H_p^{\varpi^{-1}\Phi}(X/H;\varpi^*\mathscr{B})$$

under conditions (a), (b), or (c). This leads to the transfer ν_*:

$$\boxed{H_p^\Phi(X/G;\mathscr{B}) \underset{\varpi_*}{\overset{\nu_*}{\rightleftarrows}} H_p^{\varpi^{-1}\Phi}(X/H;\varpi^*\mathscr{B})}$$

with $\varpi_*\nu_* = \operatorname{ord}(G/H)$.

19.3. Theorem. *If X is an orientable n-hm$_\mathbb{Z}$ and if G is an effective finite group of orientation preserving transformations of X, then the homology sheaf $\mathscr{H}_n(X/G;\mathbb{Z})$ is constant with stalks $\mathbb{Z}$.*

Proof. We may assume that X is connected. We will use some items from the next section. Let $S = \{\breve{x} \in X/G \,|\, G_x \neq \{e\}\}$. We claim that $\dim_\mathbb{Z} S \leq n-2$. If, on the contrary, $\dim_\mathbb{Z} S = n$, then $S^\bullet$ contains an open set U. Since U is a finite union of the fixed-point sets of elements $g \in G$ of prime order, some such g must leave fixed an open set $U \subset S^\bullet$. Since g has prime order, Smith theory implies that g leaves all of X fixed, and so G is not effective. If $\dim_\mathbb{Z} S = n-1$, then similarly there is a g of prime order whose fixed-point set F separates X locally at some point x. Then there is a connected invariant open set U containing x such that $U - F$ is disconnected. By Smith theory, F is an $(n-1)$-*hm*$_{\mathbb{Z}_2}$ and $g^2 = e$. It follows easily that locally at x, $U - F$ consists of exactly two components, say V and W. If g preserves a component of $U - F$ then, putting $h(x) = x$ for $x \in W$ and $h(x) = g(x)$ for $x \notin W$, h is a nontrivial transformation of period two fixing an open subset of U. This is contrary to Smith theory. Thus g must permute V and W. It follows from an easy Mayer-Vietoris argument that g reverses orientation, contrary to assumption. This proves our contention that $\dim_\mathbb{Z} S \leq n-2$.

If $x \in X - S^\bullet$, then there is a connected open neighborhood U of x such that $G(U)$ is the *disjoint* union of the sets gU, $g \in G$. Consider the transfer

$$\mu_* : H_n(\pi U) \to H_n(G(U)) = \bigoplus_{g\in G} H_n(gU).$$

Let $b \in H_n(\pi U) \approx \mathbb{Z}$ be a generator and let $\mu_*(b) = \bigoplus b_g$, where $b_g \in H_n(gU)$. Since $g_*\mu_* = \mu_*$, we have that $b_g = g_*(b_e)$. Then $\operatorname{ord}(G)b = \pi_*\mu_*(b) = \pi_*(\sum b_g) = \pi_*(\sum g_*(b_e)) = \operatorname{ord}(G)\pi_*(b_e)$. Thus $\pi_*(b_e) = b$. Consequently, b_e is a generator of $H_n(U)$. That is, the composition

$$H_n(\pi U) \xrightarrow{\mu_*} H_n(G(U)) \xrightarrow{j_*} H_n(U)$$

is an isomorphism, where j_* is restriction.

Now let $y \in X/G$ be arbitrary and let W be a connected open neighborhood of y. Since $\dim_{\mathbb{Z}} S \le n-2$, the map $H_c^n(W-S) \to H_c^n(W)$ is an isomorphism, and it follows that the restriction $H_n(W) \to H_n(W-S)$ is an isomorphism. Let $x \in X - S^\bullet$ and U be as above with $\pi(x) \in W$. Let W' be the component of $W^\bullet$ containing U. We have the commutative diagram

$$\begin{array}{ccccccc} H_n(W-S) & \xleftarrow{\approx} & H_n(W) & \xrightarrow{\mu_*} & H_n(W^\bullet) & \xrightarrow{r_*} & H_n(W') \\ \downarrow{\approx} & & \downarrow & & \downarrow & & \downarrow{\approx} \\ H_n(\pi U) & \xleftarrow{=} & H_n(\pi U) & \xrightarrow{\mu_*} & H_n(G(U)) & \xrightarrow{j_*} & H_n(U) \end{array}$$

in which the isomorphism on the left follows from 8.7 since $W-S$ is a connected orientable n-$hm_{\mathbb{Z}}$.[53] Since $j_*\mu_*$ is an isomorphism, as just shown, so is $r_*\mu_*$. Therefore the composition of $r_*\mu_*$ with the inverse of the isomorphism $\mathbb{Z} \approx H_n(X) \xrightarrow{\approx} H_n(W')$ gives isomorphisms $H_n(W) \xrightarrow{\approx} \mathbb{Z}$ compatible with restrictions. □

20 Smith theory in homology

We shall now apply the constructions of Section 19 to the special case in which G is cyclic of prime order p with generator g, and L is a field of characteristic p. These assumptions are made throughout this section. We shall also retain the notation of Section 19 and assume that at least one of the conditions (a), (b), or (c) of 19.2 is satisfied when we are dealing with *global* homology [as in (127), (132), and (134)] rather than with homology sheaves. Let F denote the fixed-point set of the action.

As in II-19, we put

$$\tau = 1 - g$$

and

$$\sigma = 1 + g + g^2 + \cdots + g^{p-1} = \tau^{p-1}.$$

Let $\mathscr{L}^* = \mathscr{I}^*(X/G; L)$, and put

$$\mathscr{A}^* = \pi^*\mathscr{L}^*.$$

Note that $\mathscr{D}(\pi\mathscr{A}^*) = \pi\mathscr{D}(\mathscr{A}^*)$ by (121).

By (49) on page 141 we have the exact sequence

$$0 \to \rho(\pi\mathscr{A}^*) \xrightarrow{i} \pi\mathscr{A}^* \xrightarrow{\bar\rho \oplus j} \bar\rho(\pi\mathscr{A}^*) \oplus \mathscr{L}_F^* \to 0. \tag{124}$$

Since $\mathscr{L}^*$ is a $\mathscr{C}^0(X/G;L)$-module, it follows that $\pi\mathscr{A}^* = \pi\pi^*\mathscr{L}^*$ is also a $\mathscr{C}^0(X/G;L)$-module by IV-Exercise 8, and hence it is c-fine. The operations of G on $\pi\mathscr{A}^*$ are clearly $\mathscr{C}^0(X/G;L)$-module isomorphisms, and it follows that $\rho(\pi\mathscr{A}^*)$ is c-fine for both $\rho = \sigma$ and $\rho = \tau$.

[53] It is orientable since $\mu_*\pi_* = \operatorname{ord} G$, whence $H_n(W-S) \neq 0$.

Thus, applying $\mathscr{D}$ to (124), we obtain the exact sequence

$$0 \to \mathscr{D}(\bar{\rho}(\pi\mathscr{A}^*)) \oplus \mathscr{D}(\mathscr{L}_F^*) \to \mathscr{D}(\pi\mathscr{A}^*) \to \mathscr{D}(\rho(\pi\mathscr{A}^*)) \to 0, \tag{125}$$

which remains exact upon tensoring with the sheaf $\mathscr{B}$ of L-modules on X/G [since L is a field here], and upon applying Γ_Φ [since either Φ is paracompactifying or $\mathscr{B}$ is elementary by assumption (a), (b), or (c)].

Note that $\mathscr{D}(\mathscr{L}_F^*) \otimes \mathscr{B} \approx (\mathscr{D}(\mathscr{L}^*|F) \otimes \mathscr{B}|F)^{X/G}$ by 5.4, so that

$$\mathscr{H}_p(\mathscr{D}(\mathscr{L}_F^*) \otimes \mathscr{B}) \approx \mathscr{H}_p(F;\mathscr{B})^{X/G} \tag{126}$$

by (16) on page 294 and

$$H_p(\Gamma_\Phi(\mathscr{D}(\mathscr{L}_F^*) \otimes \mathscr{B})) \approx H_p^{\Phi|F}(F;\mathscr{B}) \tag{127}$$

under one of the conditions (a) through (c) of 19.2 by 3.5 or 5.6.

We *define*

$$\boxed{{}_\rho\mathscr{H}_p(X;\mathscr{B}) = \mathscr{H}_p(\mathscr{D}(\rho(\pi\mathscr{A}^*)) \otimes \mathscr{B})} \tag{128}$$

(which is a sheaf on X/G) and

$$\boxed{{}_\rho H_p^\Phi(X;\mathscr{B}) = H_p(\Gamma_\Phi(\mathscr{D}(\rho(\pi\mathscr{A}^*)) \otimes \mathscr{B})).} \tag{129}$$

Note that ${}_\rho\mathscr{H}_p(X;\mathscr{B}) = \mathscr{S}\!\mathit{heaf}(U \mapsto {}_\rho H_p(\pi^{-1}U;\mathscr{B}))$.

Since π is proper and finite-to-one, the direct image functor $\mathscr{F} \mapsto \pi\mathscr{F}$ is exact by IV-Exercise 9. Thus we have

$$\begin{aligned} \mathscr{H}_p(\mathscr{D}(\pi\mathscr{A}^*) \otimes \mathscr{B}) &\approx \mathscr{H}_p(\pi(\mathscr{D}(\mathscr{A}^*) \otimes \pi^*\mathscr{B})) && \text{by (122)} \\ &\approx \pi\mathscr{H}_p(\mathscr{D}(\mathscr{A}^*) \otimes \pi^*\mathscr{B}) && \text{exactness of } \pi \\ &\approx \pi\mathscr{H}_p(X;\pi^*\mathscr{B}) && \text{by (16), p. 294.} \end{aligned} \tag{130}$$

[Note also that $\pi\mathscr{H}_p(X;\pi^*\mathscr{B}) \approx \mathscr{H}_p(\pi;\pi^*\mathscr{B})$ directly from the definitions and the exactness of π.]

Using (126), (128), and (130), the sequence (125) induces the exact sequence

$$\boxed{\begin{aligned} \cdots \to {}_\rho\mathscr{H}_{p+1}(X;\mathscr{B}) &\xrightarrow{\partial_*} {}_{\bar\rho}\mathscr{H}_p(X;\mathscr{B}) \oplus \mathscr{H}_p(F;\mathscr{B})^{X/G} \\ &\xrightarrow{\bar\rho_* \oplus j_*} \pi\mathscr{H}_p(X;\pi^*\mathscr{B}) \xrightarrow{i_*} {}_\rho\mathscr{H}_p(X;\mathscr{B}) \to \cdots \end{aligned}} \tag{131}$$

of sheaves on X/G. This is called the "local Smith sequence."

Under any of the conditions (a) through (c) of 19.2, we also obtain the "global" exact Smith sequence

$$\boxed{\begin{aligned} \cdots \to {}_\rho H_{p+1}^\Phi(X;\mathscr{B}) &\xrightarrow{\partial_*} {}_{\bar\rho}H_p^\Phi(X;\mathscr{B}) \oplus H_p^{\Phi|F}(F;\mathscr{B}) \\ &\xrightarrow{\bar\rho_* \oplus j_*} H_p^{\pi^{-1}\Phi}(X;\pi^*\mathscr{B}) \xrightarrow{i_*} {}_\rho H_p^\Phi(X;\mathscr{B}) \xrightarrow{\delta_*} \cdots \end{aligned}} \tag{132}$$

By II-(50) on page 142 we have that $\sigma(\pi\mathscr{A}^*) \approx \mathscr{L}^*_{(X-F)/G}$. Thus, by (94) on page 361, we have the natural isomorphism

$$\boxed{{}_\sigma\mathscr{H}_*(X;\mathscr{B}) \approx \mathscr{H}_*(X/G,F;\mathscr{B}).} \tag{133}$$

Moreover, under any of the conditions (a) through (c) of 19.2, we have

$$\boxed{{}_\sigma H_*^\Phi(X;\mathscr{B}) \approx H_*^\Phi(X/G,F;\mathscr{B})} \tag{134}$$

by (95) on page 361 [or by the remark below 5.10 in case (c)]. Note that in case (c) this is isomorphic to $H_*^{\Phi\cap(X-F)/G}((X-F)/G;\mathscr{B})$ by 5.10.

We shall now confine our attention to the case of coefficients in L, and we shall take $L = \mathbb{Z}_p$ for the most part. The following result is the "local" Smith Theorem. By using the Smith sequence (131) of local homology groups, its proof is no harder than the proof of the corresponding "global" theorem.

20.1. Theorem. *Let X be an n-hm_L where $L = \mathbb{Z}_p$. Then F is the disjoint union of open and closed subsets F_r of F for $r \le n$ such that F_r is an r-hm_L. Moreover, the orientation sheaf $\mathscr{O}_{F_r}$ of F_r is canonically isomorphic to the restriction $\mathscr{O}_X|F_r$ of the orientation sheaf $\mathscr{O}_X$ of X. Similarly, $\mathscr{H}_k(X/G,F;L)|F_r$ is isomorphic to $\mathscr{O}_X|F_r$ for $r < k \le n$ and is zero for $k \le r$ and for $k > n$.*

Proof. Since $\dim_L X < \infty$, it follows that $\dim_L X/G < \infty$ and $\dim_L F < \infty$.[54] Thus, by (133), ${}_\sigma\mathscr{H}_k(X;L) = 0$ for k large. Using (131) inductively for both $\rho = \sigma$, τ, we see that ${}_\rho\mathscr{H}_k(X;L) = 0 = \mathscr{H}_k(F;L)$ for $k > n$. Now restrict the sequence (131) to F and use the notation $\mathscr{H}_k(\rho)$ for ${}_\rho\mathscr{H}_k(X;L)|F$. Then (131) becomes the exact sequence

$$\cdots \to \mathscr{H}_{k+1}(\rho) \xrightarrow{\partial_*} \mathscr{H}_k(\bar\rho) \oplus \mathscr{H}_k(F) \xrightarrow{\bar\rho_* \oplus j_*} \mathscr{H}_k(X)|F \xrightarrow{i_*} \mathscr{H}_k(\rho) \xrightarrow{\partial_*} \cdots \tag{135}$$

Note that the composition

$$\mathscr{H}_n(X)|F \xrightarrow{i_*} \mathscr{H}_n(\rho) \xrightarrow{\rho_*} \mathscr{H}_n(X)|F \tag{136}$$

is just operation by ρ_* (which is $1 - g_*$ or $1 + g_* + \cdots + g_*^{p-1}$). Now, g_* is an automorphism of period p, and the stalks of $\mathscr{H}_n(X)|F$ (which are isomorphic to $L = \mathbb{Z}_p$) have no nontrivial automorphisms of period p. Thus the composition (136) is zero for both $\rho = \sigma$, τ. It follows that

$$\mathscr{H}_n(\bar\rho) \oplus \mathscr{H}_n(F) \xrightarrow[\approx]{\bar\rho_* \oplus j_*} \mathscr{H}_n(X)|F = \mathscr{O}_X|F \tag{137}$$

[54] See II-Exercise 11.

is an isomorphism. Also

$$\mathscr{H}_{k+1}(\rho) \xrightarrow{\partial_*} \mathscr{H}_k(\bar{\rho}) \oplus H_k(F) \tag{138}$$

is an isomorphism for $k < n$, since $\mathscr{H}_k(X)|F = 0$ for $k \neq n$.

Using the elementary fact that if $\mathscr{A} \oplus \mathscr{B}$ is a locally constant sheaf with stalks L on F then the sets $\{x \mid \mathscr{A}_x = 0\}$ and $\{x \mid \mathscr{B}_x = 0\}$ are disjoint closed sets with union F, we see immediately from (137) and (138) that $F = \bigcup_{r=0}^{n} F_r$, where $F_r = \{x \in F \mid \mathscr{H}_r(F)_x \neq 0\}$, and that the F_r are disjoint closed sets. Moreover, restricting the sheaves to F_r, we see that the homomorphisms

$$\mathscr{O}_X|F_r \xleftarrow{\bar{\rho}_*} \mathscr{H}_n(\bar{\rho}) \xrightarrow{\partial_*} \mathscr{H}_{n-1}(\rho) \xrightarrow{\partial_*} \cdots \xrightarrow{\partial_*} \mathscr{H}_{r+1}(\eta) \xrightarrow{\partial_*} \mathscr{H}_r(F_r) \tag{139}$$

are isomorphisms for $r < n$ [and $\mathscr{H}_n(F_n) \xrightarrow{\approx} \mathscr{O}_X|F_n$], where $\eta = \rho$ or $\eta = \bar{\rho}$ according as $n - r$ is even or odd. Moreover, it also follows that $\mathscr{H}_k(\rho) = 0$ for $k \leq r$ and $\mathscr{H}_k(F_r) = 0$ for $k \neq r$. The theorem follows. □

With the same notation and assumptions we have the following additional information:

20.2. Proposition. *The isomorphism $\mathscr{O}_X|F_r \approx \mathscr{O}_{F_r}$ resulting from* (139) *is independent of ρ $(= \sigma, \tau)$. Moreover, $n - r$ is even if $p \neq 2$.*

Proof. This is trivial for $p = 2$, so that we may assume that $p > 2$. The commutative diagram (55) on page 143 for $\mathscr{A} = \mathscr{A}^*$ and $\mathscr{B} = \mathscr{L}^*$ induces a commutative diagram in homology that in particular takes the form

$$\begin{array}{ccccccc}
\cdots \to & \mathscr{H}_k(\sigma) \oplus \mathscr{H}_k(F_r) & \xrightarrow{\sigma_* \oplus j_*} & \mathscr{H}_k(X)|F_r & \xrightarrow{i_*} & \mathscr{H}_k(\tau) & \xrightarrow{\partial_*} \cdots \\
& \downarrow{\scriptstyle t_* \oplus 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle s_*} & \\
\cdots \to & \mathscr{H}_k(\tau) \oplus \mathscr{H}_k(F_r) & \xrightarrow{\tau_* \oplus j_*} & \mathscr{H}_k(X)|F_r & \xrightarrow{i_*} & \mathscr{H}_k(\sigma) & \xrightarrow{\partial_*} \cdots \\
& \downarrow{\scriptstyle s_* \oplus 0} & & \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle t_*} & \\
\cdots \to & \mathscr{H}_k(\sigma) \oplus \mathscr{H}_k(F_r) & \xrightarrow{\sigma_* \oplus j_*} & \mathscr{H}_k(X)|F_r & \xrightarrow{i_*} & \mathscr{H}_k(\tau) & \xrightarrow{\partial_*} \cdots
\end{array}$$

where s_* is induced by the inclusion $\sigma(\pi\mathscr{A}^*) \hookrightarrow \tau(\pi\mathscr{A}^*)$ and t_* is induced by operation by $\tau^{p-2} : \tau(\pi\mathscr{A}^*) \to \sigma(\pi\mathscr{A}^*)$ (and $\pi\mathscr{A}^* \to \pi\mathscr{A}^*$). Since $p > 2$ and $\mathscr{H}_n(X)|F_r$ has stalks $L = \mathbb{Z}_p$, we see that t_* is zero on $\mathscr{H}_n(X)|F_r$. Obviously, t_* is also zero on $\mathscr{H}_k(F_r)$. We have the commutative diagram

$$\begin{array}{ccccccccccc}
\mathscr{O}_X|F_r & \xleftarrow{\sigma_*} & \mathscr{H}_n(\sigma) & \xrightarrow{\partial_*} & \mathscr{H}_{n-1}(\tau) & \xrightarrow{\partial_*} \cdots \xrightarrow{\partial_*} & \mathscr{H}_n(\rho) & \xrightarrow{\partial_*} & \mathscr{H}_r(F_r) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\
\mathscr{O}_X|F_r & \xleftarrow{\tau_*} & \mathscr{H}_n(\tau) & \xrightarrow{\partial_*} & \mathscr{H}_{n-1}(\sigma) & \xrightarrow{\partial_*} \cdots \xrightarrow{\partial_*} & \mathscr{H}_n(\bar{\rho}) & \xrightarrow{\partial_*} & \mathscr{H}_r(F_r)
\end{array}$$

where $h = s_*$ or $h = t_*$ according as $\rho = \tau$ or $\rho = \sigma$, and $k = 1$ or $k = 0$ according as $h = s_*$ or $h = t_*$. Since the horizontal maps are isomorphisms, we must have that $k = 1$. Thus $\rho = \tau$, and $n - r$ is even. □

Note that if X is clc_L^∞ (and hence is a cm_L), then so is F by II-Exercise 44.

Exercises

1. If $\cdots \to \mathfrak{A}_{p+1} \to \mathfrak{A}_p \to \mathfrak{A}_{p-1} \to \cdots$ is a sequence of cosheaves on X, we say that it is *locally exact* if it is of order two as a sequence of precosheaves and if the precosheaf $U \mapsto H_p(\mathfrak{A}_*(U))$ is locally zero for all p; see 12.1.

   If $0 \to \mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}'' \to 0$ is a locally exact sequence of cosheaves on X, then show that $\mathfrak{A}'(U) \to \mathfrak{A}(U) \to \mathfrak{A}''(U) \to 0$ is exact for all open $U \subset X$. If, moreover, $\mathfrak{A}''$ is a flabby cosheaf, show that $\mathfrak{A}'(U) \to \mathfrak{A}(U)$ is a monomorphism. If $\mathfrak{A}$ and $\mathfrak{A}''$ are both flabby cosheaves, then show that $\mathfrak{A}'$ is flabby.

2. For any c-soft sheaf $\mathscr{A}$ and any injective L-module M define the natural homomorphism

$$\mathscr{A} \to \mathscr{H}om(\Gamma_c\mathscr{H}om(\Gamma_c\mathscr{A}, M), M)$$

   of 1.13 directly without the use of 1.9. [*Hint*: Use (19) on page 300.]

3. Ⓢ Suppose that $\mathfrak{A}$ is a covariant functor on the open sets of X to L-modules that satisfies the following two conditions:

   (a) If U and V are open, then the sequence

$$\mathfrak{A}(U\cap V) \xrightarrow{\alpha} \mathfrak{A}(U) \oplus \mathfrak{A}(V) \xrightarrow{\beta} \mathfrak{A}(U\cup V) \to 0$$

   is exact, where $\alpha = i_{U,U\cap V} - i_{V,U\cap V}$ and $\beta = i_{U\cup V,U} + i_{U\cup V,V}$.

   (b) If $\{U_\alpha\}$ is an upward-directed family of open sets with $U = \bigcup U_\alpha$, then $\mathfrak{A}(U) = \varinjlim \mathfrak{A}(U_\alpha)$.

   Show that $\mathfrak{A}$ is a cosheaf and conversely [12].

4. Ⓢ Show that the precosheaf $\mathfrak{S}_*$ of 1.3 is a flabby cosheaf.

5. Ⓢ Let $\mathfrak{A}$ and $\mathfrak{B}$ be precosheaves on X. Show that $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent in the sense of 16.4 $\Leftrightarrow$ there exists a precosheaf $\mathfrak{C}$ and local isomorphisms $\mathfrak{A} \to \mathfrak{C}$ and $\mathfrak{B} \to \mathfrak{C}$.

6. Ⓢ If $\dim_L X = n$, show that $\mathfrak{H}_c^n(X;\mathscr{A})$ [that is, $U \mapsto H_c^n(U;\mathscr{A})$] is a cosheaf for any sheaf $\mathscr{A}$ of L-modules.

7. Ⓢ Let $f : X \to Y$ and let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and Y respectively. Define an f-homomorphism $h : \mathscr{A} \to \mathscr{B}$ to be a map of spaces such that the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{h} & \mathscr{B} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

   commutes and such that the induced maps $h_x : \mathscr{A}_x \to \mathscr{B}_{f(x)}$ are all homomorphisms. Show that there is a natural homomorphism

$$h_* : H_p^{\Psi(c)}(X;\mathscr{A}) \to H_p^{\Psi}(Y;\mathscr{B})$$

   and investigate its properties.

8. Ⓢ Let $\mathscr{A}$ be any sheaf on X and Φ a paracompactifying family of supports on X. Let U_1 and U_2 be open subsets of X with $U = U_1 \cup U_2$ and $V = U_1 \cap U_2$. Derive the following Mayer-Vietoris sequences (with coefficients in $\mathscr{A}$):

$$\cdots \to H_p^{\Phi|V}(V) \to H_p^{\Phi|U_1}(U_1) \oplus H_p^{\Phi|U_2}(U_2) \to H_p^{\Phi|U}(U) \to H_{p-1}^{\Phi|V}(V) \to \cdots,$$

$$\cdots \to H_p^{\Phi}(X, V) \to H_p^{\Phi}(X, U_1) \oplus H_p^{\Phi}(X, U_2) \to H_p^{\Phi}(X, U) \to \cdots$$

9. Ⓢ Let $\mathscr{M}$ be an *elementary* sheaf on X and Φ an arbitrary family of supports on X. If U_1 and U_2 are open, with $X = U_1 \cup U_2$ and $V = U_1 \cap U_2$, derive the Mayer-Vietoris sequence

$$\cdots \to H_p^{\Phi}(X; \mathscr{M}) \to H_p^{\Phi \cap U_1}(U_1; \mathscr{M}) \oplus H_p^{\Phi \cap U_2}(U_2; \mathscr{M}) \to H_p^{\Phi \cap V}(V; \mathscr{M}) \to \cdots$$

If F_1 and F_2 are closed, with $F = F_1 \cap F_2$ and $X = F_1 \cup F_2$, derive the Mayer-Vietoris sequence

$$\cdots \to H_p^{\Phi|F}(F; \mathscr{M}) \to H_p^{\Phi|F_1}(F_1; \mathscr{M}) \oplus H_p^{\Phi|F_2}(F_2; \mathscr{M}) \to H_p^{\Phi}(X; \mathscr{M}) \to \cdots$$

provided that $(\mathscr{M}, \Phi)$ is elementary.

10. Use 5.3 to define homology groups of a general (i.e., not locally compact) space X with supports in some family Φ consisting of locally compact subspaces of X (e.g., $\Phi = c$) and develop properties of these groups.

11. Ⓢ Let $\{A_\alpha\} \supset \Phi$ be an upward-directed system of locally closed subspaces of X. If Φ is arbitrary and $(\mathscr{A}, \Phi)$ is elementary or if Φ is paracompactifying and $\mathscr{A}$ is arbitrary, show that the canonical map

$$\varinjlim H_*^{\Phi|A_\alpha}(A_\alpha; \mathscr{A}) \to H_*^{\Phi}(X; \mathscr{A})$$

is an isomorphism.

12. Call a sheaf or a cosheaf weakly torsion-free (respectively, weakly divisible) if its value on each open set is torsion-free (respectively, divisible). If $\mathfrak{A}_*$ is a weakly torsion-free differential cosheaf, show that $\mathscr{D}(\mathfrak{A}_*)$ is weakly divisible. If $\mathscr{A}^*$ is a c-soft and weakly divisible differential sheaf, show that $\mathfrak{D}(\mathscr{A}^*)$ is weakly torsion-free. [*Hint*: Use (19) on page 300.]

13. If X is clc_L^∞ and $\mathfrak{L}_*$ is a flabby, weakly torsion free quasi-coresolution of L on X, show that

$$H_p^c(X; M) \approx H_p(\mathfrak{L}_*(X) \otimes M)$$

for any L-module M. [*Hint*: Consider the proof of 12.11 and use the algebraic universal coefficient theorem.]

14. Define, under suitable conditions, a relative cap product

$$\cap : H_m^{\Phi}(X, A; \mathscr{A}) \otimes H_\Psi^p(X; \mathscr{B}) \to H_{m-p}^{\Phi \cap \Psi}(X, A; \mathscr{A} \otimes \mathscr{B})$$

(for $A \subset X$ locally closed) and discuss the conditions under which it is defined.

15. Define a cap product

$$\cap : {}_sH_m^{\Phi}(X, A; \mathscr{A}) \otimes H_\Psi^p(X; \mathscr{B}) \to {}_sH_{m-p}^{\Phi \cap \Psi}(X, A; \mathscr{A} \otimes \mathscr{B})$$

when Φ and $\Phi \cap \Psi$ are paracompactifying.

16. For $x \in X$, show that $s\mathscr{H}_p(X;\mathscr{A})_x \approx sH_p(X, X - \{x\};\mathscr{A})$.

17. Ⓢ With the notation of 4.2 prove the following statements:

    (a) $\Phi \subset \Psi \Rightarrow \Psi^{\#} \subset \Phi^{\#}$.

    (b) $\Phi \subset \Phi^{\#\#}$; $\Phi^{\#} = \Phi^{\#\#\#}$.

    (c) If $f : X \to Y$ and $\Phi \supset c$ is a family of supports on Y, then $(f^{-1}\Phi)^{\#} = \Phi^{\#}(c)$.

18. Let $f : X \to Y$ and let Ψ be a family of supports on X. Show that there exists a spectral sequence of *sheaves* with

$$\mathscr{E}_2^{p,q} = \mathscr{H}^p_\Psi(f;\mathscr{H}_{-q}(X;L)) \Longrightarrow \mathscr{H}^\Psi_{-p-q}(f;L),$$

and generalize to arbitrary coefficient sheaves.

19. Ⓢ Determine the sheaves $\mathscr{H}_p(X;\mathbb{Z})$ for the one-point union $X = \mathbb{S}^1 \vee \mathbb{S}^2$ and compute the E_2 terms of the (absolute) spectral sequence of 8.4 for it.

20. Let X be an n-whm_L and let Φ be paracompactifying. Let $A \subset B \subset X$ be closed subspaces. Then show that

$$H^p_{\Phi|B}(B, A; \mathscr{O} \otimes \mathscr{A}) \approx H^{\Phi|X-A}_{n-p}(X - A, X - B;\mathscr{A}),$$

naturally, for any sheaf $\mathscr{A}$ on X.

21. Ⓢ If $\{\mathscr{A}_\lambda\}$ is a direct system of sheaves on X and $\mathscr{A} = \varinjlim \mathscr{A}_\lambda$, then show that there is a natural isomorphism

$$\varinjlim H^c_p(X;\mathscr{A}_\lambda) \approx H^c_p(X;\mathscr{A}).$$

22. Let X be an HLC space and let Φ and Ψ be paracompactifying families of supports on X. Show that the cap product (52) on page 336 coincides with the singular cap product

$$sH^\Phi_m(X;\mathscr{A}) \otimes sH^p_\Psi(X;\mathscr{B}) \to sH^{\Phi\cap\Psi}_{m-p}(X;\mathscr{A}\otimes\mathscr{B})$$

(defined either in the classical manner or similarly to (52)) via the isomorphisms of III-1 and 12.21 or 12.17 when either $\dim_L X < \infty$ or $\Psi \supset \Phi = c$.

23. Ⓢ Let the finite group G act on the locally connected, orientable n-$hm_{\mathbb{Q}}$ X, preserving orientation. Then show that X/G is also an orientable n-$hm_{\mathbb{Q}}$.

24. Show that there is a natural isomorphism

$${}_\Delta H^*_\Phi(X;M) \approx H^*(\Gamma_\Phi(\mathscr{D}^*(\mathfrak{S}_*;M)))$$

for Φ paracompactifying and with the left-hand side defined as in I-7. [*Hint*: Show that the natural maps

$$\mathrm{Hom}(S_*(U), M) \to \mathrm{Hom}(S_*(U), M^*) \leftarrow \mathrm{Hom}(\mathfrak{S}_*(U), M^*)$$

of presheaves induce homology isomorphisms. Then pass to the associated sheaves.]

25. Let X and Y be locally compact and clc_L^∞ and assume that $H_p^c(X;L)$ is finitely generated for each p. Let M be any L-module. Show that there is a split exact Künneth sequence

$$\bigoplus_{p+q=n} H^p(X;L)\otimes H^q(Y;M) \rightarrowtail H^n(X\times Y;M) \twoheadrightarrow \bigoplus_{p+q=n+1} H^p(X;L) * H^q(Y;M).$$

[*Hint*: Use 12.7 and Section 14.] Also show that this does not hold when X and Y are interchanged in the sequence. In addition, show that the condition of finite generation is necessary even for $M = L$. [Note that if $L = \mathbb{Z}$ or is a field, the condition on $H_*^c(X;L)$ is equivalent to the analogous condition on $H^*(X;L)$; see 14.7.]

In particular, there is a split exact sequence

$$0 \to H^p(X;L)\otimes M \to H^p(X;M) \to H^{p+1}(X;L) * M \to 0$$

under the above hypotheses.

26. Ⓢ If X is a separable metric, locally connected, and locally compact space, show that $H_0(X;L) \approx \prod_\alpha L_\alpha$, where $L_\alpha \approx L$ and α ranges over the *compact* components of X.

[Note that this implies that if X is a separable metric n-cm_L with orientation sheaf $\mathscr{O}$, then $H^n(X;\mathscr{O}) \approx \prod_\alpha L_\alpha$, as above.]

27. Let (X,A) be a locally compact pair, let $\mathscr{A}$ be a sheaf on X, and let Φ be a family of supports on X. Let K range over the members of $\Phi|A$. Show that the canonical map

$$\varinjlim H_*^\Phi(X,K;\mathscr{A}) \to H_*^\Phi(X,A;\mathscr{A})$$

is an isomorphism $\Leftrightarrow$ the canonical map

$$\varinjlim H_*(K;\mathscr{A}) \to H_*^{\Phi|A}(A;\mathscr{A})$$

is an isomorphism. [See 5.3 and 5.6 for cases to which this applies, and also note that by 5.10 the homology of (X,K) can sometimes be replaced by that of $X-K$.]

28. Ⓢ Suppose that X is a separable metric n-cm_L. Show that $H^p(X,U;L)=0$ for $p>n$ and for any open set $U\subset X$. Conclude that $\mathscr{Z}^n(X;L)$ is flabby. [*Hint*: Use the fact, from Exercise 26, that $H^n(V;L)=0$ when $V\subset X$ is open and has no compact components.]

29. Show that the exact sequences

$$0 \to \operatorname{Ext}(H_c^{p+1}(X;L),L) \to H_p(X;L) \to \operatorname{Hom}(H_c^p(X;L),L) \to 0$$

and the corresponding sequences for F and for $U = X - F$, where F is closed, are compatible with the homology and cohomology sequences of the pair (X,F) (i.e., the sequences (36) on page 313 with $\Phi = cld$ and II-10.3 with $\Phi = c$).

30. Let $(\mathscr{M},\Phi)$ be elementary on X. Let U and V be open sets in X and put $F = X-U$ and $G = X-V$. Show that there is a diagram (with coefficients

in $\mathscr{M}$ and where the supports for each subspace A appearing are taken in $\Phi \cap A$)

$$\begin{array}{ccccccc}
H_p(F\cap G) & \to & H_p(F) & \to & H_p(F\cap V) & \xrightarrow{\partial} & H_{p-1}(F\cap G)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H_p(G) & \to & H_p(X) & \to & H_p(V) & \xrightarrow{\partial} & H_{p-1}(G)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H_p(G\cap U) & \to & H_p(U) & \to & H_p(U\cap V) & \xrightarrow{\partial} & H_{p-1}(G\cap U)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H_{p-1}(F\cap G) & \to & H_{p-1}(F) & \to & H_{p-1}(F\cap V) & \xrightarrow{\partial} & H_{p-2}(F\cap G)
\end{array}$$

in which the rows and columns are the homology sequences (36) on page 313 and that commutes except for the lower right-hand corner square, which anticommutes.

31. Let X be an n-$hm_{\mathbb{Z}}$ with orientation sheaf $\mathscr{O} = \mathscr{H}_n(X;\mathbb{Z})$ that has stalks $\mathbb{Z}$. Let $G = \mathbb{Z}_p$, p prime, act on M with fixed set $F = \mathop{+}\limits_r F_r$, where F_r is an r-$hm_{\mathbb{Z}_p}$ as in 20.1. Show that

$$\mathscr{H}_k(X/G;\mathbb{Z})|F_r \approx \begin{cases} \mathscr{O}\otimes\mathbb{Z}_p, & \text{for } r+2\le k<n,\ k-r \text{ even},\\ \mathscr{O}, & \text{for } k=n \text{ if } n-r \text{ is even},\\ 0, & \text{otherwise.}\end{cases}$$

(In particular, in the neighborhood of F_{n-1} (for $p = 2$) X/G is an n-$hm_{\mathbb{Z}}$ with boundary F_{n-1}, and in the neighborhood of F_{n-2}, X/G is an n-$hm_{\mathbb{Z}}$.) [*Hint*: Compare II-19.11.]

32. Ⓢ Assume that $\dim_\Phi X = n < \infty$ and either that Φ is paracompactifying or that $\mathscr{A}$ is elementary. Show that

$$H_p^\Phi(X;\mathscr{A}) \approx \begin{cases} 0, & \text{for } p>n,\\ \Gamma_\Phi(\mathscr{H}_n(X;\mathscr{A})), & \text{for } p=n.\end{cases}$$

Moreover, if $\mathscr{A}$ is torsion-free, show that $H_n^\Phi(X;\mathscr{A})$ is torsion-free.

33. Ⓢ If $\dim_L X < \infty$ and $X \neq \varnothing$, show that $\mathscr{H}_p(X;L) \neq 0$ for some p. [Also note Example 8.12.]

34. Let L be a field and suppose that the point $x \in X$ has a countable fundamental system of neighborhoods. Show that $\mathscr{H}_p(X;L)_x = 0 \Leftrightarrow$ for each neighborhood U of x there is an open neighborhood $V \subset U$ of x such that the homomorphism $H_c^p(V;L) \to H_c^p(U;L)$ is trivial.

35. Ⓢ Let X be a hereditarily paracompact n-hm_L, let Φ be a paracompactifying family of supports on X, let $\mathscr{M}$ be an elementary sheaf on X, and let $A \subset X$ be an arbitrary subspace, *not* necessarily locally compact. Then show that $H_\Phi^p(X, X-A;\mathscr{M}) = 0$ for $p < n - \underline{\dim}_{\Phi,L}^X A$. (This shows, for example, that if M^n is a paracompact n-manifold and $A \subset M^n$ is totally disconnected, then the restriction $H_\Phi^p(M^n;\mathbb{Z}) \to H_{\Phi\cap M-A}^p(M^n - A;\mathbb{Z})$ is an isomorphism for $p < n-1$ and a monomorphism if $p = n-1$.)

36. Let L be a field and assume that X is first countable. Consider the precosheaf $\mathfrak{H}_c^p(X;L)$ of Exercise 6. Show that

$$\mathscr{H}_p(X;L) = 0 \ \Leftrightarrow\ \mathfrak{H}_c^p(X;L) \text{ is locally zero.}$$

If $\dim_L X \leq n$, then show that

$$\mathscr{H}_n(X;L) = 0 \Leftrightarrow \mathfrak{H}_c^n(X;L) = 0 \Leftrightarrow \dim_L X < n,$$

and hence that $\dim_L X = n \Leftrightarrow \mathscr{H}_n(X;L) \neq 0$.

For $\mathbb{Z}$ coefficients then as far as the author knows, it may be possible that $\dim_{\mathbb{Z}} X = n$ but $\mathscr{H}_n(X;\mathbb{Z}) = 0$; see 18.11.

37. Ⓢ Suppose that $\dim_L Y < \infty$, that Y is clc_L^∞, and that each $\mathscr{H}_p(Y;L)$ has finitely generated stalks. Let $f : X \to Y$ be a covering map (generally with infinitely many sheets), let Φ be a paracompactifying family of supports on Y, and let $\mathscr{A}$ be a sheaf on X. Then show that there is a natural isomorphism
$$H_p^\Phi(Y; f\mathscr{A}) \xrightarrow{\approx} H_p^{f^{-1}\Phi}(X;\mathscr{A}).$$

38. Ⓢ Let $f : X \to Y$ be a cfs_L^k where L is a field. Assume that X is clc_L^∞ and that $\dim_L Y = n < \infty$. Then show that Y is an n-hm_L $\Leftrightarrow$ the homology sheaves $\mathscr{H}_p(f;L)$ are all locally constant. In this case, also show that the stalks of $\mathscr{H}^q(f;L)$ and $\mathscr{H}_{n+q}(f;L)$ are of the same finite rank for all q.

39. Ⓢ Let $f : X \to Y$ be a $cfs_{\mathbb{Z}}^k$ with connected fibers, and assume that the stalks $H^i(F;\mathbb{Z})$ of $\mathscr{H}^i(f;\mathbb{Z})$ are finitely generated for all i. If X is an n-$cm_{\mathbb{Z}}$, then show that the following statements are equivalent and imply that Y is an $(n-k)$-$hm_{\mathbb{Z}}$:
    (a) Some fiber has an orientable neighborhood.
    (b) $H^k(F;\mathbb{Z}) \approx \mathbb{Z}$.
    (c) Every fiber has an orientable neighborhood.

40. Show that the map
$$\mathfrak{A}(X) \to \operatorname{Hom}(\Gamma_c(\mathscr{H}om(\mathfrak{A}, M)), M)$$
of 1.13 is a monomorphism.

41. Ⓢ If X is an n-$whm_{\mathbb{Z}}$, then show that $\dim_{\mathbb{Z}} X \leq n+1$.

42. Ⓢ Suppose that $\dim_L X < \infty$ and that $A \subset X$ is closed with $\dim_L A = 1$, e.g., A an arc. Show that there is an exact sequence
$$0 \to H_c^1(A; \mathscr{H}_{n+1}(X;L)|A) \to H_n^c(X, X-A; L) \to \Gamma_c(\mathscr{H}_n(X;L)|A) \to 0.$$

Chapter VI

Cosheaves and Čech Homology

In this short chapter we study the notion of cosheaves on general topological spaces and we go into it a bit deeper than was done in Chapter V. Our main purpose, in this chapter, is to obtain isomorphism criteria connecting various homology theories. With the minor exceptions of some definitions, and excepting the sections (10 and 11) concerning Borel-Moore homology, this chapter does not depend on Chapter V.

Basic definitions and simple results are given in Section 1. The notions of local triviality and local exactness are treated in Section 2. Local isomorphisms and the notion of "equivalence" of precosheaves are discussed in Section 3.

Section 4 is the backbone of this chapter. It initiates the study of Čech homology with coefficients in a precosheaf. This is used in Section 5 to produce a functor $\mathfrak{Cosheaf}$ that is a reflector from the category of precosheaves that are locally isomorphic to cosheaves to the category of cosheaves. This is analogous to the functor $\mathcal{Sheaf}$ in the theory of sheaves and presheaves.

The basic spectral sequence attached to an open covering of a space and a differential cosheaf on the space is developed in Section 6. These spectral sequences are of central importance in the remainder of this chapter.

Section 7 treats the notion of a coresolution and a semi-coresolution. Also in this section the spectral sequences of Section 6 are applied to prove theorems for coresolutions analogous to the fundamental theorems of sheaves in Chapter IV. This is generalized to relative Čech homology in Section 8.

In Section 9, a method of Mardešić is used to remove a (global) paracompactness assumption in the fundamental theorems when dealing with locally paracompact spaces.

In Sections 10, 11, and 12, the general theory is applied to get some comparison results for Borel-Moore homology, singular homology, and a modified version of Borel-Moore homology. The results in these sections complement the uniqueness theorems of V-12 and V-13.

An application to acyclic open coverings is given in Section 13. In Section 14 the main spectral sequences are generalized to apply to maps, and some consequences of this are given.

Throughout this chapter, L will denote a given principal ideal domain, which will be taken as the base ring.

1 Theory of cosheaves

Here we shall present a more elaborate discussion of the theory of cosheaves than the short exposition in Chapter V. We do not require the spaces here to be locally compact. In particular, there is no known fact about flabby cosheaves analogous to that on locally compact spaces that they are the cosheaves of sections with compact supports of c-soft sheaves.

1.1. Definition. *A precosheaf* $\mathfrak{A}$ *is called an "epiprecosheaf" if*

$$\bigoplus_{\alpha} \mathfrak{A}(U_\alpha) \xrightarrow{f} \mathfrak{A}(U) \to 0$$

is exact for every collection of open sets U_α *with* $U = \bigcup_\alpha U_\alpha$ *and where* $f = \sum_\alpha i_{U,U_\alpha}$.

The following result shows that the cokernel of a homomorphism of cosheaves is a cosheaf. The proof is an elementary diagram chase, which will be omitted, and similarly with the second proposition.

1.2. Proposition. *Let* $\mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}'' \to 0$ *be an exact sequence of precosheaves. If* $\mathfrak{A}'$ *is an epiprecosheaf and* $\mathfrak{A}$ *is a cosheaf, then* $\mathfrak{A}''$ *is a cosheaf.* □

1.3. Proposition. *Let* $0 \to \mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}'' \to 0$ *be an exact sequence of precosheaves. If* $\mathfrak{A}$ *is an epiprecosheaf and* $\mathfrak{A}''$ *is a cosheaf then* $\mathfrak{A}'$ *is an epiprecosheaf.* □

1.4. Proposition. *Let* $\mathfrak{A}$ *be a precosheaf. Then* $\mathfrak{A}$ *is a cosheaf* $\Leftrightarrow$ *the following two conditions are satisfied:*

(a) $\mathfrak{A}(U \cap V) \xrightarrow{g} \mathfrak{A}(U) \oplus \mathfrak{A}(V) \xrightarrow{f} \mathfrak{A}(U \cup V) \to 0$ *is exact for all open* U *and* V*, where* $g = \langle i_{U,U\cap V}, -i_{V,U\cap V}\rangle$ *and* $f = i_{U\cup V,U} + i_{U\cup V,V}$.

(b) *If* $\{U_\alpha\}$ *is directed upwards by inclusion, then the canonical map* $\varinjlim_\alpha \mathfrak{A}(U_\alpha) \to \mathfrak{A}(\bigcup_\alpha U_\alpha)$ *is an isomorphism.*

Proof. It suffices to prove that if (a) is satisfied then for any *finite* collection $\{U_0, \dots, U_n\}$ of open sets, the sequence

$$\bigoplus_{\langle i,j\rangle} \mathfrak{A}(U_i \cap U_j) \xrightarrow{g} \bigoplus_{i} \mathfrak{A}(U_i) \xrightarrow{f} \mathfrak{A}(U) \to 0$$

is exact, where $U = \bigcup_i U_i$. Exactness on the right is clear by an easy induction. The proof of exactness in the middle will also be by induction on n. Let $U' = U_1 \cup \cdots \cup U_n$, $U = U_0 \cup U'$, and $V = U_0 \cap U'$. Let $s_j \in \mathfrak{A}(U_j)$, $0 \le j \le n$, be such that $\sum_{j=0}^n i_{U,U_j}(s_j) = 0$ in $\mathfrak{A}(U)$. Let

$t' = \sum_{j=1}^{n} i_{U',U_j}(s_j) \in \mathfrak{A}(U')$. Then $i_{U,U_0}(s_0) + i_{U,U'}(t') = 0$, so that (a) implies that there exists an element $v \in \mathfrak{A}(U_0 \cap U')$ with

$$i_{U_0,V}(v) = s_0, \text{ and } i_{U',V}(v) = -t'.$$

Now, $V = (U_0 \cap U_1) \cup \cdots \cup (U_0 \cap U_n)$, so that there exist $v_j \in \mathfrak{A}(U_0 \cap U_j)$, for $1 \le j \le n$, with

$$v = \sum_{j=1}^{n} i_{V,U_0 \cap U_j}(v_j).$$

Therefore

$$g\left(\bigoplus_{\langle 0,j\rangle} v_j\right) = \left\langle \sum_{j=1}^{n} i_{U_0,U_0\cap U_j}(v_j), -i_{U_1,U_0\cap U_1}(v_1), \ldots, -i_{U_n,U_0\cap U_n}(v_n)\right\rangle,$$

and a short computation shows that the element

$$\bigoplus_{i=0}^{n} s_i - g\left(\bigoplus_{\langle 0,j\rangle} v_j\right) \in \bigoplus_{i=0}^{n} \mathfrak{A}(U_i)$$

has zero component in $\mathfrak{A}(U_0)$ and projects to zero in $\mathfrak{A}(U_1 \cup \cdots \cup U_n)$. Thus the result follows from the inductive assumption. □

The following simple result is basic.

1.5. Proposition. *Let $0 \to \mathfrak{A}' \xrightarrow{f} \mathfrak{A} \xrightarrow{g} \mathfrak{A}'' \to 0$ be an exact sequence of precosheaves. Assume that $\mathfrak{A}$ is a cosheaf and that $\mathfrak{A}''$ is a flabby cosheaf. Then $\mathfrak{A}'$ is a cosheaf.*

Proof. We will verify (a) and (b) of 1.4 for $\mathfrak{A}'$. Part (b) is an immediate consequence of the exactness of the direct limit functor. Part (a) follows from a diagram chase in the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & \mathfrak{A}'(U\cap V) & \to & \mathfrak{A}(U\cap V) & \to & \mathfrak{A}''(U\cap V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{A}'(U)\oplus\mathfrak{A}'(V) & \to & \mathfrak{A}(U)\oplus\mathfrak{A}(V) & \to & \mathfrak{A}''(U)\oplus\mathfrak{A}''(V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{A}'(U\cup V) & \to & \mathfrak{A}(U\cup V) & \to & \mathfrak{A}''(U\cup V) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns (except for the $\mathfrak{A}'$ column). □

1.6. Proposition. *Let $\mathfrak{A}$ be a precosheaf. Then the set of subprecosheaves of $\mathfrak{A}$ that are epiprecosheaves contains a maximum element, which will be denoted by $\widetilde{\mathfrak{A}}$. Any homomorphism $\mathfrak{B} \to \mathfrak{A}$, with $\mathfrak{B}$ an epiprecosheaf, factors through $\widetilde{\mathfrak{A}}$.*

Proof. This follows from results of the general theory of categories, but we give a direct proof. Define, by transfinite induction,

$$\begin{aligned}\mathfrak{A}_0 &= \mathfrak{A},\\ \mathfrak{A}_{\alpha+1} &= \{s \in \mathfrak{A}_\alpha(U) \mid \forall \text{ coverings } \{U_\beta\} \text{ of } U,\ s \in \operatorname{Im}(\textstyle\bigoplus \mathfrak{A}_\alpha(U_\beta) \to \mathfrak{A}(U))\},\\ \mathfrak{A}_\beta &= \textstyle\bigcap_{\alpha<\beta} \mathfrak{A}_\alpha(U) \text{ for } \beta \text{ a limit ordinal.}\end{aligned}$$

Then there exists an ordinal α with $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$, and we let $\widetilde{\mathfrak{A}} = \mathfrak{A}_\alpha$. The claimed properties are immediate. □

Note that the last statement of 1.6 implies that every homomorphism $\mathfrak{A} \to \mathfrak{B}$ restricts to a homomorphism $\widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$, and hence $\mathfrak{A} \mapsto \widetilde{\mathfrak{A}}$ is a *functor.*

2 Local triviality

Recall that a precosheaf $\mathfrak{A}$ is said to be locally zero if for each $x \in X$ and open neighborhood U of x there is an open neighborhood $V \subset U$ of x with $i_{V,U} : \mathfrak{A}(V) \to \mathfrak{A}(U)$ zero. An epiprecosheaf is locally zero $\Leftrightarrow$ it is zero by Exercise 2.

2.1. Definition. *A precosheaf $\mathfrak{A}$ is said to be "semilocally zero" if each point $x \in X$ has a neighborhood U with $i_{X,U} : \mathfrak{A}(U) \to \mathfrak{A}(X)$ zero.*

2.2. Definition. *A sequence $\mathfrak{A}' \xrightarrow{f} \mathfrak{A} \xrightarrow{g} \mathfrak{A}''$ of precosheaves is said to be "locally exact" if $g \circ f = 0$ and if the precosheaf* $\operatorname{Ker} g/\operatorname{Im} f$ *is locally zero.*

2.3. Proposition. *If $\mathfrak{A}''$ is a cosheaf and $\mathfrak{A}$ is an epiprecosheaf, then the sequence $\mathfrak{A}' \xrightarrow{f} \mathfrak{A} \xrightarrow{g} \mathfrak{A}'' \to 0$ of cosheaves is locally exact $\Leftrightarrow$ it is exact.*

Proof. The precosheaf $\operatorname{Coker} g$ is a cosheaf by 1.2 and so it is zero since it is locally zero by hypothesis. This gives exactness on the right. By 1.3, $\operatorname{Ker} g$ is an epiprecosheaf, and it follows that $\operatorname{Ker} g/\operatorname{Im} f$ is an epiprecosheaf. Since $\operatorname{Ker} g/\operatorname{Im} f$ is locally zero, it must be zero, whence $\operatorname{Ker} g = \operatorname{Im} f$. □

2.4. Proposition. *If $0 \to \mathfrak{A}' \xrightarrow{f} \mathfrak{A} \xrightarrow{g} \mathfrak{A}'' \to 0$ is a locally exact sequence of cosheaves with $\mathfrak{A}''$ flabby, then this sequence is exact. If $\mathfrak{A}$ is also flabby, then so is $\mathfrak{A}'$.*

Proof. This sequence is exact at $\mathfrak{A}$ and at $\mathfrak{A}''$ by 2.3. By 1.5, $\operatorname{Ker} g = \operatorname{Im} f$ is a cosheaf. Then $0 \to \mathfrak{A}' \to \operatorname{Im} f \to 0$ is locally exact and hence exact by 2.3. The last statement follows from an easy diagram chase. □

2.5. Corollary. *If $\mathfrak{A}_N \xrightarrow{d_N} \cdots \xrightarrow{d_3} \mathfrak{A}_2 \xrightarrow{d_2} \mathfrak{A}_1 \xrightarrow{d_1} \mathfrak{A}_0 \to 0$ is a locally exact sequence of flabby cosheaves, then it is exact, and* $\operatorname{Ker} d_N$ *is a flabby cosheaf.*

Proof. Let $\mathfrak{Z}_n = \operatorname{Ker} d_n$. Then

$$\cdots \to \mathfrak{A}_{n+3} \to \mathfrak{A}_{n+2} \to \mathfrak{A}_{n+1} \to \mathfrak{Z}_n \to 0$$

is locally exact. Also, by 2.3,

$$0 \to \mathfrak{Z}_1 \to \mathfrak{A}_1 \to \mathfrak{A}_0 \to 0$$

is exact. By 1.5, $\mathfrak{Z}_1$ is a cosheaf, and by 2.4 it is flabby. By induction we see that each $\mathfrak{Z}_n$ is a flabby cosheaf and that each sequence

$$0 \to \mathfrak{Z}_n \to \mathfrak{A}_n \to \mathfrak{Z}_{n-1} \to 0$$

is exact. □

2.6. Lemma. *The class of locally zero precosheaves is closed under the formation of subprecosheaves, quotient precosheaves, and extensions.*

Proof. All three parts may be handled simultaneously. Let $\mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}''$ be exact with $\mathfrak{A}'$ and $\mathfrak{A}''$ locally zero. Let U be open and $x \in U$. Let $V \subset U$ be a neighborhood of x such that $\mathfrak{A}''(V) \to \mathfrak{A}''(U)$ is zero and let $W \subset V$ be such that $\mathfrak{A}'(W) \to \mathfrak{A}'(V)$ is zero. Then $\mathfrak{A}(W) \to \mathfrak{A}(U)$ is zero by II-17.3. The cases of subprecosheaves and quotient precosheaves are given by taking $\mathfrak{A}' = 0$ or $\mathfrak{A}'' = 0$ respectively. □

3 Local isomorphisms

Recall from Chapter V that a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ of precosheaves is said to be a *local isomorphism* if $\operatorname{Ker} h$ and $\operatorname{Coker} h$ are locally zero.

If $\mathfrak{A}$ is an epiprecosheaf and $\mathfrak{B}$ is a cosheaf, then a local isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is necessarily an isomorphism by 1.2, 1.3, and Exercise 2.

Consider commutative diagrams of precosheaves of the form

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{g} & \mathfrak{B} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ \mathfrak{C} & \xrightarrow{k} & \mathfrak{D}. \end{array} \qquad (1)$$

3.1. Proposition. *If* (1) *is a pushout diagram and if the map f is a local isomorphism, then so is h. Dually, if* (1) *is a pullback diagram and if the map h is a local isomorphism, then so is f.*

Proof. We shall give the proof of the second part only since the two parts are analogous. If (1) is a pullback, then we may as well assume that $\mathfrak{A}$ is given by

$$\mathfrak{A}(U) = \{\langle b, c\rangle \in \mathfrak{B}(U) \times \mathfrak{C}(U) \mid h(b) = k(c)\}.$$

Moreover, g and f are given by the projections to the first and second factors respectively. We see that $\operatorname{Ker} f = \{\langle b, 0\rangle \mid h(b) = 0\}$, so that

$$0 \to \operatorname{Ker} f \xrightarrow{g} \operatorname{Ker} h$$

is exact. Similarly, if $c \in \mathfrak{C}(U)$ and if $k(c) \in \operatorname{Im} h$, then $c \in \operatorname{Im} f$, and it follows that

$$0 \to \operatorname{Coker} f \xrightarrow{k} \operatorname{Coker} h$$

is exact. The contention follows from 2.6 applied to these exact sequences. □

3.2. Corollary. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be precosheaves. Then the following two statements are equivalent:*

(a) *There exist a precosheaf $\mathfrak{A}$ and local isomorphisms $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{C}$.*

(b) *There exist a precosheaf $\mathfrak{D}$ and local isomorphisms $\mathfrak{B} \to \mathfrak{D} \leftarrow \mathfrak{C}$.* □

If one, hence both, of the conditions in 3.2 are satisfied, then $\mathfrak{B}$ and $\mathfrak{C}$ are said to be *equivalent*. That this is an equivalence relation, and hence coincides with the definition given in Chapter V, follows from 3.2 and the next lemma.

3.3. Lemma. *Composites of local isomorphisms are local isomorphisms.*

Proof. Suppose that $\mathfrak{A} \xrightarrow{f} \mathfrak{B} \xrightarrow{g} \mathfrak{C}$ are local isomorphisms. Then we have the exact sequences

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \xrightarrow{f} \operatorname{Ker} g,$$

$$\operatorname{Coker} f \xrightarrow{g} \operatorname{Coker} gf \to \operatorname{Coker} g \to 0,$$

and the result follows from 2.6. □

As we have remarked, locally isomorphic *cosheaves* are isomorphic. It is not so clear that this is true of *equivalent* cosheaves, but in fact, we will prove that in 5.7

3.4. Definition. *A precosheaf is said to be "smooth" if it is equivalent to a cosheaf.*

Later, we shall show that for any smooth precosheaf $\mathfrak{A}$ there is an associated cosheaf $\mathfrak{Cosheaf}(\mathfrak{A})$, unique up to isomorphism, and a *canonical local isomorphism* $\mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{A}$. Also, the functor $\mathfrak{Cosheaf}$ will be shown to be a reflector[1] from the category of smooth precosheaves to the category of cosheaves. This is analogous to the functor $\mathcal{S}heaf$.

[1]This means that any homomorphism $\mathfrak{A} \to \mathfrak{B}$ of smooth precosheaves factors as $\mathfrak{A} \to \mathfrak{Cosheaf}(\mathfrak{B}) \to \mathfrak{B}$.

3.5. Proposition. *Suppose that we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{h} & \mathfrak{B} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathfrak{A}' & \xrightarrow{h'} & \mathfrak{B}' \end{array}$$

of precosheaves such that α and β are local isomorphisms. Then the induced maps $\operatorname{Ker} h \to \operatorname{Ker} h'$, $\operatorname{Im} h \to \operatorname{Im} h'$, *and* $\operatorname{Coker} h \to \operatorname{Coker} h'$ *are all local isomorphisms.*

Proof. Denote kernels, images, and cokernels by $\mathfrak{K}$, $\mathfrak{I}$, and $\mathfrak{C}$ respectively. Then we have the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{K} & \to & \mathfrak{A} & \to & \mathfrak{I} & \to & 0 \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\iota} & & \\ 0 & \to & \mathfrak{K}' & \to & \mathfrak{A}' & \to & \mathfrak{I}' & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{I} & \to & \mathfrak{B} & \to & \mathfrak{C} & \to & 0 \\ & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & \mathfrak{I}' & \to & \mathfrak{B}' & \to & \mathfrak{C}' & \to & 0, \end{array}$$

which induce the exact sequences

$$0 \to \operatorname{Ker}\kappa \to \operatorname{Ker}\alpha \to \operatorname{Ker}\iota \to \operatorname{Coker}\kappa \to \operatorname{Coker}\alpha \to \operatorname{Coker}\iota \to 0,$$
$$0 \to \operatorname{Ker}\iota \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\iota \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0.$$

By hypothesis, $\operatorname{Ker}\alpha$, $\operatorname{Ker}\beta$, $\operatorname{Coker}\alpha$, and $\operatorname{Coker}\beta$ are locally zero. It follows that $\operatorname{Ker}\kappa$, $\operatorname{Ker}\iota$, $\operatorname{Coker}\iota$, and $\operatorname{Coker}\gamma$ are locally zero and that $\operatorname{Ker}\iota \to \operatorname{Coker}\kappa$ and $\operatorname{Ker}\gamma \to \operatorname{Coker}\iota$ are local isomorphisms. Since $\operatorname{Ker}\iota$ and $\operatorname{Coker}\iota$ are locally zero, it follows from 2.6 that their local isomorphs $\operatorname{Coker}\kappa$ and $\operatorname{Ker}\gamma$ are also locally zero. □

3.6. Corollary. *Suppose that we have a commutative diagram*

$$\begin{array}{ccccc} \mathfrak{A}' & \xrightarrow{\alpha'} & \mathfrak{A} & \xrightarrow{\alpha''} & \mathfrak{A}'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B}' & \xrightarrow{\beta'} & \mathfrak{B} & \xrightarrow{\beta''} & \mathfrak{B}'' \end{array}$$

of precosheaves in which the verticals are local isomorphisms and the compositions in the rows are zero. Then the induced map

$$\operatorname{Ker}\alpha''/\operatorname{Im}\alpha' \to \operatorname{Ker}\beta''/\operatorname{Im}\beta'$$

of precosheaves is a local isomorphism. □

4 Čech homology

This section begins the major topic of this chapter. We shall define Čech homology with coefficients in a precosheaf and investigate its properties.

Let $\mathfrak{A}$ be a precosheaf on a space X and let $\mathfrak{U} = \{U_\alpha\}$ be any open covering of X. For a p-simplex $\sigma = \langle \alpha_0, \dots, \alpha_p \rangle$ of the nerve $N(\mathfrak{U})$ (i.e., $U_\sigma = U_{\alpha_0,\dots,\alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \varnothing$) let $\sigma^{(i)} = \langle \alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p \rangle$. We define the group of Čech p-chains of the covering $\mathfrak{U}$ to be

$$\check{C}_p(\mathfrak{U};\mathfrak{A}) = \bigoplus_\sigma \mathfrak{A}(U_\sigma),$$

where the sum ranges over all p-simplices $\sigma = \langle \alpha_0, \dots, \alpha_p \rangle$. Thus a p-chain c is a finite formal sum

$$c = \sum a_\sigma \sigma$$

of p-simplices, where $a_\sigma \in \mathfrak{A}(U_\sigma)$. Note that $\check{C}_p(\mathfrak{U};\mathfrak{A})$ is an exact functor of precosheaves $\mathfrak{A}$. The differential

$$\partial : \check{C}_p(\mathfrak{U};\mathfrak{A}) \to \check{C}_{p-1}(\mathfrak{U};\mathfrak{A})$$

is defined by

$$\partial(a_\sigma \sigma) = \sum_{i=0}^{p} (-1)^i (a_\sigma \lfloor U_{\sigma^{(i)}}) \sigma^{(i)},$$

where $\lfloor U$ indicates operation by the map $\mathfrak{A}(V) \to \mathfrak{A}(U)$ when $V \subset U$, i.e., it is the dual of restriction. Also, there is the augmentation

$$\varepsilon : \check{C}_0(\mathfrak{U};\mathfrak{A}) \to \mathfrak{A}(X)$$

given by

$$\varepsilon = \sum_\alpha i_{X,U_\alpha} : \bigoplus_\alpha \mathfrak{A}(U_\alpha) \to \mathfrak{A}(X).$$

Note that $\varepsilon(\sum_\sigma a_\sigma \sigma) = \sum_\sigma (a_\sigma \lfloor X)$.

As usual, the homology of the chain complex $\check{C}_*(\mathfrak{U};\mathfrak{A})$ is denoted by

$$\boxed{\check{H}_p(\mathfrak{U};\mathfrak{A}) = H_p(\check{C}_*(\mathfrak{U};\mathfrak{A})),}$$

and the augmentation ε induces a homomorphism

$$\boxed{\varepsilon_* : \check{H}_0(\mathfrak{U};\mathfrak{A}) \to \mathfrak{A}(X).}$$

Recall that for an open set $V \subset X$, $\mathfrak{U} \cap V$ denotes the covering $\{U_\alpha \cap V\}$ of V. Thus $V \mapsto \check{H}_n(\mathfrak{U} \cap V;\mathfrak{A})$ defines a precosheaf on X denoted by

$$\boxed{\check{\mathfrak{H}}_n(\mathfrak{U};\mathfrak{A}) : V \mapsto \check{H}_n(\mathfrak{U} \cap V;\mathfrak{A}).}$$

Also, there is the homomorphism

$$\boxed{\varepsilon_* : \check{\mathfrak{H}}_0(\mathfrak{U};\mathfrak{A}) \to \mathfrak{A}}$$

of precosheaves. The following is an easy observation:

4.1. Proposition. *The homomorphism*

$$\varepsilon_* : \check{\mathfrak{H}}_0(\mathfrak{U};\mathfrak{A}) \to \mathfrak{A},$$

induced by the augmentation, is epimorphic for all $\mathfrak{U}$ *if* $\mathfrak{A}$ *is an epiprecosheaf and is isomorphic for all* $\mathfrak{U}$ *if* $\mathfrak{A}$ *is a cosheaf.* □

If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ and $\mathfrak{V} \to \mathfrak{U}$ is a refinement projection, then there is the induced chain map $\check{C}_*(\mathfrak{V};\mathfrak{A}) \to \check{C}_*(\mathfrak{U};\mathfrak{A})$, which induces the homomorphism

$$\check{H}_p(\mathfrak{V};\mathfrak{A}) \to \check{H}_p(\mathfrak{U};\mathfrak{A}).$$

The latter is independent of the particular refinement projection used; see the proof of the corresponding fact for cohomology in Chapter I. Thus we may define the Čech homology of X with coefficients in the precosheaf $\mathfrak{A}$ by

$$\boxed{\check{H}_n(X;\mathfrak{A}) = \varprojlim_{\mathfrak{U}} \check{H}_n(\mathfrak{U};\mathfrak{A}).}$$

Let $\check{\mathfrak{H}}_n(X;\mathfrak{A})$ denote the precosheaf

$$\boxed{\check{\mathfrak{H}}_n(X;\mathfrak{A}) : U \mapsto \check{H}_n(U;\mathfrak{A}).}$$

From 4.1 we deduce:

4.2. Proposition. *If* $\mathfrak{A}$ *is a cosheaf, then the augmentation homomorphism*

$$\varepsilon_* : \check{\mathfrak{H}}_0(X;\mathfrak{A}) \to \mathfrak{A}$$

is an isomorphism. □

For a covering $\mathfrak{U}$ of X, the precosheaf

$$V \mapsto \check{C}_n(\mathfrak{U} \cap V;\mathfrak{A})$$

will be denoted by $\check{\mathfrak{C}}_n(\mathfrak{U};\mathfrak{A})$. Thus

$$\boxed{\check{\mathfrak{C}}_n(\mathfrak{U};\mathfrak{A})(V) = \check{C}_n(\mathfrak{U} \cap V;\mathfrak{A}).}$$

Since $\check{C}_p(\mathfrak{U};\mathfrak{A})$ is an exact functor of precosheaves, any short exact sequence

$$0 \to \mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}'' \to 0$$

induces a long exact sequence

$$\cdots \to \check{H}_n(\mathfrak{U};\mathfrak{A}') \to \check{H}_n(\mathfrak{U};\mathfrak{A}) \to \check{H}_n(\mathfrak{U};\mathfrak{A}'') \to \check{H}_{n-1}(\mathfrak{U};\mathfrak{A}') \to \cdots$$

but, of course, this generally fails for the groups $\check{H}_*(X;\mathfrak{A})$ since the inverse limit functor is not generally exact.

4.3. Lemma. *If $\mathfrak{A}$ is a cosheaf, then so is $\check{\mathfrak{C}}_n(\mathfrak{U};\mathfrak{A})$. The latter is flabby when $\mathfrak{A}$ is flabby.*

Proof. It is clear that the direct sum of a family of cosheaves is a cosheaf. But $\check{\mathfrak{C}}_n(\mathfrak{U};\mathfrak{A})$ is the direct sum of precosheaves of the form $V \mapsto \mathfrak{A}(U \cap V)$ (where $U = U_{\alpha_0,\dots,\alpha_n}$), and these are easily seen to be cosheaves when $\mathfrak{A}$ is a cosheaf. The last statement holds by similar reasoning. □

4.4. Theorem. *For a precosheaf $\mathfrak{A}$ on X the sequence*

$$\cdots \to \check{\mathfrak{C}}_2(\mathfrak{U};\mathfrak{A}) \to \check{\mathfrak{C}}_1(\mathfrak{U};\mathfrak{A}) \to \check{\mathfrak{C}}_0(\mathfrak{U};\mathfrak{A}) \to \mathfrak{A} \to 0$$

of precosheaves is locally exact for any covering $\mathfrak{U}$ of X.

Proof. Since this is a statement about small neighborhoods of a point, it suffices to prove that the sequence is exact if $U_\gamma = X$ for some index γ. In that case, we can define a chain contraction $D : \check{C}_n(\mathfrak{U};\mathfrak{A}) \to \check{C}_{n+1}(\mathfrak{U};\mathfrak{A})$ by

$$D(a_\sigma \sigma) = a_\sigma \gamma\sigma,$$

where $\gamma\sigma = \gamma\langle \alpha_0,\dots,\alpha_p\rangle = \langle \gamma,\alpha_0,\dots,\alpha_p\rangle$. Then $\partial D + D\partial = 1$, as the reader may check, if we interpret ∂ as ε in degree 0. The naturality of this chain contraction implies that it induces one on the precosheaves $\check{\mathfrak{C}}_*(\mathfrak{U};\mathfrak{A})$ when $X \in \mathfrak{U}$. □

4.5. Corollary. *If $\mathfrak{A}$ is a flabby cosheaf, then for any open covering $\mathfrak{U}$ of X we have $\check{\mathfrak{H}}_n(\mathfrak{U};\mathfrak{A}) = 0$ for $n > 0$, whence $\check{\mathfrak{H}}_n(X;\mathfrak{A}) = 0$ for $n > 0$.*

Proof. This is an immediate consequence of 2.5, 4.3, and 4.4. □

4.6. Theorem. *Let X be a paracompact space and let $\mathfrak{A}$ be a locally zero precosheaf on X. Then, for any open covering $\mathfrak{U}$ of X there is a refinement $\mathfrak{W} \to \mathfrak{U}$ such that the induced map $\check{C}_n(\mathfrak{W};\mathfrak{A}) \to \check{C}_n(\mathfrak{U};\mathfrak{A})$ is zero for all $n \geq 0$.*

Proof. We may assume that $\mathfrak{U}$ is locally finite and "self-indexing." Let $\mathfrak{U}'$ be a shrinking of $\mathfrak{U}$, i.e., a covering that assigns to each $U \in \mathfrak{U}$ an open set U' with $\overline{U}' \subset U$. For each $U \in \mathfrak{U}$ and $x \in U'$ we choose an open set $W = W(x,U) \subset U'$ with the property that whenever $U_1,\dots,U_n$ are in $\mathfrak{U}$ and $W \cap U_1' \cap \cdots \cap U_n' \neq \varnothing$ then $W \subset U_1 \cap \cdots \cap U_n$ and the map $\mathfrak{A}(W) \to \mathfrak{A}(U_1 \cap \cdots \cap U_n)$ is trivial. The existence of such sets follows easily from the local finiteness of $\mathfrak{U}$ and local triviality of $\mathfrak{A}$. We take the refinement projection $W(x,U) \mapsto U$.

We claim that this covering satisfies the conclusion of the theorem. In fact, let

$$W_i = W(x_i, U_i),\ i = 0,\dots,n,$$

and suppose that $W_0 \cap \cdots \cap W_n \neq \varnothing$. Then since $W_0 \cap \cdots \cap W_n \subset U_0' \cap \cdots \cap U_n'$, we must have $W_i \subset U_0 \cap \cdots \cap U_n$ for each $i = 0, \dots, n$ and that $\mathfrak{A}(W_i) \to \mathfrak{A}(U_0 \cap \cdots \cap U_n)$ is zero. Since $\mathfrak{A}(W_0 \cap \cdots \cap W_n) \to \mathfrak{A}(U_0 \cap \cdots \cap U_n)$ factors through $\mathfrak{A}(W_0)$, it is zero. □

4.7. Corollary. *If X is paracompact and if $\mathfrak{A}$ is a locally zero precosheaf on X then $\check{H}_n(X; \mathfrak{A}) = 0$ for all $n \geq 0$.* □

4.8. Corollary. *Let X be paracompact and let $h : \mathfrak{A} \to \mathfrak{B}$ be a local isomorphism. Then $h_* : \check{H}_n(X; \mathfrak{A}) \to \check{H}_n(X; \mathfrak{B})$ is an isomorphism.*

Proof. Clearly, this reduces to the two cases in which $\operatorname{Coker} h = 0$ or $\operatorname{Ker} h = 0$. These are sufficiently alike that we will deal only with the first. Thus let $0 \to \mathfrak{K} \to \mathfrak{A} \to \mathfrak{B} \to 0$ be exact with $\mathfrak{K}$ locally zero. For each open covering $\mathfrak{U}$ of X we have the exact homology sequence

$$\cdots \to \check{H}_n(\mathfrak{U}; \mathfrak{K}) \to \check{H}_n(\mathfrak{U}; \mathfrak{A}) \to \check{H}_n(\mathfrak{U}; \mathfrak{B}) \to \check{H}_{n-1}(\mathfrak{U}; \mathfrak{K}) \to \cdots$$

Using 4.6 and the following lemma, we see that the induced map h_* is an isomorphism. □

4.9. Lemma. *Let $\{A_\alpha, f_{\alpha,\beta}\}$, $\{B_\alpha, g_{\alpha,\beta}\}$, $\{C_\alpha, h_{\alpha,\beta}\}$, and $\{D_\alpha, k_{\alpha,\beta}\}$ be inverse systems of abelian groups and let*

$$A_\alpha \xrightarrow{\lambda_\alpha} B_\alpha \xrightarrow{\mu_\alpha} C_\alpha \xrightarrow{\nu_\alpha} D_\alpha$$

be exact sequences commuting with the projections. Assume that for each α there is a $\beta > \alpha$ such that $f_{\alpha,\beta} : A_\beta \to A_\alpha$ and $k_{\alpha,\beta} : D_\beta \to D_\alpha$ are zero. Then the induced map

$$\mu : \varprojlim B_\alpha \to \varprojlim C_\alpha$$

is an isomorphism.

Proof. Let $\{b_\alpha\} \in \varprojlim B_\alpha$ be in $\operatorname{Ker} \mu$. Then $\mu_\alpha(b_\alpha) = 0$ for all α, and so $b_\alpha = \lambda_\alpha(a_\alpha)$ for some $a_\alpha \in A_\alpha$. Given α, there is a $\beta > \alpha$ with $f_{\alpha,\beta} = 0 = k_{\alpha,\beta}$. Thus $b_\alpha = g_{\alpha,\beta}(b_\beta) = g_{\alpha,\beta}(\lambda_\beta(a_\beta)) = \lambda_\alpha(f_{\alpha,\beta}(a_\beta)) = 0$, which shows that μ is monomorphic.

Now let $\{c_\alpha\} \in \varprojlim C_\alpha$. With β as above, we have

$$\nu_\alpha(c_\alpha) = \nu_\alpha(h_{\alpha,\beta}(c_\beta)) = k_{\alpha,\beta}(\nu_\beta(c_\beta)) = 0,$$

so that $c_\alpha = \mu_\alpha(b'_\alpha)$ for some $b'_\alpha \in B_\alpha$. Let β be as above in relation to α and let $\gamma > \beta$ be arbitrary. Note that $b'_\beta \equiv g_{\beta,\gamma}(b'_\gamma) \pmod{\operatorname{Im} \lambda_\beta}$ since μ_β takes them to the same thing. Applying $g_{\alpha,\beta}$, we obtain $g_{\alpha,\beta}(b'_\beta) = g_{\alpha,\gamma}(b'_\gamma)$ since $g_{\alpha,\beta}\lambda_\beta = \lambda_\alpha f_{\alpha,\beta} = 0$. Let b_α be this common element $g_{\alpha,\beta}(b'_\beta)$ for $\beta > \alpha$ sufficiently large. Then we have $g_{\gamma,\delta}(b_\delta) = b_\gamma$ for any $\delta > \gamma$, whence $\{b_\alpha\} \in \varprojlim B_\alpha$. Also $\mu_\alpha(b_\alpha) = \mu_\alpha(g_{\alpha,\beta}(b'_\beta)) = h_{\alpha,\beta}(\mu_\beta(b'_\beta)) = h_{\alpha,\beta}(c_\beta) = c_\alpha$, which shows that μ is epimorphic. □

5 The reflector

For a smooth precosheaf $\mathfrak{A}$ on X we define the precosheaf

$$\boxed{\mathfrak{Cosheaf}(\mathfrak{A}) = \check{\mathfrak{H}}_0(X; \widetilde{\mathfrak{A}}),}$$

where $\widetilde{\mathfrak{A}}$ is the maximal epiprecosheaf in $\mathfrak{A}$ of 1.6. (This makes sense for all precosheaves $\mathfrak{A}$, but $\widetilde{\mathfrak{A}}$ will be a cosheaf only if $\mathfrak{A}$ is smooth, as we shall show presently.) Note that $\mathfrak{Cosheaf}(\widetilde{\mathfrak{A}}) = \mathfrak{Cosheaf}(\mathfrak{A})$. There is the natural homomorphism

$$\boxed{\theta : \mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{A}}$$

given by the composition

$$\theta : \check{\mathfrak{H}}_0(X; \widetilde{\mathfrak{A}}) \xrightarrow{\varepsilon_*} \widetilde{\mathfrak{A}} \hookrightarrow \mathfrak{A}.$$

By 4.2, θ is an isomorphism if $\mathfrak{A}$ is a cosheaf. Also, if $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism of precosheaves, then there is an induced homomorphism

$$\bar{h} : \mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{Cosheaf}(\mathfrak{B}).$$

The following is a basic result:

5.1. Theorem. *If $h : \mathfrak{A} \to \mathfrak{B}$ is a local isomorphism of precosheaves and if either $\mathfrak{A}$ or $\mathfrak{B}$ is a cosheaf, then $\bar{h}$ is an isomorphism of $\mathfrak{Cosheaf}(\mathfrak{A})$ onto $\mathfrak{Cosheaf}(\mathfrak{B})$.*

Proof. First assume that $\mathfrak{A}$ is a cosheaf. Then $h(\mathfrak{A}) \subset \widetilde{\mathfrak{B}}$ since $h(\mathfrak{A})$ is an epiprecosheaf by Exercise 1. It is also clear that $h : \mathfrak{A} \to \widetilde{\mathfrak{B}}$ is a local isomorphism. Thus we may as well assume that $\mathfrak{B} = \widetilde{\mathfrak{B}}$, i.e., that $\mathfrak{B}$ is an epiprecosheaf. Then $\operatorname{Coker} h$ is also an epiprecosheaf by Exercise 1, and being locally zero, it is zero by Exercise 2. Let $\mathfrak{K} = \operatorname{Ker} h$.

Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of an open set $U \subset X$ so fine that $\mathfrak{K}(U_\alpha) \to \mathfrak{K}(U)$ is zero for all α. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \bigoplus \mathfrak{A}(U_{\alpha,\beta}) & \longrightarrow & \bigoplus \mathfrak{B}(U_{\alpha,\beta}) & \to & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle k} & & \\
0 & \to & \bigoplus \mathfrak{K}(U_\alpha) & \to & \bigoplus \mathfrak{A}(U_\alpha) & \xrightarrow{g} & \bigoplus \mathfrak{B}(U_\alpha) & \to & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle j} & \downarrow & & \\
0 & \to & \mathfrak{K}(U) & \to & \mathfrak{A}(U) & \longrightarrow & \mathfrak{B}(U) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0, & & 0 & &
\end{array}$$

which is exact except for the $\mathfrak{B}$ column. We see that $\operatorname{Ker} f \supset \operatorname{Ker} g$, so that the additive relation $j = fg^{-1}$ is single-valued and hence is a

homomorphism. Clearly, j is onto, and diagram chasing shows that $\operatorname{Ker} j = g(\operatorname{Ker} f) = \operatorname{Im} k$. Thus we have the induced isomorphism

$$\varphi : \check{H}_0(\mathfrak{U}; \mathfrak{B}) \xrightarrow{\approx} \check{H}_0(\mathfrak{U}; \mathfrak{A}).$$

Note that φ is an inverse of the natural map

$$h_*(\mathfrak{U}) : \check{H}_0(\mathfrak{U}; \mathfrak{A}) \to \check{H}_0(\mathfrak{U}; \mathfrak{B}).$$

Therefore, $h_*(\mathfrak{U})$ is an isomorphism, and upon passage to the limit over $\mathfrak{U}$, we see that $\bar{h}(U) = \varprojlim_{\mathfrak{U}} h_*(\mathfrak{U})$ is an isomorphism, as claimed.

Now suppose that $\mathfrak{B}$ is a cosheaf and that $\mathfrak{A}$ is arbitrary. Again $\operatorname{Coker} h$ is an epiprecosheaf, and hence it is zero. Let $\mathfrak{K} = \operatorname{Ker} h$ as before. Fix an open set $U \subset X$ for the time being, and choose an open covering $\{U_\alpha\}$ of U such that each $\mathfrak{K}(U_\alpha) \to \mathfrak{K}(U)$ is zero. Let $A = \operatorname{Im}\{\bigoplus \mathfrak{A}(U_\alpha) \to \mathfrak{A}(U)\}$. The maps $\bigoplus \mathfrak{A}(U_\alpha) \to \bigoplus \mathfrak{B}(U_\alpha) \to \mathfrak{B}(U)$ are onto, whence $A \to \mathfrak{B}(U)$ is onto. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \bigoplus \mathfrak{A}(U_{\alpha,\beta}) & \to & \bigoplus \mathfrak{B}(U_{\alpha,\beta}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus \mathfrak{K}(U_\alpha) & \to & \bigoplus \mathfrak{A}(U_\alpha) & \to & \bigoplus \mathfrak{B}(U_\alpha) & \to & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{K}(U) \cap A & \to & A & \to & \mathfrak{B}(U) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

which is exact except the middle vertical at the $\bigoplus \mathfrak{A}(U_\alpha)$ term, where it is of order two. Diagram chasing reveals that the left-hand vertical map is onto, whence $\mathfrak{K}(U) \cap A = 0$. Therefore the map $A \to \mathfrak{B}(U)$ is an isomorphism. It follows that a refinement of $\{U_\alpha\}$ yields the *same* subgroup A of $\mathfrak{A}(U)$. Thus, in fact, in the notation of the proof of 1.6, $A = \mathfrak{A}_1(U)$, which is the set of elements of $\mathfrak{A}(U)$ in the image of $\bigoplus \mathfrak{A}(U_\alpha)$ for *every* open covering $\{U_\alpha\}$ of U. This shows that $\mathfrak{A}_1 \to \mathfrak{B}$ is an isomorphism, and since $\mathfrak{B}$ is a cosheaf, $\mathfrak{A}_1 = \widetilde{\mathfrak{A}}$ and it is a cosheaf. Thus

$$h : \mathfrak{Cosheaf}(\mathfrak{A}) = \widetilde{\mathfrak{A}} \xrightarrow{\approx} \mathfrak{B} = \mathfrak{Cosheaf}(\mathfrak{B}),$$

as claimed. □

The latter part of the proof of 5.1 shows more:

5.2. Theorem. *Let $h : \mathfrak{A} \to \mathfrak{B}$ be a local isomorphism, and suppose that $\mathfrak{B}$ is a cosheaf. Then there exists a local isomorphism $k : \mathfrak{B} \to \mathfrak{A}$ such that $hk = 1$.*

Proof. The map k is merely the inverse of the restriction $\widetilde{\mathfrak{A}} \to \mathfrak{B}$ of h followed by the inclusion $\widetilde{\mathfrak{A}} \hookrightarrow \mathfrak{A}$. We have the split exact sequence

$$0 \to \mathfrak{K} \to \mathfrak{A} \underset{k}{\overset{h}{\rightleftarrows}} \mathfrak{B} \to 0,$$

and it follows that $\operatorname{Ker} k = 0$ and that $\operatorname{Coker} k \approx \operatorname{Ker} h = \mathfrak{K}$ is locally zero, whence k is a local isomorphism. □

5.3. Corollary. *If the precosheaf $\mathfrak{A}$ is equivalent to the cosheaf $\mathfrak{B}$, then there exists a local isomorphism $\mathfrak{B} \to \mathfrak{A}$.* □

5.4. Corollary. *For a smooth precosheaf $\mathfrak{A}$, $\mathfrak{Cosheaf}(\mathfrak{A})$ is a cosheaf and $\theta : \mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{A}$ is a local isomorphism.*

Proof. By 5.3 there is a local isomorphism $h : \mathfrak{B} \to \mathfrak{A}$ for some cosheaf $\mathfrak{B}$. We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{Cosheaf}(\mathfrak{B}) & \xrightarrow{\bar{h}} & \mathfrak{Cosheaf}(\mathfrak{A}) \\ \downarrow{\scriptstyle\approx} & & \downarrow \\ \mathfrak{B} & \xrightarrow{h} & \mathfrak{A}. \end{array}$$

By 5.1, $\bar{h}$ is an isomorphism, and hence the map $\mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{A}$ may be identified with $h : \mathfrak{B} \to \mathfrak{A}$, and the result follows. □

5.5. Corollary. *Let $\mathscr{C}$ be the category of cosheaves on X and $\mathscr{S}$ the category of smooth precosheaves on X. Then $\mathfrak{Cosheaf} : \mathscr{S} \to \mathscr{C}$ is a reflector from $\mathscr{S}$ to $\mathscr{C}$.* □

5.6. Corollary. *If $h : \mathfrak{A} \to \mathfrak{B}$ is a local isomorphism, where $\mathfrak{A}$ and $\mathfrak{B}$ are smooth, then $\bar{h} : \mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{Cosheaf}(\mathfrak{B})$ is an isomorphism.*

Proof. By 5.3 there exists a *cosheaf* $\mathfrak{C}$ and a local isomorphism $k : \mathfrak{C} \to \mathfrak{A}$. The diagram

$$\begin{array}{ccccc} \mathfrak{A} & & \xrightarrow{\quad h \quad} & & \mathfrak{B} \\ & {\scriptstyle k}\nwarrow & & \nearrow{\scriptstyle hk} & \\ & & \mathfrak{C} & & \end{array}$$

consists of local isomorphisms. In the induced diagram

$$\begin{array}{ccccc} \mathfrak{Cosheaf}(\mathfrak{A}) & & \xrightarrow{\quad \bar{h} \quad} & & \mathfrak{Cosheaf}(\mathfrak{B}) \\ & \nwarrow & & \nearrow & \\ & & \mathfrak{Cosheaf}(\mathfrak{C}) & & \end{array}$$

the diagonals are isomorphisms by 5.1, and so $\bar{h}$ is also an isomorphism. □

5.7. Corollary. *If $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent cosheaves, then they are isomorphic.* □

5.8. Proposition. *If $\mathfrak{A}' \to \mathfrak{A} \to \mathfrak{A}''$ is a locally exact sequence of smooth precosheaves, then $\mathfrak{Cosheaf}(\mathfrak{A}') \to \mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{Cosheaf}(\mathfrak{A}'')$ is also locally exact.*

Proof. Apply 3.6 and 5.4 to the commutative diagram

$$\begin{array}{ccccc}\mathfrak{Cosheaf}(\mathfrak{A}') & \to & \mathfrak{Cosheaf}(\mathfrak{A}) & \to & \mathfrak{Cosheaf}(\mathfrak{A}'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}' & \to & \mathfrak{A} & \to & \mathfrak{A}''.\end{array}$$

□

5.9. Example. Not every precosheaf is smooth. For a simple example let $X = \mathbb{I}$ and let $\mathfrak{A}$ be the precosheaf on X that assigns to U the group of singular 1-chains of U. The associated epiprecosheaf $\widetilde{\mathfrak{A}}$ has $\widetilde{\mathfrak{A}}(U)$ equal to the subgroup generated by the *constant* singular 1-simplices. The inclusion $\widetilde{\mathfrak{A}} \hookrightarrow \mathfrak{A}$ is *not* a local isomorphism, and it follows that $\mathfrak{Cosheaf}(\mathfrak{A}) \to \mathfrak{A}$ cannot be a local isomorphism. This would contradict 5.4 if $\mathfrak{A}$ were smooth. ◇

5.10. If $\mathfrak{A}$ is a smooth precosheaf, then the inclusion $\widetilde{\mathfrak{A}} \hookrightarrow \mathfrak{A}$ is a local isomorphism. Therefore, if X is *hereditarily paracompact*, then it follows from 4.8 that the induced map $\mathfrak{Cosheaf}(\mathfrak{A}) = \check{\mathfrak{H}}_0(X; \widetilde{\mathfrak{A}}) \to \check{\mathfrak{H}}_0(X; \mathfrak{A})$ is an *isomorphism.* This probably does not hold without the paracompactness assumption.

5.11. Proposition. *If X is locally connected and M is an L-module, then the constant precosheaf M on X is smooth. The associated cosheaf $\mathfrak{M} = \mathfrak{Cosheaf}(M)$ is the constant cosheaf of Chapter* V.

Proof. Recall that the constant cosheaf $\mathfrak{M}$ is defined by letting $\mathfrak{M}(U)$ be the free L-module on the components of U. The summation map $\mathfrak{M}(U) \to M$ gives a local isomorphism of precosheaves since X is locally connected, whence $\mathfrak{M} \approx \mathfrak{Cosheaf}(M)$ by 5.7. □

6 Spectral sequences

Let $\mathfrak{A}_*$ be a *flabby* differential cosheaf that is bounded below; i.e., $\mathfrak{A}_i = 0$ for $i < i_0$ for some i_0. Given an open covering $\mathfrak{U}$ of X, consider the double complex

$$L_{p,q} = \check{C}_p(\mathfrak{U}; \mathfrak{A}_q).$$

There are two spectral sequences of this double complex. In one of them we have

$$E^1_{p,q} = {}'H_q(L_{*,p}) = \check{H}_q(\mathfrak{U}; \mathfrak{A}_p) = \begin{cases} \mathfrak{A}_p(X), & \text{for } q = 0, \\ 0, & \text{for } q \neq 0 \end{cases}$$

by 4.5. Thus

$$E^2_{p,q} = \begin{cases} H_p(\mathfrak{A}_*(X)), & \text{for } q = 0, \\ 0, & \text{for } q \neq 0. \end{cases}$$

Since this spectral sequence degenerates, we have the natural isomorphism

$$H_n(L_*) \approx H_n(\mathfrak{A}_*(X)),$$

where L_* is the total complex of $L_{*,*}$.

In the other spectral sequence we have

$$E^1_{p,q} = {}''H_q(L_{p,*}) = \check{C}_p(\mathfrak{U}; \mathfrak{H}_q(\mathfrak{A}_*)),$$

whence

$$E^2_{p,q} = \check{H}_p(\mathfrak{U}; \mathfrak{H}_q(\mathfrak{A}_*)).$$

By the assumption that $\mathfrak{A}_*$ is bounded below, this spectral sequence converges to $H_n(L_*)$. Therefore we have the spectral sequences

$$\boxed{E^2_{p,q}(\mathfrak{U}) = \check{H}_p(\mathfrak{U}; \mathfrak{H}_q(\mathfrak{A}_*)) \Longrightarrow H_{p+q}(\mathfrak{A}_*(X)),} \tag{2}$$

which are functorial in the coverings $\mathfrak{U}$ as well as in the flabby differential cosheaves $\mathfrak{A}_*$.

7 Coresolutions

Let n be an integer. By a semi-n-coresolution of a cosheaf $\mathfrak{A}$ we mean a differential cosheaf $\mathfrak{A}_*$ vanishing in negative degrees together with an augmentation homomorphism $\mathfrak{A}_0 \to \mathfrak{A}$ such that the precosheaf $\tilde{H}_p(\mathfrak{A}_*(\bullet))$ is locally zero for $p \le n-1$ and $\tilde{H}_n(\mathfrak{A}_*(\bullet))$ is semilocally zero. It is an n-coresolution if $\tilde{H}_n(\mathfrak{A}_*(\bullet))$ is locally zero. Of course, the statement that $\tilde{H}_p(\mathfrak{A}_*(\bullet))$ is locally zero for $p \le n-1$ just means that the sequence

$$\mathfrak{A}_n \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{A}_2 \xrightarrow{d} \mathfrak{A}_1 \xrightarrow{d} \mathfrak{A}_0 \xrightarrow{\varepsilon} \mathfrak{A} \to 0$$

is *locally* exact. Note that by 2.3 the portion $\mathfrak{A}_1 \to \mathfrak{A}_0 \to \mathfrak{A} \to 0$ is actually exact, and so

$$\boxed{\mathfrak{A} \approx H_0(\mathfrak{A}_*).}$$

In this section we fix the integer $n > 0$ and assume that $\mathfrak{A}_*$ is a given *flabby* semi-n-coresolution of a given cosheaf $\mathfrak{A}$. We shall study the spectral sequences of the previous section. We shall also assume that X is *paracompact*.

7.1. Lemma. *If $\mathfrak{U}$ is a sufficiently fine open covering of X, then the canonical projection $H_k(\mathfrak{A}_*(X)) \twoheadrightarrow E^\infty_{k,0}(\mathfrak{U})$ is an isomorphism for all $k \le n$.*

Proof. Let $\mathfrak{U}_0 = \{X\}$, the trivial covering. Construct a sequence of coverings $\mathfrak{U}_i$ such that

$$E^2_{i,k-i}(\mathfrak{U}_{i+1}) \to E^2_{i,k-i}(\mathfrak{U}_i)$$

is zero. If $i = 0$, then this is $\check{H}_0(\mathfrak{U}_1; \mathfrak{H}_k(\mathfrak{A}_*)) \to \check{H}_0(\{X\}; \mathfrak{H}_k(\mathfrak{A}_*)) = \mathfrak{H}_k(\mathfrak{A}_*)(X)$, which is zero for some $\mathfrak{U}_1$ since $\mathfrak{H}_k(\mathfrak{A}_*)$ is semilocally zero. If $0 < i < k$, then $\mathfrak{H}_{k-i}(\mathfrak{A}_*)$ is locally zero, so that the refinement $\mathfrak{U}_{i+1}$ exists by 4.6. The fact that $E^2_{i,k-i}(\mathfrak{U}_{i+1}) \to E^2_{i,k-i}(\mathfrak{U}_i)$ is zero implies that $E^r_{i,k-i}(\mathfrak{U}_{i+1}) \to E^r_{i,k-i}(\mathfrak{U}_i)$ is zero for all $r \geq 2$ including $r = \infty$, for $i < k$.

Now, for the spectral sequence of $\mathfrak{U}_i$ the total term $H_k(\mathfrak{A}_*(X))$ is filtered by submodules

$$H_k(\mathfrak{A}_*(X)) = J^i_{k,0} \supset J^i_{k-1,1} \supset \cdots \supset J^i_{0,k} \supset 0$$

such that $E^\infty_{p,k-p}(\mathfrak{U}_i) \approx J^i_{p,k-p}/J^i_{p-1,k-p+1}$. Since the refinements have no effect on the common total terms $H_k(\mathfrak{A}_*(X))$, each map $J^{i+1}_{k,0} \to J^i_{k,0}$ is an isomorphism. Consequently each map $J^{i+1}_{p,k-p} \to J^i_{p,k-p}$ is a monomorphism. Note that $J^0_{0,k} = J^0_{k,0}$. The fact that $E^\infty_{i,k-i}(\mathfrak{U}_{i+1}) \to E^\infty_{i,k-i}(\mathfrak{U}_i)$ is zero means that the refinement takes $J^{i+1}_{i,k-i} \rightarrowtail J^i_{i-1,k-i+1}$. Thus

$$J^k_{k-1,1} \rightarrowtail J^{k-1}_{k-2,2} \rightarrowtail \cdots \rightarrowtail J^1_{0,k} \rightarrowtail J^0_{-1,k+1} = 0,$$

whence $H_k(\mathfrak{A}_*(X)) = J^k_{k,0} = J^k_{k,0}/J^k_{k-1,1} = E^\infty_{k,0}(\mathfrak{U}_k)$. □

Now let $\mathfrak{U}$ be a given open covering sufficiently fine so that it satisfies 7.1, and put $\mathfrak{U}_1 = \mathfrak{U}$. Fix some $k \leq n$. Construct refinements $\mathfrak{U}_i$ of $\mathfrak{U}_{i-1}$ for $i = 2, \ldots, k$ such that

$$E^2_{p,q}(\mathfrak{U}_i) \to E^2_{p,q}(\mathfrak{U}_{i-1})$$

is zero for all p, q with $p + q < k$ and $q \neq 0$. This can be done by 4.6 since $\mathfrak{H}_q(\mathfrak{A}_*)$ is locally zero for $0 < q < k$.

Then the image of $E^2_{k,0}(\mathfrak{U}_n) \to E^2_{k,0}(\mathfrak{U}_{n-1})$ consists of d_2-cycles and hence induces a homomorphism

$$E^2_{k,0}(\mathfrak{U}_k) \to E^3_{k,0}(\mathfrak{U}_{k-1}).$$

Similarly, we obtain homomorphisms

$$E^3_{k,0}(\mathfrak{U}_{k-1}) \to E^4_{k,0}(\mathfrak{U}_{k-2}) \to \cdots \to E^{k+1}_{k,0}(\mathfrak{U}_1) = E^\infty_{k,0}(\mathfrak{U}),$$

whence the composition gives a homomorphism

$$E^2_{k,0}(\mathfrak{U}_k) \to E^\infty_{k,0}(\mathfrak{U}). \tag{3}$$

This provides the commutative diagram

$$\begin{array}{ccccccc} \check{H}_k(\mathfrak{U}_k;\mathfrak{A}) & \approx & E^2_{k,0}(\mathfrak{U}_k) & \hookleftarrow E^\infty_{k,0}(\mathfrak{U}_k) & \xleftarrow{\approx} & H_k(\mathfrak{A}_*(X)) \\ \downarrow & & \downarrow & \searrow \quad \downarrow & & \| \\ \check{H}_k(\mathfrak{U};\mathfrak{A}) & \approx & E^2_{k,0}(\mathfrak{U}) & \hookleftarrow E^\infty_{k,0}(\mathfrak{U}) & \xleftarrow{\approx} & H_k(\mathfrak{A}_*(X)). \end{array} \tag{4}$$

Letting $\mathfrak{U}_k = \mathfrak{V}$, this yields the diagram

$$\begin{array}{ccc} \check{H}_k(\mathfrak{U};\mathfrak{V}) & \overset{\mu}{\underset{\lambda'}{\rightleftarrows}} & H_k(\mathfrak{A}_*(X)) \\ \downarrow{\scriptstyle j} & & \| \\ \check{H}_k(\mathfrak{U};\mathfrak{A}) & \overset{\lambda}{\longleftarrow} & H_k(\mathfrak{A}_*(X)), \end{array}$$

where j is the refinement projection, the λ's are edge homomorphisms, and μ is induced by (3). Checking the definitions, we see that the following commutativity relations hold:

$$\begin{cases} \lambda = j\lambda', \\ j = \lambda\mu, \\ \mu\lambda' = 1. \end{cases}$$

Also, λ and λ' are monomorphisms. The relations show that $\operatorname{Im} j = \operatorname{Im}\lambda$ and that

$$j : \operatorname{Im}\lambda' \xrightarrow{\approx} \operatorname{Im}\lambda.$$

Thus we have proved the following result:

7.2. Theorem. *Let $\mathfrak{A}_*$ be a flabby* semi-*n-coresolution of the cosheaf $\mathfrak{A}$ on the paracompact space X. Then for $k \le n$, the edge homomorphisms $\lambda_{\mathfrak{U}} : H_k(\mathfrak{A}_*(X)) \to \check{H}_k(\mathfrak{U};\mathfrak{A})$ of the spectral sequences of the coverings $\mathfrak{U}$ induce an isomorphism in the limit over $\{\mathfrak{U}\}$:*

$$\boxed{\lambda_X : H_k(\mathfrak{A}_*(X)) \xrightarrow{\approx} \check{H}_k(X;\mathfrak{A}).}$$

□

In fact, we have shown that if $\mathfrak{U}$ is a sufficiently fine open covering of X, then $\lambda_{\mathfrak{U}} : H_k(\mathfrak{A}_*(X)) \to \check{H}_k(\mathfrak{U};\mathfrak{A})$ is a monomorphism onto the image of the canonical projection

$$\pi : \check{H}_k(X;\mathfrak{A}) \to \check{H}_k(\mathfrak{U};\mathfrak{A})$$

and, moreover, that π is a monomorphism. Moreover, given this $\mathfrak{U}$, then for any sufficiently fine refinement $\mathfrak{V}$ of $\mathfrak{U}$ we have $\operatorname{Im} j = \operatorname{Im}\lambda = \operatorname{Im}\pi$ and $j : \operatorname{Im}\lambda' \xrightarrow{\approx} \operatorname{Im}\lambda$, with the notation as in the proof.

8 Relative Čech homology

Here we shall indicate how to generalize the previous results to relative cohomology. This is relatively important since we do not have a way of introducing support families into Čech homology.

Let $A \subset X$ be an arbitrary subspace. For a precosheaf $\mathfrak{A}$ on A recall that $\mathfrak{A}^X$ denotes the precosheaf

$$\mathfrak{A}^X(U) = \mathfrak{A}(U \cap A)$$

on X. Also take note of Exercise 3.

A covering of the pair (X, A) is a pair $(\mathfrak{U}, \mathfrak{U}_\circ)$, where $\mathfrak{U}$ is a covering of X and $\mathfrak{U}_\circ \subset \mathfrak{U}$ is a covering of A in X. If $\mathfrak{A}$ is a precosheaf on A, then we have the canonical isomorphism

$$\check{C}_*(\mathfrak{U}_\circ; \mathfrak{A}^X) \approx \check{C}_*(\mathfrak{U}_\circ \cap A; \mathfrak{A}).$$

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are precosheaves on A and X respectively and that we are given a *monomorphism* of precosheaves

$$\eta : \mathfrak{A}^X \rightarrowtail \mathfrak{B}.$$

Then we obtain the induced monomorphic chain map

$$\check{C}_*(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \rightarrowtail \check{C}_*(\mathfrak{U}; \mathfrak{B}),$$

and we shall denote its cokernel by

$$\check{C}_*(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}).$$

Thus we have the natural exact sequence

$$\boxed{0 \to \check{C}_*(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \to \check{C}_*(\mathfrak{U}; \mathfrak{B}) \to \check{C}_*(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) \to 0}$$

of chain complexes. As usual, we define

$$\boxed{\check{H}_n(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) = H_n(\check{C}_*(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A})).}$$

There is the usual long exact sequence in homology

$$\cdots \to \check{H}_n(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \to \check{H}_n(\mathfrak{U}; \mathfrak{B}) \to \check{H}_n(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) \to \check{H}_{n-1}(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \to \cdots$$

We also define

$$\boxed{\check{H}_n(X, A; \mathfrak{B}, \mathfrak{A}) = \varprojlim_{(\mathfrak{U}, \mathfrak{U}_\circ)} \check{H}_n(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}).}$$

Now assume that $\mathfrak{A}$ and $\mathfrak{B}$ are *flabby cosheaves*. Then 4.5 applied to the relative homology sequence yields the conclusion that $\check{H}_n(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) = 0$ for $n > 1$ and also yields the exact sequence

$$0 \to \check{H}_1(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) \to \check{H}_0(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \to \check{H}_0(\mathfrak{U}; \mathfrak{B}) \to \check{H}_0(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) \to 0.$$

The two middle terms of this sequence are canonically isomorphic to $\mathfrak{A}^X(X) = \mathfrak{A}(A)$ and $\mathfrak{B}(X)$ respectively by 4.1, and the map between them is the monomorphism $\eta(X)$. Therefore

$$\check{H}_n(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}) = \begin{cases} 0, & \text{for } n \neq 0, \\ \operatorname{Coker} \eta(X) : \mathfrak{A}(A) \to \mathfrak{B}(X), & \text{for } n = 0 \end{cases}$$

when $\mathfrak{A}$ and $\mathfrak{B}$ are flabby cosheaves.

Now suppose that $\mathfrak{A}_*$ and $\mathfrak{B}_*$ are flabby differential cosheaves on A and X respectively that are bounded below. Also suppose that we are given an exact sequence

$$0 \to \mathfrak{A}_*^X \to \mathfrak{B}_* \to \mathfrak{C}_* \to 0,$$

defining $\mathfrak{C}_*$, of differential cosheaves.

Given the pair $(\mathfrak{U}, \mathfrak{U}_\circ)$ of coverings, we consider the double complex

$$L_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ) = \check{C}_p(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}_q, \mathfrak{A}_q).$$

As before, it follows that there is a natural spectral sequence $E^r_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ)$ with

$$E^1_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ) = {}''H_q(L_{p,*}(\mathfrak{U}, \mathfrak{U}_\circ))$$

and

$$\boxed{E^2_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ) = {}'H_p({}''H_q(L_{*,*}(\mathfrak{U}, \mathfrak{U}_\circ))) \Longrightarrow H_{p+q}(\mathfrak{C}_*(X)).} \tag{5}$$

Now suppose that $\mathfrak{B}_*$ is a flabby semi-n-coresolution of the cosheaf $\mathfrak{B}$ on X and that $\mathfrak{A}_*$ is a flabby $(n-1)$-coresolution of the cosheaf $\mathfrak{A}$ on A.[2] For each p, the exact sequence

$$0 \to \check{C}_p(\mathfrak{U}_\circ \cap A; \mathfrak{A}_*) \to \check{C}_p(\mathfrak{U}; \mathfrak{B}_*) \to \check{C}_p(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}_*, \mathfrak{A}_*) \to 0$$

induces a long exact sequence of the form

$$\begin{aligned} \cdots &\to \check{C}_p(\mathfrak{U}_\circ \cap A; \mathfrak{H}_q(\mathfrak{A}_*)) \to \check{C}_p(\mathfrak{U}; \mathfrak{H}_q(\mathfrak{B}_*)) \to E^1_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ) \to \\ \cdots &\to \check{C}_p(\mathfrak{U}; \mathfrak{H}_1(\mathfrak{B}_*)) \to E^1_{p,1}(\mathfrak{U}, \mathfrak{U}_\circ) \to \check{C}_p(\mathfrak{U}_\circ \cap A; \widetilde{\mathfrak{H}}_0(\mathfrak{A}_*)) = 0, \end{aligned} \tag{6}$$

where $\widetilde{\mathfrak{H}}_0(\mathfrak{A}_*) = \operatorname{Ker}\{\mathfrak{H}_0(\mathfrak{A}_*) \to \mathfrak{A}\} = 0$. Moreover, we have the exact sequence

$$\check{C}_p(\mathfrak{U}_\circ \cap A; \mathfrak{A}) \to \check{C}_p(\mathfrak{U}; \mathfrak{B}) \to E^1_{p,0}(\mathfrak{U}, \mathfrak{U}_\circ) \to 0.$$

This gives

$$\begin{aligned} E^1_{p,0}(\mathfrak{U}, \mathfrak{U}_\circ) &= \check{C}_p(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}), \\ E^2_{p,0}(\mathfrak{U}, \mathfrak{U}_\circ) &= \check{H}_p(\mathfrak{U}, \mathfrak{U}_\circ; \mathfrak{B}, \mathfrak{A}). \end{aligned}$$

Moreover, if X and A are both paracompact, then by (6) and 4.6 (twice) there exists a refining pair $(\mathfrak{V}, \mathfrak{V}_\circ)$ of $(\mathfrak{U}, \mathfrak{U}_\circ)$ and a refinement projection such that the induced homomorphism

$$E^1_{p,q}(\mathfrak{V}, \mathfrak{V}_\circ) \to E^1_{p,q}(\mathfrak{U}, \mathfrak{U}_\circ)$$

is zero for all $p < k$ and all $q > 0$ with $p+q = k \le n$. Also, the commutative diagram

$$\begin{array}{ccccc} & & E^1_{0,k}(\mathfrak{V}, \mathfrak{V}_\circ) & \to & \check{C}_0(\mathfrak{V}_\circ \cap A; \mathfrak{H}_{k-1}(\mathfrak{A}_*)) \\ & & \downarrow & & \downarrow{\scriptstyle 0} \\ \check{C}_0(\mathfrak{U}; \mathfrak{H}_k(\mathfrak{B}_*)) & \to & E^1_{0,k}(\mathfrak{U}, \mathfrak{U}_\circ) & \to & \check{C}_0(\mathfrak{U}_\circ \cap A; \mathfrak{H}_{k-1}(\mathfrak{A}_*)) \\ \downarrow{\scriptstyle 0} & & \downarrow & & \\ H_k(\mathfrak{B}_*(X)) & \to & H_k(\mathfrak{C}_*(X)) & & \end{array}$$

[2]Note that unless A is closed, it does not generally follow that $\mathfrak{C}_*$ is a coresolution of $\operatorname{Coker}(\mathfrak{A}^X \to \mathfrak{B})$.

shows that $(\mathfrak{U},\mathfrak{U}_\circ)$ and $(\mathfrak{V},\mathfrak{V}_\circ)$ can be arranged so that $E^1_{0,k}(\mathfrak{V},\mathfrak{V}_\circ) \to H_k(\mathfrak{C}_*(X))$ is zero.

We now have all the information needed to repeat the arguments in Section 7 in the relative case. Summing up, we have the following relative version of 7.2:

8.1. Theorem. *Let $A \subset X$ both be paracompact, let $\mathfrak{A}_*$ be a flabby $(n-1)$-coresolution of the cosheaf $\mathfrak{A}$ on A, and let $\mathfrak{B}_*$ be a flabby* semi-n-*coresolution of the cosheaf $\mathfrak{B}$ on X. Suppose that $0 \to \mathfrak{A}_*^X \to \mathfrak{B}_* \to \mathfrak{C}_* \to 0$ is an exact sequence of differential cosheaves. Then for $k \le n$ the edge homomorphisms*

$$H_k(\mathfrak{C}_*(X)) \twoheadrightarrow E^\infty_{k,0}(\mathfrak{U},\mathfrak{U}_\circ) \rightarrowtail E^2_{k,0}(\mathfrak{U},\mathfrak{U}_\circ) = \check{H}_k(\mathfrak{U},\mathfrak{U}_\circ;\mathfrak{B},\mathfrak{A})$$

of the spectral sequences (5) *induce isomorphisms*

$$H_k(\mathfrak{C}_*(X)) \xrightarrow{\approx} \check{H}_k(X,A;\mathfrak{B},\mathfrak{A})$$

in the limit over covering pairs $(\mathfrak{U},\mathfrak{U}_\circ)$. □

It is easy to see that via 8.1 and 7.2 the exact sequence

$$\cdots \to H_k(\mathfrak{A}_*(A)) \to H_k(\mathfrak{B}_*(X)) \to H_k(\mathfrak{C}_*(X)) \to H_{k-1}(\mathfrak{A}_*(A)) \to \cdots$$

can be identified with the sequence

$$\cdots \to \check{H}_k(A;\mathfrak{A}) \to \check{H}_k(X;\mathfrak{B}) \to \check{H}_k(X,A;\mathfrak{B},\mathfrak{A}) \to \check{H}_{k-1}(A;\mathfrak{A}) \to \cdots,$$

which is the inverse limit of the sequences for covering pairs. In particular, it follows that this homology sequence is *exact*, a fact that limits the possibilities for the existence of the hypothesized coresolutions, since Čech homology is not generally exact.

8.2. We wish to generalize 4.8 to the relative case. Thus assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are precosheaves on A and that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are precosheaves on X, and assume that we are given homomorphisms forming the commutative diagram

$$\begin{array}{ccc} \mathfrak{A}_1^X & \to & \mathfrak{B}_1 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ \mathfrak{A}_2^X & \to & \mathfrak{B}_2. \end{array}$$

Then we have the following result, whose proof is essentially the same as that for 4.8 and so will be omitted.

8.3. Proposition. *If (X,A) is a paracompact pair and h and k are local isomorphisms, then the induced map*

$$\check{H}_*(X,A;\mathfrak{B}_1,\mathfrak{A}_1) \to \check{H}_*(X,A;\mathfrak{B}_2,\mathfrak{A}_2)$$

is an isomorphism. □

9 Locally paracompact spaces

By a strategy introduced by Mardešić [59] in the locally compact case, one can prove a result similar to 8.1 for locally paracompact, but possibly not paracompact, spaces. This is the process of introducing *paracompact carriers.*

Suppose that Φ is a paracompactifying family Φ of supports on X with $E(\Phi) = X$. Then the family Φ° of *paracompact open* sets U with $\overline{U} \in \Phi$ will be called a *carrier family.* We shall not be concerned with any other type of support family in this chapter. Recall that for any sets $K, K' \in \Phi$ with $K \subset \operatorname{int} K'$ there is a set $U \in \Phi^\circ$ with with $K \subset U$ and $\overline{U} \subset \operatorname{int} K'$ (the union of an increasing *sequence* of members of Φ containing K, each a neighborhood of the last and contained in $K'' \in \Phi$ for some $K'' \subset \operatorname{int} K'$; see [34, p. 165]). Therefore a carrier family exists $\Leftrightarrow$ X is locally paracompact.

The most important case is that of locally compact spaces and $\Phi = c$. The next most important is the case in which X is paracompact and $\Phi = cld$.

Let (X, A) be a locally paracompact pair. Let Φ° and Ψ° be carrier families on X and A respectively. Then we define[3]

$$\check{H}^\circ_k(X, A; \mathfrak{B}, \mathfrak{A}) = \varinjlim \check{H}_k(U, V; \mathfrak{B}, \mathfrak{A}),$$

where the limit is taken over pairs (U, V), where $V \subset U$, $U \in \Phi^\circ$, and $V \in \Psi^\circ$. Note that for any cosheaf $\mathfrak{C}$,

$$\mathfrak{C}(X) = \varinjlim_{U \in \Phi^\circ} \mathfrak{C}(U),$$

because for any $c \in \mathfrak{C}(X)$ and a covering $\mathfrak{U} = \{U_\alpha \in \Phi^\circ\}$, c comes from some $\mathfrak{C}(U_{\alpha_0} \cup \cdots \cup U_{\alpha_k})$ and $U_{\alpha_0} \cup \cdots \cup U_{\alpha_k}$ is contained in some $U \in \Phi^\circ$. Consequently,

$$H_k(\mathfrak{C}_*(X)) = \varinjlim_{U \in \Phi^\circ} H_k(\mathfrak{C}_*(U)),$$

for any differential cosheaf $\mathfrak{C}_*$. Therefore we have:

9.1. Theorem. *With the hypotheses of* 8.1 *except that X and A are assumed to be locally paracompact rather than paracompact, there is a canonical isomorphism*

$$H_k(\mathfrak{C}_*(X)) \approx \check{H}^\circ_k(X, A; \mathfrak{B}, \mathfrak{A})$$

for all $k \le n$. In particular, if such a $\mathfrak{C}_*$ exists, then the groups on the right are independent of the carrier family Φ° used to define them. Also,

$$\check{H}^\circ_k(X, A; \mathfrak{B}, \mathfrak{A}) \approx \check{H}_k(X, A; \mathfrak{B}, \mathfrak{A})$$

if X and A are paracompact. □

The analogue of 8.3 also obviously holds in this case.

[3] Of course these groups may depend on the families Φ and Ψ, but in our applications they will not. Thus we chose this relatively uncluttered notation over one displaying the families explicitly.

10 Borel-Moore homology

We now apply 8.1 to the case of Borel-Moore homology. Thus we consider only locally compact spaces in this section.

First, we shall associate to any sheaf $\mathscr{A}$ on a locally connected (and locally compact) space a certain cosheaf on X:

10.1. Proposition. *Let X be locally compact. Then for any sheaf $\mathscr{A}$ on X, the precosheaf $H_0^c(\bullet\,;\mathscr{A})$ is a cosheaf.*

Proof. Let $\mathscr{L}_p = \mathscr{C}_p(X;L)$, the sheaf of germs of Borel-Moore p-chains, and let $\mathscr{K} = \operatorname{Ker}(d : \mathscr{L}_0 \to \mathscr{L}_{-1})$. Then by the proof of V-5.13, $\mathscr{K} \otimes \mathscr{A}$ is c-soft and the sequence

$$0 \to \Gamma_c(\mathscr{K} \otimes \mathscr{A}|U) \to \Gamma_c(\mathscr{L}_0 \otimes \mathscr{A}|U) \to \Gamma_c(\mathscr{L}_{-1} \otimes \mathscr{A}|U) \to 0$$

is exact. Thus $\Gamma_c(\mathscr{K} \otimes \mathscr{A}|U)$ is the group of 0-cycles of U. Therefore, by definition, we have the exact sequence

$$\Gamma_c(\mathscr{L}_1 \otimes \mathscr{A}|U) \to \Gamma_c(\mathscr{K} \otimes \mathscr{A}|U) \to H_0^c(U;\mathscr{A}) \to 0$$

of precosheaves. Since the first two terms are cosheaves by V-1.6, the last is also a cosheaf by 1.2. □

By the proof of 10.1, the Borel-Moore chain complex $C_*^c(X;\mathscr{A})$ can be replaced by a chain complex vanishing in negative degrees without changing the homology with compact supports, namely:

$$\bar{C}_p^c(U;\mathscr{A}) = \begin{cases} C_p^c(U;\mathscr{A}), & \text{for } p > 0,\\ \operatorname{Ker}\{C_0^c(U;\mathscr{A}) \to C_{-1}^c(U;\mathscr{A})\}, & \text{for } p = 0,\\ 0, & \text{for } p < 0.\end{cases}$$

Also note that by the proof of 10.1 the precosheaf

$$\overline{\mathfrak{C}}_p^c(X;\mathscr{A}) : U \mapsto \bar{C}_p^c(U;\mathscr{A})$$

is a *flabby cosheaf.*

By general results from Chapter V there is an exact sequence

$$0 \to \overline{\mathfrak{C}}_p^c(A;\mathscr{A}|A)^X \to \overline{\mathfrak{C}}_p^c(X;\mathscr{A}) \to \overline{\mathfrak{C}}_p^c(X,A;\mathscr{A}) \to 0,$$

which *defines* the right-hand term.

10.2. Definition. *A locally compact space X is said to be* semi-hlc_L^n *if the precosheaf $U \mapsto \tilde{H}_k^c(U;L)$ is locally zero for all $k < n$ and semilocally zero for $k = n$.*

10.3. Proposition. *Let X be hlc_L^0. Then X is locally connected, and for the constant sheaf M we have*

$$\boxed{H_0^c(\bullet\,;M) \approx \mathfrak{Cosheaf}(M).}$$

Proof. The universal coefficient sequence V-3.13 shows that $H_0^c(U;M) \approx H_0^c(U;L) \otimes M$ since $H_{-1}^c(U;L) = 0$. It follows that $\tilde{H}_0^c(\bullet\,;M)$ is locally zero. There is the natural exact sequence

$$0 \to \tilde{H}_0^c(U;M) \to H_0^c(U;M) \to M \to 0$$

of precosheaves, by definition of the reduced groups. Thus $H_0^c(\bullet\,;M) \to M$ is a local isomorphism. Since $H_0^c(\bullet\,;M)$ is a cosheaf by 10.1, M is smooth, and the local isomorphism $H_0^c(\bullet\,;M) \to M$ induces an isomorphism $H_0^c(\bullet\,;M) \to \mathfrak{Cosheaf}(M)$. By Exercise 5,[4] X is locally connected. □

10.4. Proposition. *If X is* semi-*hlc_L^n and if M is a constant sheaf, then $\overline{\mathfrak{C}}_*^c(X;M)$ is a flabby* semi-*n-coresolution of the cosheaf $H_0^c(\bullet\,;M)$. If X is hlc_L^n and if $\mathscr{A}$ is locally constant, then $\overline{\mathfrak{C}}_*^c(X;\mathscr{A})$ is a flabby n-coresolution of the cosheaf $H_0^c(\bullet\,;\mathscr{A})$.*

Proof. By the definition of semi-hlc_L^n, the precosheaf $U \mapsto H_p^c(U;L)$ is locally zero for $0 < p < n$, is semilocally zero for $p = n$, and is locally isomorphic to L for $p = 0$. That takes care of the case $M = L$. For general M the universal coefficient formula V-3.13 and II-17.3 show that $\tilde{H}_k^c(\bullet\,;M)$ is locally zero for $k < n$. For $k = n$ and appropriate choices of U and V they give the diagram

$$\begin{array}{ccccc}
 & & H_n^c(V;M) & \twoheadrightarrow & H_{n-1}^c(V;L) * M \\
 & & \downarrow & & \downarrow{\scriptstyle 0} \\
H_n^c(U;L)\otimes M & \rightarrowtail & H_n^c(U;M) & \twoheadrightarrow & H_{n-1}^c(U;L) * M \\
\downarrow{\scriptstyle 0} & & \downarrow & & \\
H_n^c(X;L)\otimes M & \rightarrowtail & H_n^c(X;M), & &
\end{array}$$

again giving the result via II-17.3. The locally constant case with the stronger condition obviously reduces to the constant case. □

Finally, from 10.4, 8.1, and 9.1 we have:

10.5. Theorem. *Let (X,A) be a locally compact pair. Let $\mathscr{A}$ be a sheaf on X and let $\mathfrak{H}_0(X,A;\mathscr{A})$ denote the cosheaf pair $H_0^c(\bullet\,;\mathscr{A}|A) \to H_0^c(\bullet\,;\mathscr{A})$ on X. Then we have a canonical isomorphism*

$$\boxed{H_k^c(X,A;\mathscr{A}) \approx \check{H}_k^c(X,A;\mathfrak{H}_0(X,A;\mathscr{A}))}$$

for all $k \le n$ under either of the following two conditions:

(a) *X is* semi-*hlc_L^n, A is hlc_L^{n-1}, and $\mathscr{A}$ is constant.*

(b) *X is hlc_L^n, A is hlc_L^{n-1}, and $\mathscr{A}$ is locally constant.* □

10.6. Corollary. (Jussila [50].) *Let (X,A) be a locally compact pair, and assume that X is* semi-*hlc_L^n and A is hlc_L^{n-1} for some $n > 0$.*[5] *Then for*

[4]Or by the implications $hlc_L^0 \Rightarrow clc_L^0 \Rightarrow$ locally connected.

[5]The condition $n > 0$ is to ensure that M is smooth.

any L-module M regarded as a constant sheaf and constant precosheaf pair,

$$\boxed{H_k^c(X,A;M) \approx \check{H}_k^c(X,A;M)}$$

for all $k \le n$.

Proof. By 10.3, M is locally isomorphic to $\mathfrak{Cosheaf}(M) \approx H_0^c(\bullet\,;M)$, and similarly on A. Thus the result follows from 10.5 and 8.3. □

10.7. Example. For $i = 1, 2, \ldots$ let K_i be a copy of $\mathbb{S}^n$. Let $f_i : K_{i+1} \to K_i$ be a base point preserving map of degree i and let M_i be the mapping cylinder of f_i. Let I_i be the generator of M_i between the base points and let $g : I_i \to \mathbb{S}^1$ wrap the arc I_i once around the circle. Let

$$X_m = \left((M_1 \cup \cdots \cup M_m) \cup_g \mathbb{D}^2\right) / (K_1 \cup K_m).$$

There is a surjection $\pi_m : X_{m+1} \twoheadrightarrow X_m$ collapsing M_{m+1} to its generator. Finally, let $X = \varprojlim X_m$. Then with integer coefficients one can compute that $H_i(X_m) = 0$ for $0 < i \le n$ and $H_{n+1}(X_m) \approx \mathbb{Z}$ for all m. Also, the inverse system

$$H_{n+1}(X_1) \leftarrow H_{n+1}(X_2) \leftarrow \cdots$$

has the form

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{5} \cdots$$

Consequently, $H^i(X_m) = 0$ for $0 < i \le n$ and the direct system

$$H^{n+1}(X_1) \to H^{n+1}(X_2) \to \cdots$$

has the form

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{5} \cdots$$

Therefore, by continuity II-14.6, we have

$$H^n(X;\mathbb{Z}) = 0 \text{ and } H^{n+1}(X;\mathbb{Z}) \approx \mathbb{Q}.$$

Thus

$$H_n(X;\mathbb{Z}) \approx \operatorname{Ext}(\mathbb{Q},\mathbb{Z}),$$

which is a rational vector space of uncountable dimension by V-14.8. Also,

$$\check{H}_n(X;\mathbb{Z}) = \varprojlim\{\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3} \cdots\} = 0.$$

Now, any point of X has arbitrarily small neighborhoods that are either contractible or homotopy-equivalent to the one-point union of n-spheres converging to a point. Thus X is $hlc_{\mathbb{Z}}^{n-1}$ and $HLC_{\mathbb{Z}}^{n-1}$. Since $\operatorname{Ext}(\mathbb{Q},\mathbb{Z}) \approx H_n(X;\mathbb{Z}) \not\approx \check{H}_n(X;\mathbb{Z}) = 0$, X cannot be semi-$hlc_{\mathbb{Z}}^n$, something for which there is no obvious direct argument. ◇

11 Modified Borel-Moore homology

Here we shall treat the theory

$$\overline{H}^c_k(X;M) = H_k(\mathfrak{D}_*(\mathscr{I}^*(X;L);M)) = H_k(\Gamma_c(\mathscr{H}om(\mathscr{I}^*(X;L);M^*))),$$

where M^* is an injective resolution of the L-module M. Recall that this was discussed briefly in V-18. Of course, this coincides with Borel-Moore homology when $M = L$. We shall confine the discussion to the absolute case. Again, this section is restricted to locally compact Hausdorff spaces.

11.1. Lemma. *If X is clc^{n+1}_L then $\mathfrak{D}_*(\mathscr{I}^*(X;L);M)$ is a flabby quasi-n-coresolution of the constant precosheaf M.*[6]

Proof. There are the exact sequences (natural in compact A by an analogue of V-4.1)

$$0 \to \operatorname{Ext}(H^{k+1}_c(A;L),M) \to \overline{H}_k(A;M) \to \operatorname{Hom}(H^k_c(A;L),M) \to 0,$$

which with II-17.3 show that for any compact neighborhood A of a point $x \in X$ there is a compact neighborhood B of x such that $\overline{H}_k(B;M) \to \overline{H}_k(A;M)$ is zero for $0 < k \le n$. Thus the precosheaf $\overline{H}^c_k(\bullet\,;M)$ is locally zero for $0 < k \le n$. An argument similar to the proof of V-5.14 shows that $\overline{H}^c_0(\bullet\,;M)$ is locally isomorphic to M. □

In the same way as for standard Borel-Moore homology, one can modify $\mathfrak{D}_*(\mathscr{I}^*(X;L);M)$ in degree zero, without changing $\overline{H}^c_*(\bullet\,;M)$, to produce a flabby n-coresolution of M. Then 7.2 gives us:

11.2. Theorem. *If X is clc^{n+1}_L, then there is a canonical isomorphism*

$$\boxed{\overline{H}^c_k(X;M) \approx \check{H}^c_k(X;M)}$$

for all $k \le n$. □

Since $clc^{n+1}_L \Rightarrow hlc^n_L$ by V-12.10, we deduce the following sufficient condition for this modified theory to be equivalent to the standard Borel-Moore homology theory, using 10.5 and 11.2.

11.3. Corollary. *If X is clc^{n+1}_L, then there is a canonical isomorphism*

$$\boxed{\overline{H}^c_k(X;M) \approx H^c_k(X;M)}$$

for all $k \le n$. □

[6] If X is compact and $H^{n+1}(X;L) \to H^{n+1}(K;L)$ is zero for some compact neighborhood K of each point, then this is a flabby semi-$(n+1)$-coresolution. This is true, for example if X is compact and $H^{n+1}(X;L)$ is finitely generated over L.

Remark: This result also follows fairly easily from the results of V-12 as long as M is finitely generated over L. But for general M, that approach does not seem to work. Some such result is probably true for general paracompactifying supports, but one such is not known to the author.

It is worth noting the following consequence of the last corollary.

11.4. Corollary. *If X is compact and clc_L^{n+1}, then there are the split exact sequences*

$$0 \to \operatorname{Ext}(H^{k+1}(X;L), M) \to H_k(X;M) \to \operatorname{Hom}(H^k(X;L), M) \to 0$$

for all $k \le n$. □

11.5. Corollary. *If X is clc_L^{n+1}, then change of rings is valid for Borel-Moore homology with compact supports through degree n.* □

12 Singular homology

Now we apply the results of this chapter to classical singular homology. We shall use the approach to singular homology suggested in I-7. Thus, for a sheaf $\mathscr{A}$ on X, we define

$$S_p(U;\mathscr{A}) = \bigoplus_{\sigma} \Gamma(\sigma^*(\mathscr{A})),$$

where the sum ranges over all singular p-simplices $\sigma : \Delta_p \to U$. (Although this makes sense for all sheaves $\mathscr{A}$, our main results will require $\mathscr{A}$ to be locally constant.) Then $S_*(U;\mathscr{A})$ has a boundary operator as in I-7. There is also a subdivision operator $\Upsilon : S_p(U;\mathscr{A}) \to S_p(U;\mathscr{A})$ defined in the obvious way via the map $\Delta_p' \to \Delta_p$ on the barycentric subdivision Δ_p' of the standard p-simplex Δ_p.

Consider the direct system

$$S_p(U;\mathscr{A}) \xrightarrow{\Upsilon} S_p(U;\mathscr{A}) \xrightarrow{\Upsilon} \cdots,$$

and let $\mathfrak{S}_p(X;\mathscr{A})(U)$ be its direct limit. Then $\mathfrak{S}_p(X;\mathscr{A})$ is a precosheaf on X. Now subdivision induces an isomorphism in homology, and so the canonical map

$$H_p(S_*(U;\mathscr{A})) \to H_p(\mathfrak{S}_*(X;\mathscr{A})(U))$$

is an isomorphism. We denote it by

$$\boxed{{}_{\Delta}H_p^c(U;\mathscr{A}) = H_p(\mathfrak{S}_*(X;\mathscr{A})(U)).}$$

If $\mathscr{A}$ is constant, then this is the classical singular homology group, and for $\mathscr{A}$ locally constant it is the classical singular homology group with twisted coefficients as defined by Steenrod.[7]

12.1. Proposition. *The precosheaf $\mathfrak{S}_p(X;\mathscr{A})$ is a flabby cosheaf.*

[7]The superscript c is used to maintain notational consistency with other parts of the book.

Proof. For $U \subset X$ open, $S_p(U;\mathscr{A}) \to S_p(X;\mathscr{A})$ is a monomorphism. Since the direct limit functor is exact, $\mathfrak{S}_p(X;\mathscr{A})(U) \to \mathfrak{S}_p(X;\mathscr{A})(X)$ is a monomorphism. This will show that $\mathfrak{S}_p(X;\mathscr{A})$ is flabby once we show it to be a cosheaf. To see the latter, note that for open sets U and V and any n-chain $s \in S_p(U \cup V;\mathscr{A})$, there is an integer k such that the kth subdivision $\Upsilon^k(s)$ is the sum $s = s_U + s_V$, $s_U \in S_p(U;\mathscr{A})$ and $s_V \in S_p(V;\mathscr{A})$. This means that $f\langle s,t\rangle = s+t$ of

$$\mathfrak{S}_p(X;\mathscr{A})(U) \oplus \mathfrak{S}_p(X;\mathscr{A})(V) \xrightarrow{f} \mathfrak{S}_p(X;\mathscr{A})(U \cup V)$$

is onto. It is also clear that its kernel is the image of

$$\mathfrak{S}_p(X;\mathscr{A})(U \cup V) \xrightarrow{g} \mathfrak{S}_p(X;\mathscr{A})(U) \oplus \mathfrak{S}_p(X;\mathscr{A})(V),$$

where $g(s) = \langle s, -s\rangle$. Thus $\mathfrak{S}_p(X;\mathscr{A})$ satisfies condition (a) of 1.4. It also satisfies (b) since direct limits commute with one another. □

If $A \subset X$ is any subspace, then there is the exact sequence

$$0 \to \mathfrak{S}_p(A;\mathscr{A}|A) \to \mathfrak{S}_p(X;\mathscr{A}) \to \mathfrak{S}_p(X,A;\mathscr{A}) \to 0,$$

where the relative precosheaf $\mathfrak{S}_p(X,A;\mathscr{A})$ is defined in a manner similar to that of the absolute groups. By 1.2, $\mathfrak{S}_p(X,A;\mathscr{A})$ is a cosheaf. It is flabby for the same reason that the absolute ones are.

12.2. Proposition. *For any sheaf $\mathscr{A}$ on X, the precosheaf ${}_\Delta H_0^c(\bullet\,;\mathscr{A})$ is a cosheaf. If X is locally arcwise connected and if $\mathscr{A} = M$ is constant, then this cosheaf is the constant cosheaf*

$$\boxed{{}_\Delta H_0^c(\bullet\,;M) = \mathfrak{Cosheaf}(M).}$$

Proof. By definition, we have the exact sequence

$$\mathfrak{S}_1(X;\mathscr{A}) \to \mathfrak{S}_0(X;\mathscr{A}) \to {}_\Delta H_0(\bullet\,;\mathscr{A}) \to 0$$

of precosheaves on X. By 1.2, ${}_\Delta H_0^c(\bullet\,;\mathscr{A})$ is a cosheaf.

Also, ${}_\Delta H_0^c(U;M)$ is the direct sum of copies of M over the arc components of U. Thus the augmentation ${}_\Delta H_0^c(U;M) \to M$ is an isomorphism if U is arcwise connected, and this means that ${}_\Delta H_0^c(\bullet\,;M) \to M$ is a local isomorphism of cosheaves when X is locally arcwise connected. □

12.3. Definition. *A space X is said to be* semi-HLC_L^n *if the precosheaf $U \to {}_\Delta\tilde{H}_k^c(U;L)$ is locally zero for all $k < n$ and semilocally zero for $k = n$.*

Using the usual universal coefficient formula for singular homology, the proof of the following fact is completely analogous to that of 10.4:

12.4. Proposition. *If X is* semi-HLC^n_L *and if M is a constant sheaf, then $\mathfrak{S}_*(X, A; M)$ is a flabby* semi-n-*coresolution of the cosheaf ${}_\Delta H^c_0(\bullet\,; M)$. If X is HLC^n_L and if $\mathscr{A}$ is locally constant, then $\mathfrak{S}_*(X, A; \mathscr{A})$ is a flabby n-coresolution of the cosheaf ${}_\Delta H^c_0(\bullet\,; \mathscr{A})$.* □

Finally, from 12.4, 8.1, and 9.1 we have:

12.5. Theorem. *Let $A \subset X$ be locally paracompact, let $\mathscr{A}$ be a sheaf on X, and let ${}_\Delta\mathfrak{H}_0(X, A; \mathscr{A})$ denote the cosheaf pair ${}_\Delta H^c_0(\bullet\,; \mathscr{A}|A) \to {}_\Delta H^c_0(\bullet\,; \mathscr{A})$ on X. Then there is a canonical isomorphism*

$$\boxed{{}_\Delta H^c_k(X, A; \mathscr{A}) \approx \check{H}^\diamond_k(X, A; {}_\Delta\mathfrak{H}_0(X, A; \mathscr{A}))}$$

for $k \le n$, under either of the following two conditions:

(a) *X is* semi-HLC^n_L, *A is HLC^{n-1}_L, and $\mathscr{A}$ is constant.*

(b) *X is HLC^n_L, A is HLC^{n-1}_L, and $\mathscr{A}$ is locally constant.* □

12.6. Corollary. (Mardešić [59].) *Let (X, A) be a locally paracompact pair and assume that X is* semi-HLC^n_L *and A is HLC^{n-1}_L. Then for any L-module M regarded as a constant sheaf and constant precosheaf pair,*

$$\boxed{{}_\Delta H^c_k(X, A; M) \approx \check{H}^\diamond_k(X, A; M)}$$

for all $k \le n$. □

13 Acyclic coverings

Let $\mathfrak{A}_*$ be a flabby differential cosheaf vanishing in negative degrees, and put $\mathfrak{A} = \text{Coker}\{\mathfrak{A}_1 \to \mathfrak{A}_0\}$. Consider the spectral sequence (2). We have the edge homomorphism

$$\lambda_{\mathfrak{U}} : H_n(\mathfrak{A}_*(X)) \twoheadrightarrow E^\infty_{n,0} \rightarrowtail E^2_{n,0} = \check{H}_n(\mathfrak{U}; \mathfrak{A}).$$

The following is immediate:

13.1. Theorem. *If each intersection U of at most $m+1$ members of $\mathfrak{U}$ has*

$$H_q(\mathfrak{A}_*(U)) = 0 \textit{ for all } 0 \ne q < n,$$

then

$$\lambda_{\mathfrak{U}} : H_k(\mathfrak{A}_*(X)) \to \check{H}_k(\mathfrak{U}; \mathfrak{A})$$

is an isomorphism for all $k < \min\{n, m\}$ and a monomorphism for $k = \min\{n, m\}$. □

Applying this to Borel-Moore homology, we have:

13.2. Corollary. *Let $\mathscr{A}$ be an arbitrary sheaf on the locally compact Hausdorff space X. Let $\mathfrak{U}$ be an open covering of X such that for each intersection U of at most $m+1$ members of $\mathfrak{U}$, $H_q^c(U;\mathscr{A}) = 0$ for all $0 < q < n$. Then the canonical map*

$$H_k^c(X;\mathscr{A}) \to \check{H}_k(\mathfrak{U}; H_0^c(\bullet\,;\mathscr{A}))$$

is an isomorphism for all $k < \min\{n,m\}$ and a monomorphism for $k = \min\{n,m\}$. □

Similarly, for singular homology we have:

13.3. Corollary. *Let $\mathscr{A}$ be an arbitrary sheaf on the space X. Let $\mathfrak{U}$ be an open covering of X such that for each intersection U of at most $m+1$ members of $\mathfrak{U}$, ${}_\Delta H_q^c(U;\mathscr{A}) = 0$ for all $0 < q < n$. Then the canonical map*

$${}_\Delta H_k^c(X;\mathscr{A}) \to \check{H}_k(\mathfrak{U}; {}_\Delta H_0^c(\bullet\,;\mathscr{A}))$$

is an isomorphism for all $k < \min\{n,m\}$ and a monomorphism for $k = \min\{n,m\}$. □

14 Applications to maps

In this section we consider a map

$$f : E \to X.$$

Recall that for a precosheaf $\mathfrak{B}$ on E there is the direct image $f\mathfrak{B}$ on X defined by

$$(f\mathfrak{B})(U) = \mathfrak{B}(f^{-1}U).$$

Obviously, $f\mathfrak{B}$ is a cosheaf when $\mathfrak{B}$ is a cosheaf, and $f\mathfrak{B}$ is flabby when $\mathfrak{B}$ is flabby.

If $\mathfrak{B}_*$ is a flabby differential cosheaf on E, then $\mathfrak{A}_* = f\mathfrak{B}_*$ is a flabby differential cosheaf on X. Also $\mathfrak{H}_q(\mathfrak{A}_*) = f\mathfrak{H}_q(\mathfrak{B}_*)$ since f is an exact functor of *pre*cosheaves, and so the spectral sequence (2) has

$$\boxed{E^2_{p,q}(\mathfrak{U}) = \check{H}_p(\mathfrak{U}; f\mathfrak{H}_q(\mathfrak{B}_*)) \Longrightarrow H_{p+q}(\mathfrak{A}_*(X)) = H_{p+q}(\mathfrak{B}_*(E)),}$$

natural in coverings $\mathfrak{U}$ of X.

In particular, for Borel-Moore homology (with X locally compact Hausdorff), we have the spectral sequences

$$\boxed{E^2_{p,q}(\mathfrak{U}) = \check{H}_p(\mathfrak{U}; H_q^c(f^{-1}(\bullet);\mathscr{A}) \Longrightarrow H^c_{p+q}(E;\mathscr{A}),}$$

natural in coverings $\mathfrak{U}$ and in sheaves $\mathscr{A}$ on E.

Similarly, for singular homology, we have the spectral sequences

$$\boxed{E^2_{p,q}(\mathfrak{U}) = \check{H}_p(\mathfrak{U}; {}_\Delta H^c_q(f^{-1}(\bullet); \mathscr{A}) \Longrightarrow {}_\Delta H^c_{p+q}(E; \mathscr{A}),}$$

natural in coverings $\mathfrak{U}$ and in sheaves $\mathscr{A}$ on E.

Now we shall prove a generalization of 7.2. Suppose that $\mathfrak{A}_*$ is a flabby differential cosheaf with $\mathfrak{A}_p = 0$ for $p < 0$ (or generally just bounded below). Also suppose that we are given integers $0 \le k < n$ such that

(A) $\mathfrak{H}_q(\mathfrak{A}_*)$ is locally zero for all $q \neq k$, $q < n$, and semilocally zero for $q = n$.

Let $\mathfrak{Z}_p = \operatorname{Ker}\{d_p : \mathfrak{A}_p \to \mathfrak{A}_{p-1}\}$. Then, by 2.5,

$$\mathfrak{A}_k \to \mathfrak{A}_{k-1} \to \cdots \to \mathfrak{A}_0 \to 0$$

is exact and

$$\mathfrak{Z}_k \text{ is a flabby cosheaf.}$$

Moreover, it is clear that

$$\mathfrak{A}_{n+1} \to \cdots \to \mathfrak{A}_{k+1} \to \mathfrak{Z}_k \to \mathfrak{H}_k(\mathfrak{A}_*) \to 0$$

is locally exact. Thus, under the assumption (A), we see that $\mathfrak{H}_k(\mathfrak{A}_*)$ is a cosheaf and that the differential cosheaf $\mathfrak{A}'_*$, defined by

$$\mathfrak{A}'_q = \begin{cases} 0, & \text{for } q < 0, \\ \mathfrak{Z}_k, & \text{for } q = 0, \\ \mathfrak{A}_{k+q} & \text{for } q > 0 \end{cases}$$

is a flabby semi-$(n-k)$-coresolution of $\mathfrak{H}_k(\mathfrak{A}_*)$.

By 7.2 and 9.1 we deduce:

14.1. Theorem. *Let X be locally paracompact and let $\mathfrak{A}_*$ be a flabby differential cosheaf on X that is bounded below and is such that condition* (A) *is satisfied. Then there is the canonical isomorphism*

$$\boxed{H_p(\mathfrak{A}_*(X)) \approx \check{H}^{\diamond}_{p-k}(X; \mathfrak{H}_k(\mathfrak{A}_*))}$$

for all $p \le n$. □

Returning to the case of a map $f : E \to X$, let $E_\circ \subset E$ be any subspace. Applying 14.1 to the flabby differential cosheaves $f\mathfrak{C}^c_*(E, E_\circ; \mathscr{A})$ and $f\mathfrak{S}_*(E, E_\circ; \mathscr{A})$, we have:

14.2. Corollary. *Suppose that X and $(E, E_\circ)$ are locally compact, that $\mathscr{A}$ is a sheaf on E, and that there are integers $k < n$ such that the precosheaf*

$$H^c_q(f^{-1}(\bullet), f^{-1}(\bullet) \cap E_\circ; \mathscr{A})$$

is locally zero for $k \neq q < n$ and semilocally zero for $q = n$. Then this precosheaf is zero for $q < k$ and is a cosheaf for $q = k$. Moreover, there is a canonical isomorphism

$$\boxed{H_p^c(E, E_\circ; \mathscr{A}) \approx \check{H}_{p-k}^\diamond(X; H_k^c(f^{-1}(\bullet), f^{-1}(\bullet) \cap E_\circ; \mathscr{A}))}$$

for all $p \leq n$. □

14.3. Corollary. *Suppose that X is a locally paracompact space, that $\mathscr{A}$ is a sheaf on E, and that there are integers $k < n$ such that the precosheaf*

$$_\Delta H_q^c(f^{-1}(\bullet), f^{-1}(\bullet) \cap E_\circ; \mathscr{A})$$

is locally zero for $k \neq q < n$ and semilocally zero for $q = n$. Then this precosheaf is zero for $q < k$ and is a cosheaf for $q = k$. Moreover, there is a canonical isomorphism

$$\boxed{_\Delta H_p^c(E, E_\circ; \mathscr{A}) \approx \check{H}_{p-k}^\diamond(X; {_\Delta H_k^c}(f^{-1}(\bullet), f^{-1}(\bullet) \cap E_\circ; \mathscr{A}))}$$

for all $p \leq n$. □

Exercises

1. If $h : \mathfrak{A} \to \mathfrak{B}$ is an epimorphism of precosheaves and $\mathfrak{A}$ is an epiprecosheaf, then show that $\mathfrak{B}$ is an epiprecosheaf.
2. If $\mathfrak{A}$ is a locally zero epiprecosheaf, then show that $\mathfrak{A} = 0$.
3. For an inclusion $A \subset X$ and a precosheaf $\mathfrak{A}$ on A, show that $\mathfrak{A}^X$ is a cosheaf $\Leftrightarrow$ $\mathfrak{A}$ is a cosheaf and that $\mathfrak{A}^X$ is flabby $\Leftrightarrow$ $\mathfrak{A}$ is flabby.
4. Ⓢ Let $\mathfrak{A}_*$ be a flabby n-coresolution of the cosheaf $\mathfrak{A}$ on the paracompact space X. Show that for every open covering $\mathfrak{U}$ of X, the canonical maps $H_{n+1}(\mathfrak{A}_*(X)) \to \check{H}_{n+1}(\mathfrak{U}; \mathfrak{A}) \leftarrow \check{H}_{n+1}(X; \mathfrak{A})$ have equal images.
5. Ⓢ If the constant precosheaf $M \neq 0$ on X is smooth, show that X is locally connected.
6. Ⓢ If X is connected and semi-hlc_L^0, then show that $\tilde{H}_0^c(X; L) = 0$. (This shows, for example, that the solenoid is not semi-$hlc_{\mathbb{Z}}^0$, something that is not immediately obvious.)

Appendix A

Spectral Sequences

This appendix is not intended as an exposition of the theory of spectral sequences, but rather as a vehicle for the establishment of the notation and terminology that we adopt and as an outline of the basic theory. Most of the proofs, which are not difficult, are omitted. We assume that a base ring L is given.

1 The spectral sequence of a filtered complex

Let C^* be a complex, that is, a graded module with differential $d: C^* \to C^*$ of degree $+1$. A (decreasing) *filtration* $\{F_pC^*\}$ of C^* is defined to be a collection of submodules F_pC^n of C^n (for each n) such that the following three conditions are satisfied:

$$\cdots \supset F_{p-1}C^n \supset F_pC^n \supset F_{p+1}C^n \supset \cdots; \tag{1}$$

$$C^n = \bigcup_p F_pC^n; \tag{2}$$

$$d(F_pC^n) \subset F_pC^{n+1}. \tag{3}$$

The filtration is said to be *regular* if $F_pC^n = 0$ for $p > f(n)$, for some function f. (This definition of regularity is stronger than the usual one, but it is convenient for our purposes.)

The *graded module* associated with the filtration $\{F_pC^*\}$ is defined to be $\{G_pC^*\}$, where

$$G_pC^* = F_pC^*/F_{p+1}C^*. \tag{4}$$

We define the graded modules

$$Z_r^p = \{c \in F_pC^* \mid dc \in F_{p+r}C^*\} \tag{5}$$

and

$$E_r^p = \frac{Z_r^p}{dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}} = \frac{Z_r^p}{B_r^p} \tag{6}$$

(for $r \geq 0$), and we let $Z_r^{p,q}$ and $E_r^{p,q}$ denote the summands of Z_r^p and E_r^p, respectively, consisting of terms of total degree (that degree induced from the degree in C^*) equal to $p+q$. That is,

$$Z_r^{p,q} = Z_r^p \cap F_pC^{p+q} = \{c \in F_pC^{p+q} \mid dc \in F_{p+r}C^{p+q+1}\}$$

and

$$E_r^{p,q} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

The index p is called the *filtration degree* and q is called the *complementary* degree.

The differential d on C^* induces a differential d_r on $E_r^{*,*}$, which increases the filtration degree by r by (5) and hence decreases the complementary degree by $r-1$ since the total degree of d is $+1$. That is,

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}. \tag{7}$$

It is easy to check that the homology of E_r with respect to d_r is naturally isomorphic to E_{r+1}. That is,

$$E_{r+1} \approx H(E_r, d_r) \quad \text{preserving both degrees.} \tag{8}$$

It may also be checked that

$$E_1^{p,q} \approx H^{p+q}(G_pC^*),$$

with d_1 corresponding to the connecting homomorphism associated with the short exact sequence

$$0 \longrightarrow \frac{F_{p+1}C^*}{F_{p+2}C^*} \longrightarrow \frac{F_pC^*}{F_{p+2}C^*} \longrightarrow \frac{F_pC^*}{F_{p+1}C^*} \longrightarrow 0$$

of chain complexes.

The collection of bigraded modules $E_r^{p,q}$ for $r \geq 2$ (and sometimes for $r \geq 1$) together with the differentials d_r as in (7) and satisfying (8) is what is known abstractly as a *spectral sequence*. We have described the construction of *the* spectral sequence of a filtered complex. We continue with the discussion of this special case.

The definitions (5) and (6) can be extended in a reasonable manner to make sense for $r = \infty$ if we define $F_\infty C^* = 0$ and $F_{-\infty}C^* = C^*$. Thus we define

$$Z_\infty^p = \{c \in F_pC^* \mid dc = 0\} \tag{9}$$

and

$$E_\infty^p = \frac{Z_\infty^p}{(F_pC^* \cap dC^*) + Z_\infty^{p+1}} = \frac{Z_\infty^p}{B_\infty^p}. \tag{10}$$

We introduce a filtration on $H^*(C^*)$ by setting

$$F_pH^n(C^*) = \operatorname{Im}(H^n(F_pC^*) \to H^n(C^*)). \tag{11}$$

Then there is a natural isomorphism

$$E_\infty^{p,q} \approx G_pH^{p+q}(C^*) = \frac{F_pH^{p+q}(C^*)}{F_{p+1}H^{p+q}(C^*)}. \tag{12}$$

Note that there are inclusions

$$\cdots \subset B_r^p \subset B_{r+1}^p \subset \cdots \subset B_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_{r+1}^p \subset Z_r^p \subset \cdots \tag{13}$$

Suppose now that the filtration $\{F_pC^*\}$ of C^* is regular. Then $Z_r^{p,q} = Z_{r+1}^{p,q} = \cdots = Z_\infty^{p,q}$ for r sufficiently large, p and q being fixed. (Equivalently, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is zero for large r.) Thus there are natural epimorphisms (for r sufficiently large)

$$E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,q}. \tag{14}$$

One can check that in fact,

$$E_\infty^{p,q} = \varinjlim E_r^{p,q} \tag{15}$$

when the filtration is regular. (Usually, in the applications we even have that the $E_r^{p,q}$ are constant for large r and fixed p and q.)

Let $\{E_r^{p,q}; d_r\}$ be *any* spectral sequence such that for fixed p and q we have that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is zero for sufficiently large r. Then we have the epimorphisms $E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q}$ for r large as in (14), taking a d_r-cycle in E_r into its homology class in E_{r+1}, and we *define* $E_\infty^{p,q}$ by (15). Suppose that we are given a graded module A^* and a filtration $\{F_pA^*\}$ of A^* such that

$$E_\infty^{p,q} \approx G_pA^{p+q}.$$

Then this fact will be abbreviated by the notation

$$E_2^{p,q} \Longrightarrow A^{p+q}.$$

For example, in the case of a regularly filtered complex C^* we have, by (12),

$$E_2^{p,q} \Longrightarrow H^{p+q}(C^*).$$

2 Double complexes

Suppose that $C^{*,*}$ is a double complex, that is, a family of modules doubly indexed by the integers and with differentials

$$d' : C^{p,q} \to C^{p+1,q} \quad \text{and} \quad d'' : C^{p,q} \to C^{p,q+1}$$

such that $(d')^2 = 0$, $(d'')^2 = 0$ and $d'd'' + d''d' = 0$. Let C^* be the "total" complex, with

$$C^n = \bigoplus_{p+q=n} C^{p,q}$$

and differential $d = d' + d''$.

We introduce two filtrations into the complex C^*. The first filtration $'F$ is defined by

$$'F_tC^n = \bigoplus_{p\geq t} C^{p,q} \qquad p+q=n.$$

Similarly, the second filtration $''F$ is given by

$$''F_tC^n = \bigoplus_{q \geq t} C^{p,q} \qquad p + q = n.$$

From these filtrations we obtain spectral sequences denoted by $\{'E_r^{p,q}\}$ and $\{''E_r^{p,q}\}$ respectively.

Denoting homology with respect to d' and d'' by $'H$ and $''H$ respectively, it can be seen that

$$'E_1^{p,q} \approx {''H^q}(C^{p,*}) \quad \text{and} \quad ''E_1^{p,q} \approx {'H^q}(C^{*,p}), \tag{16}$$

with differentials d_1 corresponding to the differentials induced by d' and d'' respectively. Therefore

$$'E_2^{p,q} \approx {'H^p}({''H^q}(C^{*,*})) \quad \text{and} \quad ''E_2^{p,q} \approx {''H^p}({'H^q}(C^{*,*})). \tag{17}$$

Note that if $C^{p,q} = 0$ for $p < p_0$ (p_0 fixed), then the second filtration is regular, while if $C^{p,q} = 0$ for $q < q_0$, the first spectral sequence is regular. Another useful condition implying regularity of both filtrations is that there exist integers p_0 and p_1 such that $C^{p,q} = 0$ for $p < p_0$ and for $p > p_1$.

Assume, for the remainder of this section, that $C^{*,*}$ is a double complex, with $C^{p,q} = 0$ for $q < 0$. For $r \geq 2$, $'E_r^{p,0}$ consists entirely of d_r-cycles, so that there is a natural epimorphism

$$'E_r^{p,0} \twoheadrightarrow {'E_{r+1}^{p,0}}$$

assigning to a cycle its homology class. Also,

$$'E_\infty^{p,0} = {'G_pH^p}(C^*) = {'F_pH^p}(C^*) \subset H^p(C^*)$$

[since $'F_{p+1}H^p(C^*) = 0$ in the present situation]. Thus we obtain the homomorphisms

$$'H^p({''H^0}(C^{*,*})) = {'E_2^{p,0}} \twoheadrightarrow {'E_\infty^{p,0}} \rightarrowtail H^p(C^*), \tag{18}$$

whose composition is called an *edge homomorphism*. Since $''H^0(C^{*,*})$ may be identified with the d''-cycles of $C^{*,0}$, there is a monomorphism

$$''H^0(C^{*,*}) \rightarrowtail C^*, \tag{19}$$

which is seen to be a chain map with respect to the differentials $d' = d_1$ and d respectively. The homology homomorphism induced by (19) is easily checked to be identical with (18). [The reader should note that the homomorphism (18) is defined under more general circumstances. For example, it suffices that $C^{p,q} = 0$ for $q < q_0$ and that $'E_2^{p,q} = 0$ for $q < 0$.]

If $C^{p,q} = 0$ for $p < 0$ as well as for $q < 0$, then there is an analogous edge homomorphism

$$H^p(C^*) = {'F_0H^p}(C^*) \twoheadrightarrow {'G_0H^p}(C^*) = {'E_\infty^{0,p}} \rightarrowtail {'E_2^{0,p}} = {'H^0}({''H^p}(C^{*,*})).$$

There are also, of course, similar edge homomorphisms for the second spectral sequence.

3 Products

Let ${}_1C^*$, ${}_2C^*$, and ${}_3C^*$ be complexes, and assume we are given homomorphisms

$$h : {}_1C^p \otimes {}_2C^q \to {}_3C^{p+q} \tag{20}$$

such that

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{\deg \alpha}\alpha(d\beta), \tag{21}$$

where we denote $h(\alpha \otimes \beta)$ by the juxtaposition $\alpha\beta$.

Assume further that we are given filtrations of each of these complexes such that

$$\alpha \in F_s({}_1C^p) \quad \text{and} \quad \beta \in F_t({}_2C^q) \;\Rightarrow\; \alpha\beta \in F_{s+t}({}_3C^{p+q}). \tag{22}$$

Denote the spectral sequence of the filtered complex ${}_iC^*$ by $\{{}_iE_r^{p,q}\}$. It follows easily from the definition (6) that there are induced products

$$h_r : {}_1E_r^{p,q} \otimes {}_2E_r^{s,t} \to {}_3E_r^{p+s,q+t} \tag{23}$$

and that the differentials d_r satisfy the analogue of (21). Also, the product h_{r+1} is induced from h_r via the isomorphism $E_{r+1} \approx H(E_r)$. It is also clear that h_∞ is induced from the h_r via the isomorphism $E_\infty \approx \varinjlim E_r$ when the filtrations in question are regular.

On the other hand, because of (21), h induces a product

$$H^p({}_1C^*) \otimes H^q({}_2C^*) \to H^{p+q}({}_3C^*)$$

satisfying the analogue of (22), for the filtrations given by (11). Because of this analogue of (22) we obtain an induced product

$$G_pH^{p+q}({}_1C^*) \otimes G_sH^{s+t}({}_2C^*) \to G_{p+s}H^{p+s+q+t}({}_3C^*).$$

By (12) this is equivalent to a product

$${}_1E_\infty^{p,q} \otimes {}_2E_\infty^{s,t} \to {}_3E_\infty^{p+s,q+t},$$

which can be seen to coincide with h_∞.

A special case of importance is that in which the ${}_iC^*$ are the total complexes of double complexes ${}_iC^{*,*}$ and in which h arises from a product

$${}_1C^{p,q} \otimes {}_2C^{s,t} \to {}_3C^{p+s,q+t}.$$

In this case the condition (22) is satisfied for both the first and second filtrations.

4 Homomorphisms

Let ${}_1C^*$ and ${}_2C^*$ be complexes with given filtrations and assume that we are given a chain map

$$h : {}_1C^* \to {}_2C^* \tag{24}$$

such that

$$h(F_p({}_1C^*)) \subset F_p({}_2C^*). \tag{25}$$

This situation is actually a special case of that considered in Section 3.

In brief, h induces homomorphisms

$$h_r : {}_1E_r^{p,q} \to {}_2E_r^{p,q}, \tag{26}$$

which commute with the differentials d_r; h_{r+1} is induced from h_r via the isomorphism $E_{r+1} \approx H(E_r)$; and in the case of regularly filtered complexes, h_∞ is induced from the h_r via the isomorphism $E_\infty \approx \varinjlim E_r$. Moreover, h_∞ is compatible, in the obvious sense, with the homomorphism

$$h^* : H^n({}_1C^*) \to H^n({}_2C^*).$$

It is sometimes useful to note that h^* cannot decrease the filtration degree of an element, which is the same as to say that it cannot increase the complementary degree.

A basic and often used fact concerning this situation is that if h is a chain map of regularly filtered complexes such that for some k,

$$h_k : {}_1E_k^{p,q} \to {}_2E_k^{p,q}$$

is an isomorphism for all p and q, then so is $h^* : H^n({}_1C^*) \to H^n({}_2C^*)$. This is proved by a standard spectral sequence argument as follows: The fact that h_k is a d_k-chain map and that h_{k+1} is the induced homomorphism in homology implies that h_{k+1} is an isomorphism. Inductively, h_r is an isomorphism for $r \geq k$, and consequently, h_∞ is also an isomorphism, by regularity. Then a repeated 5-lemma argument, using the regularity of the filtrations of $H^n({}_iC^*)$, shows that

$$h^* : F_pH^n({}_1C^*) \to F_pH^n({}_2C^*)$$

is an isomorphism for each p. The contention then follows from the fact that $H^n({}_iC^*) = \bigcup_p F_pH^n({}_iC^*)$.

When h_r is an isomorphism for all $r \geq k$ including $r = \infty$ and when h^* is also an isomorphism, then we say that this map of spectral sequences is an "isomorphism from E_k on." Thus, for regularly filtered complexes, this is equivalent to the hypothesis that h_k be an isomorphism.

Appendix B

Solutions to Selected Exercises

This appendix contains the solutions to a number of the exercises. Those exercises chosen for inclusion are the more difficult ones, or more interesting ones, or were chosen because of the importance of their usage in the main text.

Solutions for Chapter I:

1. By definition, $i^*\mathscr{A} = \{\langle b, a\rangle \in B \times \mathscr{A} \mid b = \pi(a)\}$ and $\mathscr{A}|B = \pi^{-1}(B)$. The functions $\langle b, a\rangle \mapsto a$ of $i^*\mathscr{A} \to \mathscr{A}|B$ and $a \mapsto \langle \pi(a), a\rangle$ of $\mathscr{A}|B \to i^*\mathscr{A}$ are both continuous and mutually inverse.
2. We will use the fact from I-Exercise 1 that $(i\mathscr{B})|B \approx i^*i\mathscr{B}$. The canonical homomorphism $\alpha : i^*i\mathscr{B} \to \mathscr{B}$ of Section I-4 is the composition

$$(i^*i\mathscr{B})_b = (i\mathscr{B})_{i(b)} = \varinjlim_{i(b)\in U} (i\mathscr{B})(U) = \varinjlim_{i(b)\in U} \mathscr{B}(i^{-1}U) \to \varinjlim_{b\in V} \mathscr{B}(V) = \mathscr{B}_b$$

   on the stalks at $b \in B$, this holding for *any map* $i : B \to X$ of arbitrary spaces. In our case, in which $i : B \hookrightarrow X$ is an inclusion of a subspace, the sets $i^{-1}U = U \cap B$ form a neighborhood basis of b in B. It follows that the arrow in the displayed composition is an isomorphism. Thus α is an isomorphism on each stalk and hence is an isomorphism.
4. Let X be a locally connected Hausdorff space without isolated points and let $\mathscr{P}$ be a projective sheaf on X. Suppose that $G = \mathscr{P}_{x_0} \neq 0$ for some $x_0 \in X$. Let $\mathscr{G}$ be the constant sheaf on X with stalk G. Let $A = \{x_0\}$, and note that $\mathscr{G}_A \approx \mathscr{P}_A$. Let $k : \mathscr{P} \to \mathscr{G}_A$ be the composition of the canonical epimorphism $\mathscr{P} \to \mathscr{P}_A$ with an isomorphism $\mathscr{P}_A \to \mathscr{G}_A$. The diagram

$$\begin{array}{ccccc} \mathscr{G} & \longrightarrow & \mathscr{G}_A & \longrightarrow & 0 \\ & \nwarrow^{h} & \uparrow{\scriptstyle k} & & \\ & & \mathscr{P} & & \end{array}$$

   can be completed since $\mathscr{P}$ is projective. Let s be a section, nonzero at x_0, of $\mathscr{P}$ over a connected neighborhood U of x_0. Then $h(s(x_0)) \neq 0$, and this implies that $h \circ s$ is nonzero on some connected neighborhood $V \subset U$ of x_0 since $\mathscr{G}$ is constant. Let $x \in V$, $x \neq x_0$, and put $B = \{x_0, x\}$. There is the inclusion map $i : \mathscr{G}_A \hookrightarrow \mathscr{G}_B$. Consider the diagram

$$\begin{array}{ccccc} \mathscr{G} & \xrightarrow{j} & \mathscr{G}_B & \longrightarrow & 0 \\ & \nwarrow^{h'} & \uparrow{\scriptstyle i \circ k} & & \\ & & \mathscr{P}, & & \end{array}$$

where j is the canonical epimorphism. Since V is connected, $h' \circ s$ is everywhere nonzero on V (since each $W_g = \{y \in V \mid h'(s)(y) = g\}$ is open for $g \in G$ and also closed since its complement is $\bigcup\{W_{g'} \mid g' \neq g\}$, which is open). Therefore $0 \neq j \circ h' \circ s(x) = i \circ k \circ s(x) = 0$, because $(\mathcal{G}_A)_x = 0$. This contradiction shows that $\mathcal{P} = 0$, as claimed.

8. Let $s \in (f\mathcal{A})(Y)$ correspond to $t \in \mathcal{A}(X)$. For $y \in Y$ we have that

$$\begin{aligned} y \notin |s| \quad &\Leftrightarrow \quad \exists \text{ open neighborhood } V \text{ of } y \text{ with } V \cap |s| = \varnothing \\ &\Leftrightarrow \quad \exists V \ni 0 = r_{V,Y}(s) \in (f\mathcal{A})(V) \\ &\Leftrightarrow \quad \exists V \ni 0 = r_{f^{-1}(V),X}(t) \in \mathcal{A}(f^{-1}V) \\ &\Leftrightarrow \quad \exists V \ni f^{-1}(V) \cap |t| = \varnothing \\ &\Leftrightarrow \quad \exists V \ni V \cap f(|t|) = \varnothing \\ &\Leftrightarrow \quad \exists V \ni V \cap \overline{f(|t|)} = \varnothing \\ &\Leftrightarrow \quad y \notin \overline{f(|t|)}. \end{aligned}$$

Therefore $|s| = \overline{f(|t|)}$. Consequently, if $|s| \in \Phi$, then $|t| \subset f^{-1}(|s|) \in f^{-1}(\Phi)$. Conversely, if $|t| \in f^{-1}(\Phi)$, then there is a closed set $K \in \Phi$ with $|t| \subset f^{-1}(K)$, and so $f(|t|) \subset K$. Hence $|s| = \overline{f(|t|)} \subset K$, and so $|s| \in \Phi$.

9. Let $y_0 \in Y$. For a sufficiently small open and connected neighborhood U of y_0 we have that $j'_{y_0,U} : \mathcal{A}'(U) \to \mathcal{A}'_{y_0}$ and $j''_{y_0,U} : \mathcal{A}''(U) \to \mathcal{A}''_{y_0}$ are isomorphisms. If $s''_1, \dots, s''_k$ are generators of $\mathcal{A}''_{y_0}$, let s_i be elements of $\mathcal{A}_{y_0}$ mapping to s''_i. Then U can also be assumed so small that $s_1, \dots, s_k$ are in the image of $j_{y_0,U} : \mathcal{A}(U) \to \mathcal{A}_{y_0}$. Then the composition $\mathcal{A}(U) \to \mathcal{A}_{y_0} \to \mathcal{A}''_{y_0}$ is onto. It follows that $\mathcal{A}(U) \to \mathcal{A}''(U)$ is onto. Since $\mathcal{A}'$ and $\mathcal{A}''$ are locally constant and U is connected, $j'_{y,U} : \mathcal{A}'(U) \to \mathcal{A}'_y$ and $j''_{y,U} : \mathcal{A}''(U) \to \mathcal{A}''_y$ are isomorphisms for all $y \in U$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}'(U) & \to & \mathcal{A}(U) & \to & \mathcal{A}''(U) & \to & 0 \\ & & \downarrow{\approx} & & \downarrow & & \downarrow{\approx} & & \\ 0 & \to & \mathcal{A}'_y & \to & \mathcal{A}_y & \to & \mathcal{A}''_y & \to & 0. \end{array}$$

The 5-lemma implies that $j_{y,U} : \mathcal{A}(U) \xrightarrow{\approx} \mathcal{A}_y$ for all $y \in U$. Let $A = \mathcal{A}_{y_0}$. Then the map $J : U \times A \to \mathcal{A}|U$, given by $J(y, a) = j_{y,U}(j^{-1}_{y_0,U}(a))$, is an isomorphism of sheaves. The continuity of J follows from the fact that each $j^{-1}_{y_0,U}(a) \in \mathcal{A}(U)$ is a section.

Note that $\mathcal{A}$ may be nonconstant even when $\mathcal{A}'$ and $\mathcal{A}''$ are constant, which is shown by the "twisted" sheaf $\mathcal{A}$ on $\mathbb{S}^1$ with stalks $\mathbb{Z}_4$ and the subsheaf $\mathcal{A}' = 2\mathcal{A} \approx \mathbb{Z}_2$.

10. Let C_n, $n \geq 1$, be the circle of diameter $1/n$ tangent to the real axis at the origin. Let $X = \bigcup C_n$. Let $\mathcal{S}_n$ be the locally constant sheaf with stalk $\mathbb{Z}_4$ on X that is "twisted" on C_n (only), and put $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{S}_n$. Then the subgroups $\mathbb{Z}_2$ in each summand of each stalk of $\mathcal{A}$ provide a subsheaf $\mathcal{A}' \subset \mathcal{A}$. The quotient sheaf $\mathcal{A}'' = \mathcal{A}/\mathcal{A}'$ is isomorphic to $\mathcal{A}'$, and both $\mathcal{A}'$ and $\mathcal{A}''$ are constant sheaves with stalks $\bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. The sheaf $\mathcal{A}$ is not locally constant since $\mathcal{A}|C_n$ is not constant because of the twisted $\mathbb{Z}_4$ in the nth factor.

11. It is only necessary to show that $\Gamma_\Phi(\mathscr{A}) \to \Gamma_\Phi(\mathscr{A}'')$ is surjective. Let G'' be the common stalk of $\mathscr{A}''$. For $s \in \Gamma_\Phi(\mathscr{A}'')$ and $g \in G''$ let $U_g = \{x \in X \mid s(x) = g\}$. This is an open set since it is essentially the intersection of $s(x)$ with the constant section equal to g. Since $X - U_g = \bigcup\{U_h \mid h \neq g\}$ is open, U_g is also closed. If G is the stalk of $\mathscr{A}$, let $f : G'' \to G$ be a function (not a homomorphism) splitting the surjection $G \to G''$ such that $f(0) = 0$. Put $t(x) = f(s(x))$. Then t is a section of $\mathscr{A}$ and $|t| = |s|$. Thus $t \in \Gamma_\Phi(\mathscr{A})$, and it maps to $s \in \Gamma_\Phi(\mathscr{A}'')$.

12. Take G as coefficient group. For a (possibly infinite) singular p-chain s, we will use the notation $s = \sum_\sigma s(\sigma)\sigma$, where σ ranges over all singular simplices $\sigma : \Delta_p \to X$ and $s(\sigma) \in G$. By abuse of notation we also set $\sigma \cap U = \sigma(\Delta_p) \cap U \subset X$. If $s \in \Delta_p(X, X - U)$ and $x \in U$, then there is an open neighborhood V of x such that $s(\sigma) \neq 0$, for at most a finite number of σ such that $\sigma \cap V \neq \varnothing$. Let $t(\sigma) = s(\sigma)$ if $\sigma \cap V \neq \varnothing$ and $t(\sigma) = 0$ if $\sigma \cap V = \varnothing$. Then $t \in \Delta_p^c(X, X - V)$, and the canonical inclusion $\Delta_p^c(X, X - V) \hookrightarrow \Delta_p(X, X - V)$ takes t to s. It follows that the induced homomorphism of generated sheaves is an isomorphism. Let $A(U) = \Delta_p(X, X - U)$. To show that A is a monopresheaf, let $U = \bigcup U_\alpha$, and suppose that $s \in A(U)$ has each $s_\alpha = s|U_\alpha = 0$ in $A(U_\alpha)$. This means that s is a chain in $X - U_\alpha$ for all α (i.e., that $s(\sigma) = 0$ whenever $\sigma \cap U_\alpha \neq \varnothing$). Therefore s is a chain in $X - \bigcup U_\alpha = X - U$, but that means that $s = 0$ in $A(U)$.

Now let $X = \bigcup U_\alpha$ and $s_\alpha \in A(U_\alpha)$. Assume that $s_\alpha|U_\alpha \cap U_\beta = s_\beta|U_\alpha \cap U_\beta$ for all α, β. Given the singular simplex σ, define $f : \Delta_p \to G$ by $f(x) = s_\alpha(\sigma)$ for any α such that $\sigma(x) \in U_\alpha$. If $\sigma(x) \in U_\alpha \cap U_\beta$, then $s_\alpha(\sigma) = s_\beta(\sigma)$ since $s_\alpha - s_\beta$ is a chain in $X - (U_\alpha \cap U_\beta) \subset X - \{x\}$. Thus f is well-defined. Now, f is locally constant and hence continuous, where G has the discrete topology. Since Δ_p is connected, f is constant. Therefore the definition $s(\sigma) = f(x)$ for $x \in \sigma(\Delta_p)$ makes sense and defines a chain s. Since $s(\sigma) = s_\alpha(\sigma)$ for all σ such that $\sigma \cap U_\alpha \neq \varnothing$, it follows that s is locally finite and that $s|U_\alpha = s_\alpha$. Thus A is conjunctive for coverings of X. By I-6.2, $\theta : A(X) \to \Gamma(\varDelta_*)$ is isomorphic when X is paracompact. If $s \in A(X)$ is a locally finite chain such that $\theta(s) \in \Gamma_c(\varDelta_*)$ and $K = |\theta(s)|$, then $s(\sigma) = 0$ whenever $\sigma \cap K \neq \varnothing$. Since K is compact and s is locally finite, it follows that s is finite. Therefore, θ restricts to an isomorphism $\Delta_*^c(X) \xrightarrow{\approx} \Gamma_c(\varDelta_*)$.[1]

To see that $A(U)$ is not generally fully conjunctive, consider two open sets U_1 and U_2 and a singular simplex σ intersecting both U_1 and U_2 but not intersecting $U_1 \cap U_2$. Then $\sigma \in A(U_1)$ and $0 \in A(U_2)$ restrict to the same element $0 \in A(U_1 \cap U_2)$ but do not come from a common element of $A(U_1 \cup U_2)$. Also, if $X = [0, 1]$ and $U_n = (1/n, 1)$, then one can find locally finite chains $s_n \in A(U_n)$ that do come from a (unique) chain s in $(0, 1)$, but with s not locally finite in X.[2]

14. That d is onto is the statement that a continuous function is locally integrable. That $\operatorname{Im} i = \operatorname{Ker} d$ is the statement that a differentiable function on an open neighborhood U of $x \in X$ whose derivative is zero on U is

[1]Note, however, that c is not paracompactifying unless X is locally compact.

[2]Note that the boundary operator does not make sense for infinite non–locally finite chains in general.

constant on some smaller neighborhood V of x, indeed V can be taken to be the component of U containing x. The statement that d is onto is that a continuous function on an open neighborhood U of x can be integrated on a smaller open neighborhood V of x. The group $\mathbb{R}(X)$ of global sections is just the group of constant functions on X since X is connected. Also, $\mathscr{D}(X)$ is the group of continuously differentiable functions on X, and $\mathscr{C}(X)$ is the group of continuous functions on X. Several standard results of elementary calculus imply that

$$0 \longrightarrow \mathbb{R}(X) \xrightarrow{i_X} \mathscr{D}(X) \xrightarrow{d_X} \mathscr{C}(X) \xrightarrow{\oint} \mathbb{R} \longrightarrow 0$$

is exact, whence $\operatorname{Coker} d_X \approx \mathbb{R}$. Note also that $0 \to \mathbb{R}(U) \to \mathscr{D}(U) \to \mathscr{C}(U) \to 0$ is exact for all *proper* open $U \subset X$. Also keep in mind that for example, $\mathbb{R}(U)$ is *not* the group of constant functions on U but rather the group of functions on U that are constant on each component of U.

15. First let us show that the presheaf $U \mapsto G(U) = \mathscr{F}(U)/\mathbb{Z}(U)$ is a sheaf. ($\mathbb{Z}$ is regarded as a *sheaf* here, so that $\mathbb{Z}(U)$ is the set of all locally constant functions $U \to \mathbb{Z}$; i.e., functions that are constant on each component of U.) Let $U = \bigcup U_\alpha$ and let $j : \mathscr{F}(U) \to G(U)$ be the projection. To prove (S1), let $s, t \in \mathscr{F}(U)$ be such that $j(s)|U_\alpha = j(t)|U_\alpha$ for each α. This means that $s - t$ is locally constant on U. But that implies that $j(s) = j(t)$ on U.

To show that G is conjunctive, let $s_\alpha \in \mathscr{F}(U_\alpha)$ be such that $j(s_\alpha) = j(s_\beta)$ on $U_\alpha \cap U_\beta$. This means that $s_\alpha - s_\beta$ is locally constant on $U_\alpha \cap U_\beta$. If V is a component of U, then V is an open interval. One can find an increasing, doubly infinite, discrete on V set of points $x_n \in V$, $n \in \mathbb{Z}$, such that each (x_{n-1}, x_{n+1}) is contained in some U_α.[3] Let U_n denote one such U_α. On the interval (x_n, x_{n+1}), both s_n and s_{n+1} are defined and their difference is constant on (x_n, x_{n+1}). Let $p_n = s_n - s_{n+1} \in \mathbb{Z}$ be this constant value. Define a function $s : V \to \mathbb{Z}$ by $s(x) = s_0$ on (x_{-1}, x_1), by $s(x) = s_n + p_{n-1} + \cdots + p_0$ on (x_{n-1}, x_{n+1}) for $n > 0$, and by $s(x) = s_n - (p_n + \cdots + p_{-1})$ on (x_{n-1}, x_{n+1}) for $n < 0$. We see immediately that these definitions agree on the overlaps $(x_{n-1}, x_{n+1}) \cap (x_n, x_{n+2}) = (x_n, x_{n+1})$, and so s is well-defined. Putting these together on all components of U gives a function $s \in \mathscr{F}(U)$ that differs from s_α on U_α by a locally constant function. Hence, $j(s) \in \mathscr{F}(U)/\mathbb{Z}(U)$ extends each $j(s_\alpha)$.

Thus the generated sheaf $\mathscr{G}$ is just $\mathscr{G}(U) = \mathscr{F}(U)/\mathbb{Z}(U)$, and in particular, $\mathscr{F}(X) \to \mathscr{G}(X)$ is onto.

Let $t \in \Gamma_c(\mathscr{G})$, and pull t back to some $s \in \Gamma(\mathscr{F})$. Suppose that $|t| \subset [-n, n]$. Then s is constant on $(-\infty, -n)$. We can modify s by a constant function so that the new s is zero on $(-\infty, -n)$. Then s is uniquely defined by this requirement and the fact that it maps into t by the exact sequence $0 \to \mathbb{Z} \to \mathscr{F}(X) \to \mathscr{G}(X) \to 0$. Then t is constant on (n, ∞). Set $k(s)$ equal to this constant value. Clearly $k(s) = 0 \Leftrightarrow t \in \Gamma_c(\mathscr{F})$. Therefore k induces an exact sequence $0 \to \Gamma_c(\mathbb{Z}) \to \Gamma_c(\mathscr{F}) \to \Gamma_c(\mathscr{G}) \xrightarrow{k} \mathbb{Z} \to 0$, as claimed. Note that $\Gamma_c(\mathbb{Z}) = 0$.

[3]One chooses x_n (and x_{-n}) by induction on $n \geq 0$ as follows: If x_n has been chosen, consider all intervals (u, v) with $x_n \in (u, v) \subset U_\alpha \cap V$ for some α. Let $N = \sup(v - x_n)$ over this collection of intervals and choose $x_{n+1} > x_n$ so that $x_{n+1} - x_n > N/2$.

Those who have delved into Chapters II and III will note a trivial solution of this problem coming from the exact sequences

$$0 \to \Gamma(\mathbb{Z}) \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \to H^1(\mathbb{R};\mathbb{Z})$$

and

$$0 \to \Gamma_c(\mathbb{Z}) \to \Gamma_c(\mathscr{F}) \to \Gamma_c(\mathscr{G}) \to H^1_c(\mathbb{R};\mathbb{Z}) \to H^1_c(\mathbb{R};\mathscr{F})$$

and the facts that $H^1(\mathbb{R};\mathbb{Z}) = 0$, $H^1_c(\mathbb{R};\mathbb{Z}) \approx \mathbb{Z}$, and $H^1_c(\mathbb{R};\mathscr{F}) = 0$ since $\mathscr{F}$ is (obviously) flabby. This indicates the power of the theory in Chapter II.

18. As remarked in I-4, the composition

$$f^*\mathscr{M} \xrightarrow{f^*(\beta)} f^*ff^*\mathscr{M} \xrightarrow{\alpha} f^*\mathscr{M}$$

is the identity. This implies that β is monomorphic over $f(A)$ (for *any* sheaf $\mathscr{M}$). But $\operatorname{Ker}\beta$ is a subsheaf of a constant sheaf (hence an open subset as a space) and this implies that $N = \{x \in X \mid (\operatorname{Ker}\beta)_x \neq 0\}$ is open. Since $N \cap f(A) = \varnothing$ and $f(A)$ is dense in X we must have that $N = \varnothing$, i.e., that $\operatorname{Ker}\beta = 0$. For the requested counterexample for arbitrary $\mathscr{M}$, take $A = (0,1)$, $X = [0,1]$, $f : A \hookrightarrow X$, and $\mathscr{M} = \mathbb{Z}_{\{0\}}$. Then $ff^*\mathscr{M} = 0$.

19. We know that ${}_{AS}H^n_x(X;G) \approx \check{H}^n_x(X;G)$. But for a cochain $c \in \check{C}^n(\mathfrak{U};G)$, $c(\alpha_0,\ldots,\alpha_n) \neq 0 \Rightarrow U_{\alpha_0,\ldots,\alpha_n} \subset |c|$. Thus, if $0 \neq c \in \check{C}^n_x(\mathfrak{U};G)$, then $\{x\}$ is open in X, in which case the result is trivial.

Solutions for Chapter II:

1. (Compare Section II-18.) Take $X = [0,1]$, $F = \{0,1\}$, $A = X - F = (0,1)$, $\Phi = cld|F$, G any nonzero constant sheaf on X, and $\mathscr{B} = G|A$. Then $\mathscr{B}$ is constant on A and $\mathscr{B}^X = G_A$. We have that

$$H^*_{\Phi|A}(A;\mathscr{B}) = 0$$

since $\Phi|A = 0$. Also,

$$H^*_\Phi(X;\mathscr{B}^X) = H^*_{cld|F}(X;G_A) \approx H^*(X,A;G_A)$$

by II-12.1 since an open set A is always taut. The exact sequence of the pair (X,A) with coefficients in G_A has the segment

$$H^0(X;G_A) \to H^0(A;G|A) \to H^1(X,A;G_A). \tag{1}$$

But the first term is $\Gamma(G_A) = 0$ and the second term is $\Gamma(G|A) \approx G \neq 0$, so that $H^1_\Phi(X;\mathscr{B}^X) \approx H^1(X,A;G_A) \neq 0$.

It is of interest to note that the next term in the exact sequence (1) is $H^1(X;G_A) \approx H^1(X,F;G) \approx G$ by II-12.3 and using the fact from Chapter III that this is isomorphic to singular cohomology (or using a direct computation). The next term is $H^1(A;G|A) = 0$ since A is contractible and $G|A$ is constant. It follows that for $G = \mathbb{Z}$ we have that $H^1(X,A;\mathbb{Z}_A) \approx \mathbb{Z} \oplus \mathbb{Z}$.

2. First suppose that $A \subset X$ is closed. Then we claim that A contains a smallest element. To see this, consider the family $\mathfrak{F}$ of sets of the form $F_x = \{y \in A \mid y \leq x\}$ for $x \in A$. Then $\mathfrak{F}$ satisfies the finite intersection property and so has a nonempty intersection. Obviously this intersection must be a single element, which is $\min A$. Similarly, A contains a largest element. Now assume that $0 \neq \alpha \in H^p(X)$ for some $p > 0$. Let A be a minimal closed set such that $0 \neq \alpha|A \in H^p(A)$, which exists by II-10.8. Let $x_0 = \min A$ and $x_1 = \max A$. Now, $A \neq \{x_0, x_1\}$, since $H^p(A) \neq 0$, and so there is an $x \in A$ with $x_0 < x < x_1$. Put $A_0 = \{y \in A \mid y \leq x\}$ and $A_1 = \{y \in A \mid y \geq x\}$. Then $A_0 \cap A_1 = \{x\}$ and $A_0 \cup A_1 = A$. The exact Mayer-Vietoris sequence

$$0 = \tilde{H}^{p-1}(\{x\}) \to H^p(A) \to H^p(A_0) \oplus H^p(A_1)$$

shows that α restricts nontrivially to at least one of A_0, A_1. This contradiction shows that α cannot exist.

3. Let T be the "long ray" $[0, \Omega)$ compactified by the point Ω at infinity. By II-Exercise 2, T is acyclic. Let L be the long line $(-\Omega, 0] \cup [0, \Omega)$. Define the "long contraction" $h : L \times T \to L$ by $h(a, b) = \min(a, b)$ and $h(-a, b) = -\min(a, b)$ for $a \geq 0$. Then $h(x, 0) = 0$ and $h(x, \Omega) = x$ for all $x \in L$. Let $\pi : L \times T \to L$ be the projection and $i_t : L \to L \times T$ the inclusion $i_t(a) = (a, t)$. By II-11.8, π^* is an isomorphism and $i_t^* = (\pi^*)^{-1}$. Now, $h \circ i_0(x) = h(x, 0) = 0$ and $h \circ i_\Omega(x) = h(x, \Omega) = x$. Therefore

$$0 = \text{constant}^* = (h \circ i_0)^* = (h \circ i_\Omega)^* = 1^* = 1 : H^p(L) \to H^p(L)$$

for $p > 0$, whence $H^p(L) = 0$.

4. Let $\mathscr{A}_n = \mathbb{Z}_{(-\infty,n)}$. Then $\varinjlim \mathscr{A}_n = \mathbb{Z}$, $\Gamma(\mathscr{A}_n) = 0$ and $\Gamma(\mathbb{Z}) = \mathbb{Z}$, so that θ is not onto. Let $\mathscr{B}_n = \mathbb{Z}_{[n,\infty)}$. Then $\varinjlim \mathscr{B}_n = 0$ and $\varinjlim \Gamma(\mathscr{B}_n) = \mathbb{Z}$, so that θ is not one-to-one.

6. Let A and B be two closed subspaces of X that cannot be separated by open sets. There is the canonical map $\rho : \varinjlim \Gamma(\mathbb{Z}|W) \to \Gamma(\mathbb{Z}|A \cup B)$, induced by restrictions, where W ranges over the open neighborhoods of $A \cup B$. The section $s \in \Gamma(\mathbb{Z}|A \cup B)$ taking value 0 on A and 1 on B cannot be in $\operatorname{Im} \rho$ because if $t \in \Gamma(\mathbb{Z}|W)$ has $\rho(t) = s$, then the sets $U = \{w \in W \mid t(w) = 0\}$ and $V = \{w \in W \mid t(w) = 1\}$ are open neighborhoods of A and B that are disjoint. This shows, by II-10.6, that $A \cup B$ is not taut.

   One example in which the subspace $A \cup B$ is paracompact is given by the topological 2-manifold M whose boundary consists of an uncountable number of components each homeomorphic to $\mathbb{R}$ and whose boundary can be split into two unions A and B, of its components, that cannot be separated by open sets; see [19, I-17.5].

9. Let $X^+ = X + \{\infty\}$ with open sets the open sets of X and the complements $X^+ - K$ of the members of Φ. Since every point of X has a neighborhood in Φ, X^+ is Hausdorff. We claim that X^+ is paracompact. To see this, let $\{U_\alpha\}$ be an open covering of X^+, and assume that $\infty \in U_{\alpha_0}$. Then $K = X^+ - U_{\alpha_0} \in \Phi$. Let $K' \in \Phi$ be a neighborhood of K. Then K' is covered by the sets $V_\alpha = U_\alpha \cap \operatorname{int} K'$ and $V = K' - K$. Let $\{W_\beta\}$ be a locally finite refinement of $\{V_\alpha\}$ on K'. Then the sets $W_\beta \cap \operatorname{int} K'$ and $U_{\alpha_0} = X^+ - K$ form a locally finite refinement of $\{U_\alpha\}$. For a sheaf $\mathscr{A}$

on X we have $H^*(X^+; \mathscr{A}^{X^+}) \approx H^*_\Phi(X; \mathscr{A})$ by II-10.1 since $\Phi = cld|X$ and cld is paracompactifying on X^+.

10. Let $s \in \mathscr{A}(V)$ for $V \subset X$ open. Consider the collection $\mathscr{W}$ of all pairs (W, s_W), where $W \supset V$ is open, $s_W \in \mathscr{A}(W)$, and $s_W|V = s$. Order $\mathscr{W}$ by $(W, s_W) < (W', s_{W'})$ if $W \subset W'$ and $s_{W'}|W = s_W$. The union of any chain in $\mathscr{W}$ is in $\mathscr{W}$, and so $\mathscr{W}$ is inductively ordered. Suppose that (W, s_W) is maximal in $\mathscr{W}$. If $W \neq X$, then let $x \in X - W$ and let $U = U_x$. Then there is a section $s_U \in \mathscr{A}(U)$ with $s_U|U \cap V = s|U \cap V$. Now, $s_W - s_U$ on $W \cap U$ extends to some $t \in \mathscr{A}(U)$, since $\mathscr{A}|U$ is flabby. Then $s_W - s_U - t = 0$ on $W \cap U$ so that s_W on W and $s_U + t$ on U match on $W \cap U$, and so combine to give an extension of s_W to $W \cup U$. This contradicts the maximality of (W, s_W) and shows that $W = X$ as desired.
11. This an immediate consequence of the cohomology sequence of (X, F), II-10.1, and II-10.2.
12. Since Φ is paracompactifying, K has a neighborhood $K' \in \Phi$ with $K' \subset U$. Suppose that $\mathscr{A}$ is Φ-fine. Then, by definition, the sheaf $\mathscr{H}om(\mathscr{A}, \mathscr{A})$ is Φ-soft. Thus the section $f \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A})|K \cup \partial K')$, which is 1 on K and 0 on $\partial K'$, can be extended to K and then can be extended by 0 to all of X. The resulting extension $g \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A})) = \operatorname{Hom}(\mathscr{A}, \mathscr{A})$ has the desired properties. Conversely, suppose that the stated property holds. Let $f \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A})|K)$. By II-9.5, f extends to $g \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A})|V) = \operatorname{Hom}(\mathscr{A}|V, \mathscr{A}|V)$ for some open $V \supset K$. There is an open neighborhood U of K with $\overline{U} \subset V$ since K has a paracompact neighborhood. Let $h \in \operatorname{Hom}(\mathscr{A}, \mathscr{A})$ be 1 on K and 0 outside U as hypothesized. Then $hg \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A})|V)$ is g on K and vanishes on $V - \overline{U}$. Extending hg by 0 to all of X gives an extension of f to some $k \in \Gamma(\mathscr{H}om(\mathscr{A}, \mathscr{A}))$. Thus $\mathscr{H}om(\mathscr{A}, \mathscr{A})$ is Φ-soft, whence $\mathscr{A}$ is Φ-fine.
14. (a) This reduces immediately to the compact case by II-9.12. Let K, W, $\{U_\alpha\}$, and $\{V_\beta\}$ be as in the hint. By II-Exercise 13 there are partitions of unity $\{h_\alpha\} \subset \mathscr{H}om(\mathscr{A}, \mathscr{A})$ and $\{k_\beta\} \subset \mathscr{H}om(\mathscr{B}, \mathscr{B})$ subordinate to $\{U_\alpha\}$ and $\{V_\beta\}$ respectively. Let $s = \sum h_\alpha \widehat{\otimes} k_\beta$, the sum ranging over those pairs $\langle \alpha, \beta \rangle$ for which $U_\alpha \times V_\beta \subset W$, and let t be the same sum over all other pairs $\langle \alpha, \beta \rangle$. Then

$$s + t = \sum_\alpha \sum_\beta h_\alpha \widehat{\otimes} k_\beta = \sum_\alpha h_\alpha \widehat{\otimes} \sum_\beta k_\beta = \sum_\alpha h_\alpha \widehat{\otimes} 1 = 1 \widehat{\otimes} 1 = 1,$$

so that s is an endomorphism of $\mathscr{A} \widehat{\otimes} \mathscr{B}$ that is 1 on K and 0 outside W. Consequently, $\mathscr{A} \widehat{\otimes} \mathscr{B}$ is c-fine by II-Exercise 12.

(b) Let $\mathscr{L} = \mathscr{A} \widehat{\otimes} \mathscr{B}$ and let $\mathscr{M}$ denote the collection of all open sets W of $X \times Y$ such that $\mathscr{L}_W$ is c-acyclic. Let $\mathscr{M}_1$ denote the collection of all open subsets $W = U \times V$ where $U \subset X$ and $V \subset Y$ are open. Since $\mathscr{L}_{U \times V} = \mathscr{A}_U \widehat{\otimes} \mathscr{B}_V$, II-15.2 implies that $U \times V \in \mathscr{M}$, that is, $\mathscr{M}_1 \subset \mathscr{M}$. Let $\mathscr{M}_k$ be the collection of all unions of k members of $\mathscr{M}_1$. Since

$$(U \times V) \cap \bigcup_{i=1}^{k} (U_i \times V_i) = (U_1 \cap U) \times (V_1 \cap V) \cup \cdots \cup (U_k \cap U) \times (V_k \cap V),$$

we see that $W \in \mathscr{M}_1$, $W' \in \mathscr{M}_k \Rightarrow W \cap W' \in \mathscr{M}_k$. Suppose that $\mathscr{M}_k \subset \mathscr{M}$, and let $W \in \mathscr{M}_1$ and $W' \in \mathscr{M}_k$. Then W, W', and $W \cap W'$ are all in

$\mathscr{M}$, and it follows from the Mayer-Vietoris sequence (27) on page 94 that $W \cup W' \in \mathscr{M}$. Thus $\mathscr{M}_{k+1} \subset \mathscr{M}$. By induction, $\mathscr{M}_\infty = \bigcup_k \mathscr{M}_k \subset \mathscr{M}$. Now, $\mathscr{M}_\infty$ is directed by inclusion, and any open set $W \subset X \times Y$ is the directed union of those members of $\mathscr{M}_\infty$ contained in W. By continuity, II-14.5, it follows that $\mathscr{M}$ consists of all open sets in $X \times Y$. Thus $\mathscr{L}$ is c-soft by II-16.1.

15. We are to show that $i\mathscr{A}(W) \to i\mathscr{A}(K)$ is onto for $K \subset W$ closed. Let $s \in i\mathscr{A}(K)$. If $\infty \notin K$, then the argument in the proof of II-9.3 shows that s, as a member of $\mathscr{A}(K)$, extends to X so as to be zero outside some compact neighborhood of K. Hence it extends by 0 to a global section of $i\mathscr{A}$. If $\infty \in K$, then s extends to a compact neighborhood K' of K by II-9.5. Let $C = W - \operatorname{int}(K') \subset X$, which is compact. Since $\mathscr{A}$ is c-soft and $\mathscr{A} = (i\mathscr{A})|X$, $s|\partial C$ extends to C, and this extends s to a global section of $i\mathscr{A}$. (Note that a virtually identical argument applies to the one-point paracompactification of II-Exercise 9.)

18. The hypothesized exact sequence of (X, A) has $H^p_{\Phi|X-A}(X;\mathscr{A}) = 0 = H^p_\Phi(X;\mathscr{A})$ for $p > 0$ and $\mathscr{A}$ flabby. Thus $\Gamma_\Phi(\mathscr{A}) \to \Gamma_{\Phi\cap A}(\mathscr{A}|A)$ is onto and $H^p_{\Phi\cap A}(A;\mathscr{A}|A) = 0$ for $p > 0$ and $\mathscr{A}$ flabby, which is precisely the definition of A being Φ-taut.

21. (a) $\Rightarrow$ (b) by II-5.5.
    (b) $\Rightarrow$ (c) is tautological.
    (c) $\Rightarrow$ (d) by $H^1(X,U;\mathscr{L}) \approx H^1_{cld|X-U}(X;\mathscr{L})$ from II-12.1.
    (d) $\Rightarrow$ (a) by the exact sequence $H^0(X;\mathscr{L}) \to H^0(U;\mathscr{L}|U) \to H^1(X,U;\mathscr{L})$.

23. Let $U = \{1/n \mid n \geq 1\}$, which is open in X, and consider the sheaf $\mathscr{A} = \mathbb{Z}_U$. The section 1 of $\mathbb{Z}_U$ over U cannot be extended to X because an extension would have to be 0 at $\{0\}$, since $\mathbb{Z}_U$ has stalk 0 at $\{0\}$, and hence would have to be 0 in a neighborhood of $\{0\}$. Thus $\mathscr{A}$ is not flabby, and so $\operatorname{Dim} X > 0$ by II-Exercise 21. The inclusion $\mathscr{A} \hookrightarrow \mathscr{C}^0(X;\mathscr{A})$ is an isomorphism on U, and so $\mathscr{Z}^1(X;\mathscr{A})$ is concentrated on $\{0\}$ and hence is flabby. By II-Exercise 22, $\operatorname{Dim} X \leq 1$, and so $\operatorname{Dim} X = 1$. (Alternatively, use II-16.11.) Note that $H^1_{cld|\{0\}}(X;\mathbb{Z}_U) \approx H^1(X,U;\mathbb{Z}_U) \neq 0$ by II-12.1 and the exact sequence $H^0(X;\mathbb{Z}_U) \to H^0(U;\mathbb{Z}_U|U) \to H^1(X,U;\mathbb{Z}_U)$.

25. By II-16.8 and II-16.11, every point has a neighborhood U with $\operatorname{Dim} U \leq n+1$. By II-Exercise 22, $\mathscr{Z}^{n+1}(X;\mathscr{A})|U$ is flabby. By II-Exercise 10, $\mathscr{Z}^{n+1}(X;\mathscr{A})$ is flabby. By II-Exercise 22, $\operatorname{Dim} X \leq n+1$.

    For the second part let $F \subset \mathbb{R}^n$ be the union of the spheres about the origin $x = 0$ of radii $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Let $\mathbb{S}^n$ be the compactified $\mathbb{R}^n$, and let $U = \mathbb{S}^n - \{x\} - F$. Let Ψ be the family of supports on $\mathbb{S}^n - \{x\}$ consisting of the sets K closed in $\mathbb{S}^n - \{x\}$ and contained in U. Then we have $H^n(\mathbb{S}^n - \{x\};\mathbb{Z}_U) \approx H^n_\Psi(U;\mathbb{Z}) \approx \prod_{i=1}^\infty \mathbb{Z}$, which is uncountable. Also, regarding $\mathbb{Z}_U$ as a sheaf on $\mathbb{S}^n$, $H^n(\mathbb{S}^n;\mathbb{Z}_U) \approx H^n_c(U;\mathbb{Z}) \approx \bigoplus_{i=1}^\infty \mathbb{Z}$ is countable. The exact sequence of $(\mathbb{S}^n, \mathbb{S}^n - \{x\})$ with coefficients in $\mathbb{Z}_U$ has the segment

$$H^n(\mathbb{S}^n;\mathbb{Z}_U) \to H^n(\mathbb{S}^n - \{x\};\mathbb{Z}_U) \to H^{n+1}(\mathbb{S}^n, \mathbb{S}^n - \{x\};\mathbb{Z}_U).$$

    It follows that $H^{n+1}(\mathbb{S}^n, \mathbb{S}^n - \{x\};\mathbb{Z}_U)$ is uncountable. By excision,

$$H^{n+1}_\Theta(\mathbb{R}^n;\mathbb{Z}_U) \approx H^{n+1}(\mathbb{R}^n, \mathbb{R}^n - \{x\};\mathbb{Z}_U)$$

is uncountable, where $\Theta = \{\{x\}, \varnothing\}$. Therefore $\operatorname{Dim} M^n = n+1$ for any n-manifold M^n $(n > 0)$. This fact is due to Satya Deo [31].

27. If $\{\mathscr{A}_n, \pi_n\}$ is an inverse sequence, then the definition $\mathscr{A}(U_n) = \mathscr{A}_n$ with $\pi_n : \mathscr{A}(U_n) \to \mathscr{A}(U_{n-1})$ and $\mathscr{A}(\mathbb{N}) = \varprojlim \mathscr{A}_n$ gives a presheaf on $\mathbb{N}$ that is clearly a conjunctive monopresheaf and hence a sheaf. Conversely, if $\mathscr{A}$ is a presheaf, then putting $\mathscr{A}_n = \mathscr{A}(U_n)$ with π_n the restriction map gives an inverse sequence. Moreover, the only thing preventing $\mathscr{A}$ from being a sheaf is the requirement that $\mathscr{A}(\mathbb{N}) = \varprojlim \mathscr{A}(U_n)$. (Hence also $H^0(\mathbb{N};\mathscr{A}) = \varprojlim \mathscr{A}_n$.)

A serration over U_n is just an n-tuple $\langle a_1, \dots, a_n\rangle$ with $a_i \in \mathscr{A}_i$, whence $C^0(U_n;\mathscr{A})$ is the group of such n-tuples. The sheaf $\mathscr{C}^0(\mathbb{N};\mathscr{A})$ is equivalent to the inverse sequence in which the nth term $C^0(U_n;\mathscr{A})$ is the group of n-tuples $\langle a_1, \dots, a_n\rangle$ and the restriction $C^0(U_n;\mathscr{A}) \to C^0(U_{n-1};\mathscr{A})$ takes $\langle a_1, \dots, a_n\rangle$ to $\langle a_1, \dots, a_{n-1}\rangle$. The inclusion $\varepsilon : \mathscr{A} \hookrightarrow \mathscr{C}^0(\mathbb{N};\mathscr{A})$ takes $a_n \in \mathscr{A}(U_n)$ to the n-tuple $\langle a_1, \dots, a_n\rangle$, where $a_{i-1} = \pi_i a_i$ for all $1 < i \le n$. The presheaf cokernel of ε has value on U_n that is the quotient of the group of n-tuples $\langle a_1, \dots, a_n\rangle$ modulo those that satisfy $a_{i-1} = \pi_i a_i$. The map $\langle a_1, \dots, a_n\rangle \mapsto \langle a_1 - \pi_2 a_2, \dots, a_{n-1} - \pi_n a_n\rangle \in C^0(U_{n-1};\mathscr{A})$ induces an isomorphism on this cokernel, and it is onto $C^0(U_{n-1};\mathscr{A})$, since given $\langle a_1', \dots, a_{n-1}'\rangle$, the system of equations

$$\begin{aligned} a_1' &= a_1 - \pi_2 a_2 \\ a_2' &= a_2 - \pi_3 a_3 \\ &\dots \\ a_{n-1}' &= a_{n-1} - \pi_n a_n \end{aligned}$$

has a solution (solving backward from $a_n = 0$). Therefore $Z^1(U_n;\mathscr{A})$, $\mathscr{Z}^1(\mathbb{N};\mathscr{A})$, and $d : C^0(U_n;\mathscr{A}) \to Z^1(U_n;\mathscr{A})$ are as claimed. Now $\mathscr{A}$ is flabby $\Leftrightarrow$ each $\mathscr{A}(\mathbb{N}) \to \mathscr{A}(U_n)$ is surjective. This means that given $a_n \in \mathscr{A}_n$, there are $a_{n+1}, \dots$ with $\pi_i a_i = a_{i-1}$ for all $i > n$. Therefore $\mathscr{A}$ is flabby $\Leftrightarrow$ each π_n is surjective.

Since $\mathscr{Z}^1(\mathbb{N};\mathscr{A}) \approx \eta\mathscr{C}^0(\mathbb{N};\mathscr{A})$, where $\eta : \mathbb{N} \to \mathbb{N}$ is given by $\eta(n) = n+1$, it is flabby by II-5.7 or by direct examination. Therefore

$$0 \to \mathscr{A} \to \mathscr{C}^0(\mathbb{N};\mathscr{A}) \to \mathscr{Z}^1(\mathbb{N};\mathscr{A}) \to 0$$

is a flabby resolution of $\mathscr{A}$, and so $\dim \mathbb{N} \le 1$ (indeed $\operatorname{Dim} \mathbb{N} \le 1$; see Exercise II-22). Using this resolution, $H^1(\mathbb{N};\mathscr{A})$ is the cokernel of $d : C^0(\mathbb{N};\mathscr{A}) \to Z^1(\mathbb{N};\mathscr{A})$, and so $\mathscr{A}$ is acyclic $\Leftrightarrow$ $d : \langle a_1, \dots, a_n, \dots\rangle \mapsto \langle a_1 - \pi_2 a_2, \dots, a_n - \pi_{n+1} a_{n+1}, \dots\rangle$ is surjective, leading to the claimed criterion.

The inverse system $\mathscr{A}$ in which $\mathscr{A}_n = \mathbb{Z}$ for all n and $\pi_n : \mathscr{A}_n \to \mathscr{A}_{n-1}$ is multiplication by n gives a sheaf that is not acyclic since the system of equations

$$\begin{aligned} 2a_2 &= a_1 - 1 \\ 3a_3 &= a_2 - 1 \\ &\dots \end{aligned}$$

has no solution in integers. [For if there is a solution, then an induction shows that

$$a_1 = n!a_n + (n-1)! + \cdots + 2! + 1!.$$

From this we conclude that if $n+1$ is prime, then $a_n \neq 0$, since if $a_n = 0$ then $a_1 = (n-1)! + \cdots + 2! + 1!$ and $a_1 = (n+1)!a_{n+1} + n! + a_1$, whence $n+1$ divides $n!$. But if $a_n \neq 0$, then

$$|a_1| \geq |n! - \{(n-1)! + \cdots + 1!\}| > |n! - (n-1)(n-1)!| = (n-1)!,$$

and this cannot happen for infinitely many n.] Therefore this $H^1(\mathbb{N};\mathscr{A}) \neq 0$, and so $\dim \mathbb{N} \geq 1$, whence $\dim \mathbb{N} = 1$. (For another proof of this, and in fact, a proof that if $\mathscr{A} = \{A_1 \supset A_2 \supset \cdots\}$ is *any* strictly decreasing inverse sequence of subgroups of $\mathbb{Z}$ then $\varprojlim^1 \mathscr{A}$ is uncountable, see V-5.17.)

To prove the last statement, suppose that $\mathscr{A}$ satisfies the Mittag-Leffler condition. For each n, let $\mathscr{I}_n = \operatorname{Im} \pi_{n,m}$ for m large. Then this is an inverse sequence that is clearly a flabby sheaf on $\mathbb{N}$; i.e., each $\mathscr{I}_{n+1} \to \mathscr{I}_n$ is surjective. Also, there is the inclusion $i : \mathscr{I} \hookrightarrow \mathscr{A}$. Let $\mathscr{C}$ be the cokernel of i, so that there is the short exact sequence $0 \to \mathscr{I} \to \mathscr{A} \to \mathscr{C} \to 0$ of sheaves on $\mathbb{N}$. The induced exact cohomology sequence shows that $\varprojlim^1 \mathscr{A}_n \approx \varprojlim^1 \mathscr{C}_n$. Now $\mathscr{C}$ has the property that for each n, there is an $m > n$ with $\mathscr{C}_m \to \mathscr{C}_n$ zero. Since by II-11.6 passage to a subsequence does not affect $\varprojlim^1$, we may assume that each $\mathscr{C}_{n+1} \to \mathscr{C}_n$ is zero. But then $\mathscr{C}$ is obviously acyclic by our given criterion for acyclicity.

The reader might note that *every* sheaf on $\mathbb{N}$ is soft, so that this exercise gives another example of a soft sheaf that is not acyclic.

28. For $0 \neq n \in L$ we have the exact sequence $0 \to M \xrightarrow{n} M \to M/nM \to 0$, which induces the exact sequence

$$0 \to \Gamma_\Phi(M) \xrightarrow{n} \Gamma_\Phi(M) \to \Gamma_\Phi(M/nM) \xrightarrow{0} H^1_\Phi(X;M) \xrightarrow{n} H^1_\Phi(X;M)$$

since Γ_Φ is exact on sequences of *constant* sheaves by I-Exercise 11. This gives the first statement. The requested example is given by II-11.5. For another example, let Z^t be the twisted locally constant sheaf on $\mathbb{S}^1$ with stalk $\mathbb{Z}$; see I-Example 3.4. Take $n = 2$. Then there is the exact sequence $0 \to Z^t \xrightarrow{2} Z^t \to \mathbb{Z}_2 \to 0$, which induces the exact cohomology sequence

$$0 \to \Gamma(Z^t) \to \Gamma(Z^t) \to \Gamma(\mathbb{Z}_2) \to H^1(\mathbb{S}^1; Z^t).$$

Since $\Gamma(Z^t) = 0$, $H^1(\mathbb{S}^1; Z^t)$ has 2-torsion. Although not requested, we shall go on to show that in fact, $H^1(\mathbb{S}^1; Z^t) \approx \mathbb{Z}_2$. Consider the covering map $f : X = \mathbb{S}^1 \to \mathbb{S}^1 = Y$ of degree 2. From I-Example 3.4 we have the exact sequence $0 \to \mathbb{Z} \to f\mathbb{Z} \to Z^t \to 0$. The induced cohomology sequence has the segment

$$H^1(Y;\mathbb{Z}) \xrightarrow{h} H^1(Y; f\mathbb{Z}) \to H^1(Y; Z^t) \to H^2(Y;\mathbb{Z}) = 0.$$

By II-11.2 the composition of h with $f^\dagger : H^1(Y; f\mathbb{Z}) \xrightarrow{\approx} H^1(X;\mathbb{Z})$ is just f^*, which is multiplication by 2 (using that this is the same as the map in singular theory). The contention follows. Another, more direct, way of doing this computation is to decompose $\mathbb{S}^1$ as the union of two intervals and apply the Mayer-Vietoris sequence II-(26):

$$0 \to H^0(\mathbb{I};\mathbb{Z}) \oplus H^0(\mathbb{I};\mathbb{Z}) \xrightarrow{g} H^0(\mathbb{S}^0;\mathbb{Z}) \to H^1(\mathbb{S}^1; Z^t) \to 0.$$

A little thought about sections of Z^t over the two intervals and their intersection $\mathbb{S}^0$ shows that with appropriate choices of generators a, b of the left-hand group and u, v for $H^0(\mathbb{S}^0; \mathbb{Z})$, we have $g(a) = u - v$ and $g(b) = u + v$. Thus $H^1(\mathbb{S}^1; Z^t) \approx \operatorname{Coker} g = \{u, v \mid u = v,\ u = -v\} \approx \mathbb{Z}_2$. Also see the solution to III-Exercise 13.

29. Let $A_n = \{1, \frac{1}{2}, ..., \frac{1}{n}\}$. Then $\mathscr{L}_n = \mathbb{Z}_{A_n}$ is obviously flabby. Then we claim that the sheaf $\mathscr{L} = \varprojlim \mathscr{L}_n$ is not flabby. Indeed, the restriction $\mathscr{L}(\mathbb{I}) \to \mathscr{L}((0,1])$ is not onto because the section over $(0,1]$ which is 1 at all points $1, \frac{1}{2}, \frac{1}{3}, ...$ does not extend to $\mathbb{I}$ since it would be 0 at 0 (since the stalk at 0 is trivial), and so must coincide with the zero section on some neighborhood of $0 \in \mathbb{I}$.

31. Let $Z\text{-}\dim X = n$, and assume that the result holds for Zariski spaces K with $Z\text{-}\dim K < n$. We need only show that $H^{n+1}(X; \mathbb{Z}_U) = 0$ for all open $U \subset X$ since then II-16.12 and II-Exercise 30 would imply that $\dim_{\mathbb{Z}} X \le n$. Now, one can express $X = X_1 \cup X_2 \cup \cdots \cup X_k$ where the X_i are irreducible and no X_i contains X_j for $j \neq i$. (These are called the irreducible components of X.) Let $Y = X_1 \cup \cdots \cup X_{i-1}$. Then clearly $Z\text{-}\dim Y \cap X_i < n$, so that $\dim_{\mathbb{Z}} Y \cap X_i < n$. Then the Mayer-Vietoris sequence

$$H^n(Y \cap X_i; \mathscr{A}) \to H^{n+1}(Y \cup X_i; \mathscr{A}) \to H^{n+1}(Y; \mathscr{A}) \oplus H^{n+1}(X_i; \mathscr{A})$$

shows that by a finite induction, we need only show that $\dim_{\mathbb{Z}} X_i \le n$; i.e., we may as well assume that X is irreducible. In that case note that if U and V are nonempty open subsets of X, then $U \cap V \neq \varnothing$ since otherwise $(X - U) \cup (X - V) = X$ and so X is not irreducible. Therefore any open subspace $U \subsetneq X$ is connected, and so any section of $\mathbb{Z}$ over U is constant. That implies that $\Gamma(\mathbb{Z}) \to \Gamma(\mathbb{Z}|U)$ is surjective, whence $\mathbb{Z}$ is a flabby sheaf on X. Now, $F = X - U$ is a Zariski space with $Z\text{-}\dim F < n$. Also, $H^n(X; \mathbb{Z}_F) \approx H^n(F; \mathbb{Z})$ by II-10.2. Therefore the exact sequence

$$0 = H^n(X; \mathbb{Z}_F) \to H^{n+1}(X; \mathbb{Z}_U) \to H^{n+1}(X; \mathbb{Z}) = 0$$

proves the result. (The equality on the left is by the inductive assumption and the one on the right is because $\mathbb{Z}$ is flabby on X.)

33. (a) Let M be generated by $a_1, a_2, ...$, and put

$$N_n = \{a \in M \mid ka = \sum_{i=1}^{n} k_i a_i \text{ for some } k_i \in L,\ 0 \neq k \in L\}.$$

By hypothesis, N_n is free. By looking at $M \otimes Q$, where Q is the field of fractions of L, we see that $\operatorname{rank} N_n \le n$. We inductively construct a basis for each N_n as follows. Suppose that $\{b_1, ..., b_s\}$ is a basis of N_n. If $N_{n+1} = N_n$, then take the same basis. If $N_{n+1} \neq N_n$, then N_{n+1}/N_n is finitely generated and torsion-free, whence free, and is of rank one. Let $b_{s+1}N_n$ be a generator (whence $a_{n+1}N_n = qb_{s+1}N_n$ for some $q \in L$). Then $b_1, ..., b_{s+1}$ form a basis of N_{n+1}. Finally, $b_1, b_2, ...$ span $N = \bigcup N_n$ and are independent, so they form a free basis of N.

(b) Since $H_c^0(X; L) \approx \tilde{H}^0(X^+; L)$, it suffices to consider reduced cohomology of a compact space, and so we shall assume X to be compact.

Let $\{a_1, ..., a_n\} \subset H^0(X; L)$ be given and let N be as described. Let $\tilde{N}$ be the image of N in $\tilde{H}^0(X; L)$. It suffices to show that N and $\tilde{N}$ are finitely generated since 0-dimensional cohomology with coefficients in L is torsion-free. Also, N is finitely generated $\Leftrightarrow$ $\tilde{N}$ is finitely generated. Let $K_1, ..., K_k$ be as described in part (b). We argue, by induction, that the image of N (or $\tilde{N}$) in $H^0(K_1 \cup \cdots \cup K_i)$ is finitely generated. If $a \in N$, then $ka \mapsto 0$ in $\tilde{H}^0(K_j)$ for some $k \in L$. This implies that $a \mapsto 0$ in $\tilde{H}^0(K_j)$ since this group is torsion-free. Thus the contention is true for $i = 1$. Suppose it is true for i and put $K = K_1 \cup \cdots \cup K_i$. If $K \cap K_{i+1} \neq \varnothing$, then the Mayer-Vietoris sequence

$$0 \to \tilde{H}^0(K \cup K_{i+1}) \to \tilde{H}^0(K) \oplus \tilde{H}^0(K_{i+1}) \to \tilde{H}^0(K \cap K_{i+1})$$

proves the inductive step since the images of $\tilde{N}$ in $\tilde{H}^0(K_{i+1})$ and (hence) in $\tilde{H}^0(K \cap K_{i+1})$ are zero. If $K \cap K_{i+1} = \varnothing$, then $H^0(K \cup K_{i+1}) \approx H^0(K) \oplus H^0(K_{i+1})$, and the contention again follows from this.

(c) Let K_i be as in the hint. Assume, by induction on r, that whenever $D_i \subset \operatorname{int} K_i$, with $\{D_i\}$ a closed covering of X, the image of N in $H_c^1(D_1 \cup \cdots \cup D_{r-1})$ is finitely generated. Let E_i be compact sets with $D_i \subset \operatorname{int} E_i$, $E_i \subset \operatorname{int} K_i$. Put $D = D_1 \cup \cdots \cup D_{r-1}$, $E = E_1 \cup \cdots \cup E_{r-1}$, and $K = K_1 \cup \cdots \cup K_{r-1}$. By the inductive assumption, the image of N in $H_c^1(E)$ is finitely generated. By choice of the K_i, all $b_j | K_i = 0$. Since $H_c^1(K_i)$ is torsion-free by II-Exercise 28, the image of N in $H_c^1(K_i)$ is zero. Consider the diagram

$$\begin{array}{ccccc}
 & & H_c^1(K \cup K_r) & \to & H_c^1(K) \oplus H^1(K_r) \\
 & & \downarrow & & \downarrow{\scriptstyle k \oplus k_r} \\
H^0(E \cap E_r) & \to & H_c^1(E \cup E_r) & \to & H_c^1(E) \oplus H^1(E_r) \\
\downarrow{\scriptstyle j_r} & & \downarrow & & \\
H^0(D \cap D_r) & \to & H_c^1(D \cup D_r). & &
\end{array}$$

The image of j_r is finitely generated by II-17.5. Also, the image of N in $H_c^1(E) \oplus H_c^1(E_r)$ is finitely generated by the inductive assumption. By II-17.3, the image of N in $H_c^1(D \cup D_r)$ is finitely generated, completing the induction and the proof of (c).

(d) Since the one-point compactification of a locally compact separable metric space is separable metric, we may restrict attention to the compact case. Now X can be embedded in the Hilbert cube $\mathbb{I}^\infty$. Let $p_n : \mathbb{I}^\infty \to \mathbb{I}^n$ be the projection, and put $X_n = p_n(X)$. Then $X \approx \varprojlim X_n$, and so $H^*(X) \approx \varinjlim H^*(X_n)$. Thus it suffices to show that $H^*(X_n)$ is countably generated. But any compact subset X_n of $\mathbb{I}^n \subset \mathbb{R}^n$ is the intersection of a descending sequence of finite polyhedra, and so $H^*(X_n)$ is the direct limit of a sequence of finitely generated groups, and such a group is obviously countably generated.

34. (a) If $\mathbb{Z}$ is regarded as the constant sheaf, then $H^0(X; \mathbb{Z}) = \Gamma(\mathbb{Z}) \approx C(X)$. Now, $A^0(U; \mathbb{Z}) = \{f : U \to \mathbb{Z}\}$ is a sheaf. For $f \in A^0(X; \mathbb{Z})$ we have $df(x_0, x_1) = f(x_1) - f(x_0)$. Also, $A_0^1(X; \mathbb{Z}) = \{f : X \times X \to \mathbb{Z} \,|\, f = 0$ on a neighborhood of the diagonal $\Delta \subset X \times X\}$. Therefore ${}_{AS}H^0(X; \mathbb{Z}) = \operatorname{Ker}\{d : A^0(X) \to A^1(X)/A_0^1(X)\} = \{f : X \to \mathbb{Z} \,|\, f$ is locally constant$\} = C(X)$. The reader may handle the arbitrary coefficient case.

(b) $f \in B(S) \Rightarrow \exists$ unique extension $\hat{f} \in C(\hat{S})$. Conversely, $\hat{f} \in C(\hat{S})$ $\Rightarrow \hat{f}$ is bounded $\Rightarrow f = \hat{f}|S \in B(S)$.

(c) (i) $\Rightarrow$ (ii): Let $S \subset X$ be dense of cardinality $\leq \eta$. Restriction gives a monomorphism $C(X) \rightarrowtail B(S)$. Thus $B(S)$ free $\Rightarrow C(X)$ free, since a subgroup of a free group is free. (ii) $\Rightarrow$ (i): Let $\operatorname{card}(S) \leq \eta$ and put $X = \hat{S}$. By (ii), $C(X)$ is free. By (b), $B(S)$ is free.

(d) The proof is identical to that of (c) using that $\operatorname{Ext}(\bullet, \mathbb{Z})$ is right exact.

37. Let $\Omega' = \Omega \cup \{\Omega\}$ be the set of ordinals up to and including the least uncountable ordinal Ω and $\omega' = \omega \cup \{\omega\}$ the set of ordinals up to and including the least infinite ordinal ω. Give these the order topologies. Then let X be the "Tychonoff plank" $\Omega' \times \omega'$ and let $A = \Omega \times \{\omega\} \cup \{\Omega\} \times \omega$. For any constant sheaf $\mathscr{A}$ with stalks $L \neq 0$ on X, let $0 \neq a \in L$ and let $s \in \mathscr{A}(A)$ be defined by $s(x) = 0$ for $x \in \Omega \times \{\omega\}$ and $s(x) = a$ for $x \in \{\Omega\} \times \omega$. If s extends to $t \in \mathscr{A}(U)$ for some open $U \supset A$, then $V = \{x \in U \mid t(x) = 0\}$ and $W = \{x \in U \mid t(x) = a\}$ are disjoint open sets with $U = V \cup W$. For $n \in \omega$ there is an element $\alpha_n \in \Omega$ such that $[\alpha_n, \Omega] \times \{n\} \subset W$. There is a $\beta < \Omega$ with $\alpha_n < \beta$ for all n. Then $([\beta, \Omega] \times \omega') - \{\Omega \times \omega\} \subset W$; but this must intersect V, a contradiction. Therefore $s \in \mathscr{A}(A)$ does not come from $\varinjlim \mathscr{A}(U)$.

38. Since $\mathscr{A} = \mathscr{A}_{\bar{A}}$, we have $H^*_\Phi(X; \mathscr{A}) = H^*_\Phi(X; \mathscr{A}_{\bar{A}}) \approx H^*_{\Phi \cap \bar{A}}(\bar{A}; \mathscr{A}|\bar{A})$ by II-10.2. The desired result then comes from the 5-lemma applied to the pairs (X, A) and $(\bar{A}, A)$.

39. By construction, $\mathscr{C}^*(\bar{A}, A; \mathscr{A})$ is zero on $\operatorname{int} A$, whence $C^*_{\Phi \cap \bar{A}}(\bar{A}, A; \mathscr{A}) = 0$ by the hypothesis on Φ. The last part follows from II-Exercise 38 and the cohomology sequence of (X, A).

40. Let $M(U, \mathbb{R})$ stand for the group of continuous functions $U \to \mathbb{R}$, etc. This is a conjunctive monopresheaf and so $\mathscr{F} = M(\bullet, \mathbb{R})$. Similarly, $\mathbb{Z} = M(\bullet, \mathbb{Z})$ and $\mathscr{F}_0 = M(\bullet, T)$. Standard covering-space theory applied to the covering $\mathbb{R} \to T$ shows that

$$0 \to M(U, \mathbb{Z}) \to M(U, \mathbb{R}) \xrightarrow{j_U} M(U, T) \to [U; T] \to 0$$

is exact. Now, if $f : U \to T$ and $x \in U$, then there is a neighborhood $V \subset U$ of x such that $f|V$ is homotopic to the constant map to $0 \in T$. (Just take $V = f^{-1}(W)$, where W is an open arc about $f(x)$.) It follows that $\mathscr{S}\!heaf([\bullet; T]) = 0$, whence we have the induced exact sequence

$$0 \to \mathbb{Z} \to \mathscr{F} \to \mathscr{F}_0 \to 0$$

of sheaves. Now assume that X is paracompact. Then $\mathscr{F}$ is soft by II-9.4. Therefore the exact cohomology sequence of this coefficient sequence has the form

$$0 \to \Gamma(\mathbb{Z}) \to \Gamma(\mathscr{F}) \xrightarrow{j_\Gamma} \Gamma(\mathscr{F}_0) \to H^1(X; \mathbb{Z}) \to 0,$$

where j_Γ is equivalent to j_X. Hence

$$[X; T] \approx \operatorname{Coker} j_X \approx \operatorname{Coker} j_\Gamma \approx H^1(X; \mathbb{Z}),$$

finishing the first part. See II-13.2 for an interesting example of this.

For a paracompactifying family of supports $\Phi \neq cld$ on X, the sequence

$$0 \to \Gamma_\Phi(\mathscr{F}) \xrightarrow{j} \Gamma_\Phi(\mathscr{F}_0) \to H^1_\Phi(X;\mathbb{Z}) \to 0$$

is exact, and it is clear that $\Gamma_\Phi(\mathscr{F}) = M_\Phi(X,\mathbb{R})$ and $\Gamma_\Phi(\mathscr{F}_0) = M_\Phi(X;T)$, with M_Φ standing for maps that vanish outside some member of Φ. Again, covering space theory immediately yields that $\operatorname{Coker} j \approx [X;T]_\Phi$, where the latter denotes homotopy classes where one demands all maps and homotopies to be constant to the zero element of $\mathbb{R}$ or T outside some element of Φ. In particular, for locally compact spaces we deduce that

$$[X;T]_c \approx H^1_c(X;\mathbb{Z})$$

and in particular that

$$[\mathbb{R};T]_c \approx H^1_c(\mathbb{R};\mathbb{Z}) \approx \mathbb{Z}.$$

That this formula, in the case $\Phi = cld$, does not hold for spaces that are not paracompact is shown by the following example. Let L be the "long ray" compactified at both ends; see II-Exercise 3. Let $x \in L$ be the maximal element. Let L_1 and L_2 be two copies of L with $x_i \in L_i$ corresponding to $x \in L$. Let A be the one-point union of L_1 and L_2 obtained by identifying x_1 with x_2. We will denote this common point by x. Let $I = [-1,1]$. Define

$$\tilde{X} = A \times I, \quad X = \tilde{X} - \{x \times 0\}.$$

We claim that $H^1(X;\mathbb{Z}) \approx \mathbb{Z}$. To see this, let $X_1 = (A \times [0,1]) \cap X$ and $X_2 = (A \times [-1,0]) \cap X$. Since X_1 has $A \times \{1\}$ as a strong deformation retract, we have that $H^n(X_1;\mathbb{Z}) \approx H^n(A;\mathbb{Z})$ by II-11.12. But $H^n(A;\mathbb{Z}) = 0$ for $n > 0$ by II-Exercise 2, and $H^0(A;\mathbb{Z}) \approx \mathbb{Z}$ since A is connected. Now, $X_1 \cap X_2$ consists of two disjoint copies of the "open" long ray, and the latter is acyclic by II-Exercise 3. Thus, the Mayer-Vietoris sequence (26) on page 94 completes the computation.

Next, we claim that $[X;T] = 0$. To see this, let $f : X \to \mathbb{S}^1$ and let $r \in [-1,1] = I$. Then a well-known property of the long interval implies that there is an interval $[y_1(r), y_2(r)] \subset A$ containing x on which f is constant. A similar property of the long ray shows that there is a $y_i \in L$, $y_i \neq x$, such that $y_i > y_i(r)$ for all rational r. Then f is constant on $[y_1, y_2] \times \{r\}$ for all rational r, and continuity shows that this is also the case for irrational r. That means that on $[y_1, y_2] \times I$, f is the projection $[y_1, y_2] \times I \to I$ followed by some continuous function $I \to T$. It follows that f extends continuously to a map $\tilde{X} \to T$. Since $\tilde{X}$ is compact, the first part of the problem gives $[\tilde{X};T] \approx H^1(\tilde{X};\mathbb{Z}) \approx H^1(A;\mathbb{Z}) = 0$, since $\tilde{X}$ has A as a deformation retract.

Note that it follows that in the example, $\mathscr{F}$ is a c-soft sheaf on X which is *not* acyclic for cld supports.

41. (a) $\mathscr{H}^*(X,A;\mathscr{A}) = \mathscr{S}\!heaf(U \mapsto H^*(U, A \cap U;\mathscr{A}))$. There is the exact sequence $0 \to H^0(U, A \cap U;\mathscr{A}) \to H^0(U;\mathscr{A}) \to H^0(A \cap U;\mathscr{A})$, and the last map is just the restriction $j : \Gamma(\mathscr{A}|U) \to \Gamma(\mathscr{A}|A \cap U)$. Thus $\mathscr{H}^0(X,A;\mathscr{A}) = \mathscr{S}\!heaf(\operatorname{Ker} j) = \mathscr{S}\!heaf(U \mapsto \Gamma_{U-A}(\mathscr{A}|U))$. Since $\mathscr{A}$ is constant, a section

of $\mathscr{A}|U$ with support in $U - A$ must actually have support in $U - \bar{A}$. Hence $(\operatorname{Ker} j)(U) = \Gamma_{U-\bar{A}}(\mathscr{A}|U) \approx \Gamma(\mathscr{A}_{X-\bar{A}}|U)$, which generates the sheaf $\mathscr{A}_{X-\bar{A}}$.

(b) The stalk $\mathscr{H}^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})_0 \approx H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \approx \mathbb{Z}$.

(c) If $\mathscr{H}^n(X, A; \mathscr{A}) = 0$ for $q > 0$, then $\mathscr{C}^*(X, A; \mathscr{A})$ is a flabby resolution of $\mathscr{A}_{X-\bar{A}}$, and so

$$H^*_\Phi(X, A; \mathscr{A}) \approx H^*(\Gamma_\Phi(\mathscr{C}^*(X, A; \mathscr{A}))) \approx H^*_\Phi(X; \mathscr{A}_{X-\bar{A}}).$$

(d) Under the hypothesis, the stalk at x is

$$\begin{aligned}\mathscr{H}^p(X, A; \mathscr{A})_x &= \varinjlim H^p(U, A \cap U; \mathscr{A}) \approx \varinjlim H^p(U; \mathscr{A}_{U-\bar{A}}) \\ &= \mathscr{H}^p(X; \mathscr{A}_{X-\bar{A}})_x = 0\end{aligned}$$

for $p > 0$, since $\mathscr{C}^*(X; \mathscr{A}_{X-\bar{A}})$ is a resolution.

45. Let $x \in |s|$. Since $\mathscr{L}_{X-\{x\}}$ is flabby, there is a section $t \in \mathscr{L}(X)$ such that $t(x) = 0$ and $t(y) = s(y)$ for all $y \neq x$. But $t(x) = 0$ implies that t is zero on a neighborhood of x. Thus, also $s = 0$ on some $U - \{x\}$ with U open in X. Therefore x is isolated in $|s|$, and so $|s|$ is discrete. We note that this shows that flabbiness of a sheaf $\mathscr{A}$ *rarely* implies the flabbiness of $\mathscr{A}_U$, as distinct from the situation for soft sheaves; see II-9.13.

46. Note that $\mathscr{L} \otimes \mathbb{Z}_U = \mathscr{L}_U$. Thus $H^1_\Phi(X; \mathscr{L}_U) = 0$ for all Φ. By II-Exercise 21, $\mathscr{L}_U$ is flabby for all open U. But $M \hookrightarrow \mathscr{L}$ gives a section s which is nowhere zero. By II-Exercise 45, $X = |s|$ is discrete.

    This exercise shows dramatically that resolutions of the type $\mathscr{L}^* \otimes \mathscr{A}$ do not suffice for computing cohomology for all sheaves $\mathscr{A}$ and support families Φ, in distinction to the case of paracompactifying families, where such resolutions do suffice.

51. Following θ by a map of $\mathscr{B}^*$ to an injective resolution shows that we may as well assume that $\mathscr{B}^*$ is injective over $\mathbb{Z}_2$. Then the map h_0 of II-20 can be taken to be θ. Then $h_0\tau = 0 = h_0\sigma$, so that we can take h_1 and k_1, and hence the rest of the h_i and k_i, to be zero. It follows that $St_n(a) = 0$ for $n > 0$ and all $a \in H^q_\Phi(X; \mathbb{Z}_2)$, whence $St^j(a) = 0$ for $j = q - n < q$. Therefore, $a = St^0(a) = 0$ for $\deg(a) > 0$. For $U \subset X$ open, passage to $\mathscr{A}^*_U$ and $\mathscr{B}^*_U$ gives $H^q_\Phi(U; \mathbb{Z}_2) = 0$ for $q > 0$, whence $\dim_{\mathbb{Z}_2} X = 0$ by II-16.32.

53. Let $K = |a| \in \Phi$, where $a \in C^p_\Phi(X; \mathscr{A})$ is a cocycle representative of α. For any point $x \in X$ there is a neighborhood U_x of x such that $\beta|U_x = 0$ in $H^q(U_x; \mathscr{B})$ since a point is always taut and $q > 0$. Since K is compact, $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$ for some points x_i. Let $U_0 = X - K$, $U_i = U_{x_i}$ and $V_k = U_0 \cup \cdots \cup U_k$. Let

$$\mathscr{C}_k = \mathscr{A} \otimes \underbrace{\mathscr{B} \otimes \cdots \otimes \mathscr{B}}_{k \text{ times}}.$$

For $0 \le k \le n$ we claim that $\alpha\beta^k|V_k = 0$ in $H^{p+kq}_{\Phi \cap V_k}(V_k; \mathscr{C}_k)$. The proof is by induction on k. It is true for $k = 0$ since $a|V_0 = a|X - K = 0$. Suppose it is true for a particular value of k. Then by the exact sequence of (X, V_k), $\alpha\beta^k$

comes from an element $\gamma \in H^{p+kq}_{\Phi|X-V_k}(X;\mathscr{C}_k) \approx H^{p+kq}_{\Phi}(X,V_k;\mathscr{C}_k)$. Similarly, β comes from an element $\beta' \in H^q_{\Phi|X-U_{k+1}}(X;\mathscr{B}) \approx H^q_\Phi(X,U_{k+1};\mathscr{B})$. Thus $\alpha\beta^{k+1}$ comes from

$$\gamma \cup \beta' \in H^{p+(k+1)q}_{\Phi|X-V_{k+1}}(X;\mathscr{C}_{k+1}) \approx H^{p+(k+1)q}_{\Phi}(X,V_{k+1};\mathscr{C}_{k+1})$$

since $(\Phi|A)\cap(\Phi|B) = \Phi|(A\cap B)$. Consequently, $\alpha\beta^{k+1}|V_{k+1} = 0$, completing the induction. Since $V_n = X$, we conclude that $\alpha\beta^n = 0$ in $H^{p+nq}_\Phi(X;\mathscr{C}_n)$.

54. Using double induction on $\dim G$ and the number of components of G, we can assume the result holds for any action of a proper subgroup of G. Let $F \subset X$ be the fixed-point set of G on X. Then $\dim_L X/G = \max\{\dim_L F, \dim_L (X-F)/G\}$ by II-Exercise 11. By the inductive assumption, the existence of slices, and the local nature of dimension (see II-16.8), we have that $\dim_L (X-F)/G \le \dim_L X - F \le \dim_L X$. Since $\dim_L F \le \dim_L X$, the result follows.

55. We may as well assume that $\mathscr{A}'$ and $\mathscr{A}''$ are constant. The conditions are to assure that for any point $x\in X$ and neighborhood U of x there is a neighborhood V of x such that $H^1(U;\mathscr{A}') \to H^1(V;\mathscr{A}')$ is zero. In case (a) this follows by selecting *compact* neighborhoods and using the universal coefficient formula II-15.3 and II-17.3. In case (b) it follows from the exact sequence of the coefficient sequence $0\to\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}_n\to 0$ and II-17.3. In case (c) it is obvious. Since X is $clc^1_{\mathbb{Z}}$, it is locally connected. Then for such U and V, which can be taken to be open and connected, and any $x\in V$, consult the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{A}'(U) & \to & \mathscr{A}(U) & \to & \mathscr{A}''(U) & \to & H^1(U;\mathscr{A}') \\
 & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow 0 \\
0 & \to & \mathscr{A}'(V) & \to & \mathscr{A}(V) & \to & \mathscr{A}''(V) & \to & H^1(V;\mathscr{A}') \\
 & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \\
0 & \to & \mathscr{A}'_x & \to & \mathscr{A}_x & \to & \mathscr{A}''_x & \to & 0.
\end{array}$$

A diagram chase shows that $\mathscr{A}(V) \to \mathscr{A}_x$ is an isomorphism (for *any* $x \in V$). Now the same argument as in the last part of the proof of I-Exercise 9 shows that $\mathscr{A}$ is constant on V. For the counterexample let X be similar to the example in the solution of I-Exercise 10 but using projective planes rather than circles. This space is seen to be $clc^1_{\mathbb{Z}}$ from continuity and the fact that $H^1(\mathbb{P}^2;\mathbb{Z}) = 0$, but it is not, of course, $clc^2_{\mathbb{Z}}$, nor is it $clc^1_{\mathbb{Z}_2}$.

56. This follows immediately from II-15.2, II-16.14, and II-16.26 provided we can show that $A\otimes A = 0 = A * A \Rightarrow A = 0$ for abelian groups A (applied to $A = H^n_c(U;\mathbb{Z})$). To prove this, consider the exact sequence

$$A\otimes\mathbb{Z} \xrightarrow{p} A\otimes\mathbb{Z} \to A\otimes\mathbb{Z}_p \to 0.$$

Now $(A\otimes\mathbb{Z}_p)\otimes(A\otimes\mathbb{Z}_p) \approx (A\otimes A)\otimes\mathbb{Z}_p = 0$. But $A\otimes\mathbb{Z}_p$ is a $\mathbb{Z}_p$-vector space, and so this implies that $A\otimes\mathbb{Z}_p = 0$. The sequence then shows that

$A \approx A \otimes \mathbb{Z}$ is divisible and hence injective. If $A \neq 0$, then $\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, A) = 0$ since A is injective, so that the exact sequence

$$\operatorname{Hom}(\mathbb{Q}, A) \to \operatorname{Hom}(\mathbb{Z}, A) \to \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, A)$$

shows that A contains a nonzero subgroup of the form $\mathbb{Q}/K$ for some subgroup K of $\mathbb{Q}$. Since $\mathbb{Q}/K$ is divisible, it is injective, and that implies that $\mathbb{Q}/K$ is a direct summand of A, whence $\mathbb{Q}/K$ satisfies the same hypotheses as does A. Since $\mathbb{Q} \otimes \mathbb{Q} \neq 0$, we have $K \neq 0$. Now $\mathbb{Q} \otimes \mathbb{Q}/K = 0$ since $\mathbb{Q}/K$ is all torsion. Thus the exact sequence

$$0 = \mathbb{Q} * \mathbb{Q}/K \to \mathbb{Q}/K \otimes \mathbb{Q}/K \to \mathbb{Z} \otimes \mathbb{Q}/K \to \mathbb{Q} \otimes \mathbb{Q}/K = 0$$

shows that $0 = \mathbb{Q}/K \otimes \mathbb{Q}/K \approx \mathbb{Z} \otimes \mathbb{Q}/K \approx \mathbb{Q}/K \neq 0$, a contradiction.

We remark that Boltjanskiĭ constructed a compact metric space X with $\dim_{\mathbb{Z}} X = 2$ and $\dim_{\mathbb{Z}} X \times X = 3$.

57. For $x \in X$, the minimal neighborhood U of x must be open and unique. Also, $\mathscr{L}_x = \mathscr{L}(U)$ for any sheaf $\mathscr{L}$ on X, and it follows that $(\prod_\lambda \mathscr{L}_\lambda^*)_x = \prod_\lambda (\mathscr{L}_\lambda^*)_x$. The remainder of the proof of II-5.10 now applies.

58. If $a \in A$ then $U_a \cap A$ is the smallest neighborhood of a in A, and so A is rudimentary. If $x \in A \subset U_x$, then the only relatively open set in A containing x is A itself. Let $\mathscr{A}^*$ be a flabby resolution of a given sheaf $\mathscr{A}$ on X. Then the restriction $\mathscr{A}^*(U_x) \to \mathscr{A}_x^*$ is an isomorphism by definition. Consequently,

$$H^p(U_x; \mathscr{A}) \approx H^p(\mathscr{A}_x^*) \approx \begin{cases} \mathscr{A}_x & p = 0, \\ 0 & p \neq 0. \end{cases}$$

Therefore $H^*(U_x; \mathscr{A}) \xrightarrow{\approx} H^*(\{x\}; \mathscr{A}_x)$. This also applies to the space A and shows that $H^*(A; \mathscr{A}|A) \xrightarrow{\approx} H^*(\{x\}; \mathscr{A}_x)$, whence $\mathscr{A}|A$ is acyclic. It also follows that $H^*(U_x; \mathscr{A}) \xrightarrow{\approx} H^*(A; \mathscr{A}|A)$, whence A is taut by II-10.6.

59. Let $U_{n,m} = U_n \times U_m$. These only form a basis for the open sets, but that is sufficient for describing sheaves, and so we will ignore the other open sets. For any sheaf $\mathscr{A}$, $C^0(U_{n,m}; \mathscr{A})$ is just the group of $n \times m$ matrices $(a_{i,j})$, where $a_{i,j} \in A_{i,j}$. Using π and ϖ for all the $\pi_{i,j}$ and $\varpi_{i,j}$, there is the sequence

$$\begin{aligned} 0 \to \mathscr{A}(U_{n,m}) \to C^0(U_{n,m}; \mathscr{A}) &\xrightarrow{d^0} C^0(U_{n-1,m}; \mathscr{A}) \oplus C^0(U_{n,m-1}; \mathscr{A}) \\ &\xrightarrow{d^1} C^0(U_{n-1,m-1}; \mathscr{A}) \to 0, \end{aligned}$$

where $d^0(a_{i,j}) = (a_{i,j} - \pi a_{i+1,j}) \oplus (a_{i,j} - \varpi a_{i,j+1})$ and $d^1((a_{i,j}) \oplus (b_{i,j})) = (a_{i,j} - b_{i,j} - \varpi a_{i,j+1} + \pi b_{i+1,j})$. It is easy to see that this sequence is exact and gives the exact sequence

$$0 \to \mathscr{A} \to \mathscr{C}^0(\mathrm{M}; \mathscr{A}) \to \eta' \mathscr{C}^0(\mathrm{M}; \mathscr{A}) \oplus \eta'' \mathscr{C}^0(\mathrm{M}; \mathscr{A}) \to \eta' \eta'' \mathscr{C}^0(\mathrm{M}; \mathscr{A}) \to 0,$$

where $\eta'(n,m) = (n+1, m)$ and $\eta''(n,m) = (n, m+1)$. We omit the other details, which are straightforward.

For the last part, consider the map $f : \mathrm{M} \to \mathbb{N}$ given by $f(n,m) = \max(n,m)$. Then one sees easily that f is closed and continuous. Also,

each $f^{-1}(n)$ is taut and acyclic by II-Exercise 58 since $U_{n,n}$ is the smallest open set containing (n,n) and $(n,n) \in f^{-1}(n) \subset U_{n,n}$. Also, $(f\mathscr{A})(U_n) = \mathscr{A}(f^{-1}U_n) = \mathscr{A}(U_{n,n})$, so that $f\mathscr{A} \approx \mathscr{A}|\Delta$. By II-11.1 we have

$$\varprojlim{}_k^n A_{k,k} = H^n(\mathbb{N}; f\mathscr{A}) \approx H^n(\mathbb{M}; \mathscr{A}) = \varprojlim{}_{i,j}^n A_{i,j}.$$

Solutions for Chapter III:

4. Let $U \subset X$ be open. Since U is paracompact, $S^p(U;G) \to \Gamma(\mathscr{S}^p|U)$ is onto by I-6.2. Any singular cochain on U can be arbitrarily extended to one on X, and so $S^p(X;G) \to S^p(U;G)$ is onto. Thus the composition $S^p(X;G) \to \Gamma(\mathscr{S}^p) \to \Gamma(\mathscr{S}^p|U)$ is onto, which implies that $\mathscr{S}^p$ is flabby by definition. The analogous argument works for $\mathscr{A}^*(X;G)$.

7. A disjoint union of lens spaces converging to a point serves for X. Then

$$_\Delta H^2(X;\mathbb{Q}) \approx \mathrm{Hom}(0;\mathbb{Q}) \oplus \mathrm{Ext}(\bigoplus \mathbb{Z}_n;\mathbb{Q}) = 0.$$

Also, by II-15.1,

$$\begin{aligned} {}_sH^2(X;\mathbb{Q}) &\approx H^2(\Gamma(\mathscr{S}^* \otimes \mathbb{Q})) \approx H^2(\Gamma(\mathscr{S}^*) \otimes \mathbb{Q}) \\ &\approx H^2(\Gamma(\mathscr{S}^*)) \otimes \mathbb{Q} \\ &\approx H^2(X;\mathbb{Z}) \otimes \mathbb{Q} \approx \mathrm{Ext}(\bigoplus \mathbb{Z}_n, \mathbb{Z}) \otimes \mathbb{Q} \\ &\approx \mathrm{Hom}(\bigoplus \mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Q} \\ &\approx (\prod \mathbb{Z}_n) \otimes \mathbb{Q} \neq 0. \end{aligned}$$

10. If U_x is the minimal open set containing x, then the covering $\mathfrak{U} = \{U_x\}$ refines all other coverings of X, so that $\check{H}^n(X;A) = \check{H}^n(\mathfrak{U};A)$. Also, $A(U_x) = \mathscr{A}_x = \mathscr{A}(U_x)$ for all x. Since $y \in U_x \Rightarrow U_y \subset U_x$, A agrees with $\mathscr{A}$ on all simplices $\{x_0, \ldots, x_n\}$ of $N(\mathfrak{U})$, whence $\check{H}^n(X;A) = \check{H}^n(X;\mathscr{A})$. In particular, $\check{H}^n(X;A) = 0$ if $\mathscr{A} = 0$. Thus $\check{H}^n(X;\mathscr{A})$ is a fundamental connected sequence of functors of sheaves $\mathscr{A}$ on X with $\check{H}^0(X;\mathscr{A}) \approx \Gamma(\mathscr{A})$. Therefore $\check{H}^n(X;A) \approx \check{H}^n(X;\mathscr{A}) \approx H^n(X;\mathscr{A})$ by II-6.2.

11. Any locally finite covering of $\mathbb{N}$ contains $\{\mathbb{N}\}$. Thus the result follows from III-4.8 (or by direct examination).

13. For an arc (x,y) of $\mathbb{S}^1$ containing x_0 one sees that $\mathscr{A}_n|(x,y)$ has a subsheaf isomorphic to $\mathbb{Z}_{(x_0,y)}$ with quotient sheaf isomorphic to $\mathbb{Z}_{(x,x_0]}$. We have $H^*((x,y);\mathbb{Z}_{(x_0,y)}) \approx H^*((x,y),(x,x_0];\mathbb{Z}) = 0$ and $H^*((x,y);\mathbb{Z}_{(x,x_0]}) \approx H^*((x,x_0];\mathbb{Z}) \approx H^*(\star;\mathbb{Z})$. Thus $H^i((x,y);\mathscr{A}_n) = 0$ for $i > 0$, whence III-4.13 applies and shows that $\check{H}(X;\mathscr{A}_n)$ can be computed using a covering $\mathfrak{U}$ by three arcs (actually two arcs would also do). For an appropriate choice of bases one computes easily that the incidence matrix for $d : \check{C}^1(\mathfrak{U};\mathscr{A}_n) \to \check{C}^2(\mathfrak{U};\mathscr{A}_n)$ is

$$\begin{bmatrix} -1 & n & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

which is unimodularly equivalent to $\mathrm{diag}(1,1,n-1)$, whence $\check{H}^1(\mathbb{S}^1;\mathscr{A}_n) \approx \mathbb{Z}/(n-1)\mathbb{Z}$. Particularly note the cases $\check{H}^1(\mathbb{S}^1;\mathscr{A}_0) = 0$ and $\check{H}^1(\mathbb{S}^1;\mathscr{A}_{-1}) \approx \mathbb{Z}_2$. [The sheaf $\mathscr{A}_{-1}$ is the same as Z^t of I-3.4 and so the solution to II-Exercise 28 gives another computation of $\check{H}^1(\mathbb{S}^1;\mathscr{A}_{-1})$.]

14. There is an open covering $\mathfrak{U}$ of X such that γ is the image of some $\gamma_{\mathfrak{U}} \in \check{H}^n(\mathfrak{U})$ via the canonical map $\check{H}^n(\mathfrak{U}) \to \check{H}^n(X) \approx H^n(X)$. Let ε be a Lebesgue number for $\mathfrak{U}$ and let $f : X \twoheadrightarrow Y$ be as described. If $y \in Y$, then $\operatorname{diam} f^{-1}(y) < \varepsilon$, whence $f^{-1}(y) \subset U$ for some $U \in \mathfrak{U}$. Thus y has a neighborhood V_y with $f^{-1}(V_y) \subset U$. Let $\mathfrak{V} = \{V_y \mid y \in Y\}$. Then $f^{-1}\mathfrak{V}$ refines $\mathfrak{U}$. Since f is surjective, it maps the nerve $N(f^{-1}\mathfrak{V})$ isomorphically onto $N(\mathfrak{V})$. Thus the induced map $\check{H}^n(\mathfrak{V}) \to \check{H}^n(f^{-1}\mathfrak{V})$ is an isomorphism. The result then follows from the commutative diagram

$$\begin{array}{ccccc} \gamma_{\mathfrak{U}} \in \check{H}^n(\mathfrak{U}) & \to & \check{H}^n(f^{-1}\mathfrak{V}) & \to & \check{H}^n(X) \\ & & \uparrow \approx & & \uparrow \\ & & \check{H}^n(\mathfrak{V}) & \to & \check{H}^n(Y). \end{array}$$

16. Let X be any space with a nonisolated point x. Let A be the constant presheaf with values $M \neq 0$ and let A'' be the presheaf with $A''(U) = 0$ if $x \notin U$ and $A''(U) = M$ if $x \in U$. Then there is the epimorphism $A \twoheadrightarrow A''$ of presheaves. Let $\Phi = \{\{x\}, \varnothing\}$. Then $\check{C}^0_\Phi(\mathfrak{U}; A) = 0$ for any covering $\mathfrak{U}$ since if $0 \neq c \in \check{C}^0_\Phi(\mathfrak{U}; A)$, then $c(U) \neq 0$ for some $U \in \mathfrak{U}$, whence $\{x\} \supset |c| \supset \overline{U}$. But $\check{\mathscr{C}}^0(\mathfrak{U}; A'')$ is concentrated on $\{x\}$ so that $\check{C}^0_\Phi(\mathfrak{U}; A'') = \check{C}^0(\mathfrak{U}; A'') = \prod\{M \mid x \in U \in \mathfrak{U}\}$. Now any open covering is refined by some $\mathfrak{U} = \{U_y\}$ such that $x \notin U_y$ for all $y \neq x$. For such $\mathfrak{U}$, $\check{C}^0_\Phi(\mathfrak{U}; A'') = A''(U_x) = M$. Consequently, $\check{C}^0_\Phi(X; A) = 0$ and $\check{C}^0_\Phi(X; A'') \approx M$, so that $\check{C}^0_\Phi(X; A) \to \check{C}^0_\Phi(X; A'')$ is not surjective.

Solutions for Chapter IV:

1. Let $K \in \Phi$ and $s \in (f_\Psi\mathscr{A})(K)$. Extend s to an open set $U \supset K$ giving $t \in (f_\Psi\mathscr{A})(U) = \Gamma_{\Psi(U)}(\mathscr{A}|f^{-1}(U))$. Let $L \in \Phi$ with $K \subset \operatorname{int} L$ and $L \subset U$. As a section of $\mathscr{A}$ over $f^{-1}(U)$, t restricts to $t' \in \Gamma_{\Phi(\Psi)|f^{-1}(L)}(\mathscr{A}|f^{-1}L)$. Since $\mathscr{A}$ is $\Phi(\Psi)$-soft, t' extends to $s' \in \Gamma_{\Phi(\Psi)}(\mathscr{A})$ by II-9.3. But $\Gamma_{\Phi(\Psi)}(\mathscr{A}) = \Gamma_\Phi(f_\Psi\mathscr{A})$ by IV-5.2, and under that equivalence s' becomes an extension of the original section s. This shows that $\Gamma_\Phi(f_\Psi\mathscr{A}) \to (f_\Psi\mathscr{A})(K)$ is onto for all $K \in \Phi$, which implies that $f_\Psi\mathscr{A}$ is Φ-soft by definition.

2. By assumption there are compact sets A, B in X with $A \cap B \subset Y$ and $A \cup B \cup Y = X$. If $A \cup B \neq X$, then Y contains an open set, and we are finished. If $A \cup B = X$, then $X - (A \cap B) = U \cup V$ (disjoint), where $U = A - (A \cap B) = X - B$ and $V = B - (A \cap B) = X - A$, which shows that the compact set $A \cap B$ disconnects X. The Mayer-Vietoris sequence

$$H^{n-1}(A \cap B) \to H^n(X) \to H^n(A) \oplus H^n(B)$$

has the direct sum term zero by II-16.29, and so $\dim_L A \cap B \geq n - 1$. It follows that $\underline{\dim}_L^X Y \geq n - 1$ since $A \cap B \in c|Y$.

3. Since a compact totally disconnected space has dimension zero by II-16.19, we have $\underline{\dim}_L^X(B - A) = 0$. The result then follows from the first inequality of IV-8.10.

5. Let A be the presheaf with $A(U) = H^n(\mathscr{L}^*(U))$ and let $\mathscr{A} = \mathscr{S}\!\mathit{heaf}(A) = \mathscr{H}^n(\mathscr{L}^*)$ by definition. Then there is the canonical map $\theta : A(Y) \to \mathscr{A}(Y)$ of Chapter I. Now $A(Y) = H^n(\Gamma(\mathscr{L}^*))$ and $\mathscr{A}(Y) = \Gamma(\mathscr{H}^n(\mathscr{L}^*))$, so that

$\theta : H^n(\Gamma(\mathscr{L}^*)) \to \Gamma(\mathscr{H}^n(\mathscr{L}^*))$. If $s \in \Gamma(\mathscr{L}^*)$ is a cocycle representing $[s] \in H^n(\Gamma(\mathscr{L}^*))$, then for $y \in Y$, $s(y)$ is an n-cocycle of $\mathscr{L}_x^*$ representing the cohomology class $[s(y)] \in \mathscr{H}^n(\mathscr{L}^*)_y$. By definition $\theta[s](y) = [s(y)]$. Now, if $s(y) = 0$, then $[s(y)] = 0$, so that $y \notin |\theta(s)|$. That is,

$$|\theta(s)| \subset |s|.$$

Therefore if $s \in \Gamma_\Phi(\mathscr{L}^*)$, then $\theta(s) \in \Gamma_\Phi(\mathscr{H}^n(\mathscr{L}^*))$, so that

$$\theta : H^n(\Gamma_\Phi(\mathscr{L}^*)) \to \Gamma_\Phi(\mathscr{H}^n(\mathscr{L}^*))$$

as desired.

We shall treat the remainder more generally in the context of IV-2.1, where $\mathscr{L}^*$ is a differential sheaf on Y consisting of Φ-acyclic sheaves. (The case of the Leray spectral sequence is that for which $\mathscr{L}^q = f_\Psi(\mathscr{C}^q(X;\mathscr{A}))$.) The spectral sequence in question is the first spectral sequence of the double complex $L^{p,q} = \Gamma_\Phi(\mathscr{C}^p(Y;\mathscr{L}^q))$ together with the fact that the second spectral sequence of this degenerates to an isomorphism induced by the monomorphism

$$\varepsilon : \Gamma_\Phi(\mathscr{L}^*) \rightarrowtail \Gamma_\Phi(\mathscr{C}^0(Y;\mathscr{L}^*)),$$

which identifies $\Gamma_\Phi(\mathscr{L}^*)$ with the d' 0-cocycles of $L^{*,*} = \Gamma_\Phi(\mathscr{C}^*(Y;\mathscr{L}^*))$. Thus cohomology classes of the total complex L^* are given by the image of $\varepsilon^* : H^n(\Gamma_\Phi(\mathscr{L}^*)) \xrightarrow{\approx} H^n(\Gamma_\Phi(\mathscr{C}^*(Y;\mathscr{L}^*)))$. Let us describe this explicitly. Start with a section $s \in \Gamma_\Phi(\mathscr{L}^n)$ that is a cocycle. Then $\varepsilon(s)$ is given just by regarding s as a *serration*, and we shall continue to name it s. Then $s(y) \in \mathscr{L}_y^n$ is a cocycle of $\mathscr{L}_y^*$ representing $[s(y)] \in \mathscr{H}^n(\mathscr{L}^*)_y$. Then $y \mapsto [s(y)]$ is a serration of $\mathscr{H}^n(\mathscr{L}^*)$, that is, an element of $\Gamma_\Phi(\mathscr{C}^0(Y;\mathscr{H}^n(\mathscr{L}^*)))$. By general principles this is a cocycle, and its cohomology class in $H^0(\Gamma_\Phi(\mathscr{C}^*(Y;\mathscr{H}^n(\mathscr{L}^*)))) = H^0_\Phi(Y;\mathscr{H}^n(\mathscr{L}^*)) = {}'E_2^{0,n}$ is just $\xi[s]$, where ξ is the edge homomorphism. Hence, as a section of $\mathscr{H}^n(\mathscr{L}^*)$, we have that $\xi[s](y) = [s(y)] = \theta[s](y)$. Therefore $\xi = \theta$ as claimed.

If Y is a single point y, then it is clear that ξ is the identity. Thus naturality applied to the diagram

$$\begin{array}{ccc} y^\bullet & \hookrightarrow & X \\ \downarrow & & \downarrow \\ y & \hookrightarrow & Y \end{array}$$

gives the commutative diagram

$$\begin{array}{ccc} H^n_{\Phi(\Psi)}(X;\mathscr{A}) & \xrightarrow{\xi} & \Gamma_\Phi(\mathscr{H}^n_\Psi(f;\mathscr{A})) \\ {\scriptstyle\text{restriction}}\big\downarrow & & \big\downarrow{\scriptstyle i_y^*} \\ H^n_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}|y^\bullet) & \xrightarrow{\xi=1} & H^n_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}|y^\bullet), \end{array}$$

since we may as well assume that $\{y\} \in \Phi$. This proves the last statement. The proof could also be based on II-6.2.

6. The indicated edge homomorphism is the composition $\zeta = i_y^* \circ \xi \circ \eta^*$, where

$$\eta^* : H^n_\Phi(X/G;\mathscr{A}) \xrightarrow{\approx} H^n_\Psi(X_G;\eta^*\mathscr{A})$$

is induced by $\eta : X_G \to X/G$;

$$\xi : H^n_\Psi(X_G; \eta^*\mathscr{A}) \to \Gamma(\mathscr{H}^n_\Theta(X; f^*\mathscr{A}))$$

is the edge homomorphism of the Leray spectral sequence of $\pi : X_G \to B_G$, as in IV-Exercise 5; and

$$i_y^* : \Gamma(\mathscr{H}^n_\Theta(X; f^*\mathscr{A})) \xrightarrow{\approx} H^n_\Theta(X; f^*\mathscr{A})$$

is as in IV-Exercise 5, where $y \in B_G$. Let $j : X \approx X \times_G G \hookrightarrow X \times_G E_G = X_G$ be the inclusion of the fiber corresponding to $y \in B_G$. Then the diagram

$$\begin{array}{ccc} X \approx X \times_G G & \xrightarrow{f} & X/G \\ \downarrow{\scriptstyle j} & & \| \\ X_G = X \times_G E_G & \xrightarrow{\eta} & X/G \end{array}$$

commutes; that is, $\eta \circ j = f$. By IV-Exercise 5 we have $i_y^* \circ \xi = j^*$. Therefore $\zeta = i_y^* \circ \xi \circ \eta^* = j^* \circ \eta^* = f^*$ as claimed.

9. The first derived functor of f_Ψ is $\mathscr{A} \mapsto \mathscr{H}^1(f_\Psi\mathscr{A})$, whose stalk at $y \in Y$ is $\mathscr{H}^1_\Psi(f;\mathscr{A})_y \approx H^1_{\Psi\cap y^\bullet}(y^\bullet;\mathscr{A}) = 0$ by IV-4.2 and the assumption that $\dim_\Psi(y^\bullet) = 0$. Therefore f_Ψ is exact.

10. We have $\tau \cup \pi^*(\omega) = \tau \cup \pi^*(i^*j^*(\tau)) = \tau \cup j^*(\tau) = \tau \cup \tau$ by (23) on page 236 and II-12.2.

11. It suffices to treat the two cases $B = V$ open and $B = F$ closed. Thus let $F \subset Y$ be closed, and put $V = Y - F$. For U open in Y, the exact sequence

$$0 \to \mathscr{A}_{V^\bullet} \to \mathscr{A} \to \mathscr{A}_{F^\bullet} \to 0$$

(restricted to U) induces the exact cohomology sequence

$$0 \to H^0_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}_{V^\bullet}) \to H^0_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}) \to H^0_{\Psi\cap U^\bullet}(U^\bullet;\mathscr{A}_{F^\bullet}) \to \cdots$$

and hence the exact sequence of sheaves

$$0 \to f_\Psi(\mathscr{A}_{V^\bullet}) \xrightarrow{\varphi} f_\Psi(\mathscr{A}) \xrightarrow{\theta} f_\Psi(\mathscr{A}_{F^\bullet}) \to \mathscr{H}^1_\Psi(f;\mathscr{A}_{V^\bullet}) \to \cdots$$

For $y \in F$ there is the commutative diagram

$$\begin{array}{ccc} f_\Psi(\mathscr{A})_y & \xrightarrow{\theta_y} & f_\Psi(\mathscr{A}_{F^\bullet})_y \\ \downarrow{\scriptstyle r_y^*} & & \downarrow{\scriptstyle r_y^*} \\ H^0_\Psi(y^\bullet;\mathscr{A}|y^\bullet) & \xrightarrow{\tau} & H^0_\Psi(y^\bullet;\mathscr{A}_{F^\bullet}|y^\bullet), \end{array}$$

and τ is trivially an isomorphism since $\mathscr{A}_{F^\bullet}|y^\bullet = \mathscr{A}|y^\bullet$. Since f is Ψ-closed, the maps r_y^* are isomorphisms by IV-4.2. Therefore θ_y is an isomorphism for $y \in F$, and so

$$f_\Psi(\mathscr{A})|F \xrightarrow{\theta|F} f_\Psi(\mathscr{A}_{F^\bullet})|F$$

is an isomorphism. It is obvious that $f_\Psi(\mathscr{A}_{F^\bullet})|V = 0$, and so θ induces an isomorphism $f_\Psi(\mathscr{A})_F \xrightarrow{\approx} f_\Psi(\mathscr{A}_{F^\bullet})$, which is our result for $B = F$. On the other hand, the fact that $\theta|F$ is an isomorphism implies that $f_\Psi(\mathscr{A}_{V^\bullet})|F = 0$ by exactness of the previous sequence. Consequently, φ induces a homomorphism $f_\Psi(\mathscr{A}_{V^\bullet}) \to f_\Psi(\mathscr{A})_V$, which is an isomorphism since $f_\Psi(\mathscr{A}_{F^\bullet})|V = 0$, and this is our result for $B = V$.

12. The statement about normal Stiefel-Whitney classes follows from the obvious real analogue of IV-10.4. If Y is not the complete fixed-point set, then let F be some other component. By hypothesis, $H^i(M;\mathbb{Z}_2) \to H^i(F;\mathbb{Z}_2)$ is vacuously a monomorphism for $i < m-k+1$. Therefore, by IV-10.5 we have that $k = \dim Y \le m-(m-k+1) = k-1$, a contradiction.

13. Since $\mathscr{H}^{-1}(\mathscr{A}^*) = 0$, we have

$$\mathscr{H}^q(\mathscr{B}^*) \approx \begin{cases} \mathscr{A}^{-1}/\operatorname{Ker} h \approx \operatorname{Im} h, & \text{for } q=-1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the spectral sequence of IV-2.1 has

$$E_2^{p,q} = \begin{cases} H^p_\Phi(X;\operatorname{Im} h), & \text{for } q=-1, \\ 0, & \text{for } q \ne -1. \end{cases}$$

Consequently, there is the induced isomorphism

$$H^p_\Phi(X;\operatorname{Im} h) \approx H^{p-1}(\Gamma_\Phi(\mathscr{B}^*)),$$

which vanishes for $p \ge 1$ since $\mathscr{B}^q = 0$ for $q \ge 0$. Therefore $\operatorname{Im} h$ is Φ-acyclic. The exact cohomology sequence induced by the exact coefficient sequence $0 \to \operatorname{Im} h \to \mathscr{A}^0 \to \operatorname{Coker} h \to 0$ then shows that $\operatorname{Coker} h$ is Φ-acyclic since $\mathscr{A}^0$ and $\operatorname{Im} h$ are Φ-acyclic.

20. By IV-14.4, $H^*(X/G, P/G;\mathbb{Z}_p) = 0$. By the proof of IV-14.3, the Leray spectral sequence of $(f, f|P)$, where $f : X \to X/G$ is the orbit map, has $E_2^{s,t} = H^s(X/G, P/G; H^t(G;\mathbb{Z}_p)) = 0$ converging to $H^{s+t}(X,P;\mathbb{Z}_p)$. Thus the latter group is zero, and the result follows from the exact sequence of the pair (X,P). (This result, due to Oliver [65], is not true without the condition that the group be connected.)

22. Mapping X to a point $\star$ induces, by the naturality of θ, the commutative diagram

$$\begin{array}{ccc} H^0(\star;M) & \xrightarrow{\approx} & {}_\Delta H^0(\star;M) \\ \downarrow{\approx} & & \downarrow{\approx} \\ H^0(X;M) & \xrightarrow{\theta} & {}_\Delta H^0(X;M), \end{array}$$

so that θ is an isomorphism as claimed.

23. The map φ is induced by the augmentation $\varepsilon : M \to {}_\Delta\mathscr{H}^0(X;M)$, so that it suffices to prove that ε is an isomorphism when X is locally arcwise connected. But ${}_\Delta\mathscr{H}^0(X;M) = \mathscr{Sheaf}(U \mapsto {}_\Delta H^0(U;M))$, and ε is induced by the classical augmentation $M \rightarrowtail {}_\Delta H^0(U;M)$. Since U can be restricted to *arcwise connected* open sets, the contention follows.

24. It is clear that ${}_\Delta\mathscr{H}^q(X;M) = 0$ for $q \ne 0$, so that the spectral sequence degenerates to the isomorphism

$${}_\Delta H^p(X;M) \approx H^p(X;{}_\Delta\mathscr{H}^0(X;M)),$$

and it is elementary that the group on the left is zero for $p \ne 0$. Let $\mathscr{H}$ and $\tilde{\mathscr{H}}$ denote ${}_\Delta\mathscr{H}^0(X;M)$and ${}_\Delta\tilde{\mathscr{H}}^0(X;M)$ respectively. Note that $\tilde{\mathscr{H}}$ is

concentrated on the closed line segment where X is not locally arcwise connected. There is an exact sequence

$$0 \to M \to \mathscr{H} \to \tilde{\mathscr{H}} \to 0,$$

which induces the exact sequence

$$0 \to H^0(X;M) \xrightarrow{\theta} H^0(X;\mathscr{H}) \to H^0(X;\tilde{\mathscr{H}}) \to H^1(X;M) \to H^1(X;\mathscr{H}),$$

and $H^1(X;\mathscr{H}) \approx {}_\Delta H^1(X;M) = 0$ and similarly, $\Gamma(\mathscr{H}) = H^0(X;\mathscr{H}) \approx M$. Thus this sequence has the form

$$0 \to M \xrightarrow{\theta} M \to \Gamma(\tilde{\mathscr{H}}) \to M \to 0.$$

The map θ here is an isomorphism by IV-Exercise 22. Consequently we have that $\Gamma({}_\Delta\tilde{\mathscr{H}}^0(X;M)) \approx M$. The reader might attempt to see this directly by studying the sheaf $\tilde{\mathscr{H}}$.

25. Since X is locally arcwise connected, the sheaf ${}_\Delta\mathscr{H}^0(X;M) \approx M$ via the augmentation. It is also clear that the sheaves ${}_\Delta\mathscr{H}^q(X;M)$ are concentrated at the origin x_0 for $q > 0$. [For $q > 1$ these are probably zero, but we know no proof of that. The analogue in higher dimensions is false.] Let $\mathfrak{B} = {}_\Delta\mathscr{H}^1(X;M)_{x_0}$, which is expected to be quite large. [At least for $M = \mathbb{Q}$ this is uncountable.] Then we have $E_2^{0,0} \approx M$, $E_2^{1,0} = H^1(X;M) \approx \bigoplus_{i=1}^\infty M$ by continuity II-14.6, $E_2^{0,q} = \Gamma({}_\Delta\mathscr{H}^q(X;M)) \approx {}_\Delta\mathscr{H}^q(X;M)_{x_0}$, and all others are zero. Consequently, there is an exact sequence

$$0 \to \bigoplus_{i=1}^{\infty} M \to {}_\Delta H^1(X;M) \to \mathfrak{B} \to 0$$

and isomorphisms

$${}_\Delta H^q(X;M) \approx {}_\Delta\mathscr{H}^q(X;M)_{x_0},$$

probably all zero, for $q > 1$. Note that the term $\bigoplus_{i=1}^\infty M$ represents those classes in ${}_\Delta H^1(X;M)$ that are locally zero.

27. This follows immediately from IV-2.11 applied to the open subspaces $A_n = U_n \times \mathbb{N}$, and the fact, from II-Exercise 59, that $\varprojlim^2_{i,j} A_{i,j} = 0$, once we show that $H^q(U_n \times \mathbb{N};\mathscr{A}) \approx \varprojlim^q_j A_{n,j}$. To see this, consider the projection $f : U_n \times \mathbb{N} \to \mathbb{N}$. This is closed and each $f^{-1}(m)$ is taut and $\mathscr{A}|f^{-1}(m)$ is acyclic by II-Exercise 58. Also, $f\mathscr{A}(U_k) = \mathscr{A}(f^{-1}U_k) = \mathscr{A}(U_{n,k}) = A_{n,k}$, so that $f\mathscr{A}$ is the inverse sequence $\{A_{n,1} \leftarrow A_{n,2} \leftarrow \cdots\}$. Therefore $H^q(U_n \times \mathbb{N};\mathscr{A}) \approx H^q(\mathbb{N}; f\mathscr{A}) = \varprojlim^q_j A_{n,j}$ by II-11.1, as claimed. The spectral sequence can also be derived as the Leray spectral sequence of the projection $\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

29. Let $\mathscr{A}^* = \mathscr{C}^*(X;\mathscr{A})$. By definition, $1^\diamond$ is the composition of the edge homomorphism in the first spectral sequence of the double complex $L^{p,q} = C^p_\Psi(X;\mathscr{A}^q)$ with the inverse of that for the second spectral sequence. These edge homomorphisms are induced by the canonical monomorphisms

$$C^n_\Psi(X;\mathscr{A}) \rightarrowtail \bigoplus_{p+q=n} C^p_\Psi(X;\mathscr{A}^q) \leftarrowtail \Gamma_\Psi(\mathscr{A}^n).$$

These are all exact functors of $\mathscr{A}$. Consequently, the derived maps

$$H^n_\Psi(X;\mathscr{A}) \to H^n(L^*) \xleftarrow{\approx} H^n(\Gamma_\Psi(\mathscr{A}^*)) = H^n_\Psi(X;\mathscr{A}),$$

which are the edge homomorphisms, are maps of fundamental connected sequences of functors of $\mathscr{A}$. In degree zero this composition is the identity, as is seen by chasing the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Gamma_\Psi(\mathscr{A}) & \to & \Gamma_\Psi(\mathscr{A}^0) & \to & \Gamma_\Psi(\mathscr{A}^1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C^0_\Psi(X;\mathscr{A}) & \to & C^0_\Psi(X;\mathscr{A}^0) & \to & C^0_\Psi(X;\mathscr{A}^1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C^1_\Psi(X;\mathscr{A}) & \to & C^1_\Psi(X;\mathscr{A}^0) & \to & C^1_\Psi(X;\mathscr{A}^1).
\end{array}$$

Thus the contention follows from II-6.2.

30. Let $L^{p,q} = \check{C}^p(\mathfrak{U}; C^q(\bullet;\mathscr{A})) = \check{C}^p(\mathfrak{U}; \mathscr{C}^q(X;\mathscr{A}))$. In the second spectral sequence of this double complex we have

$$''E_1^{p,q} = \check{H}^p(\mathfrak{U}; \mathscr{C}^q(X;\mathscr{A})) = \begin{cases} \Gamma(\mathscr{C}^q(X;\mathscr{A})), & \text{for } p = 0, \\ 0, & \text{for } p \neq 0 \end{cases}$$

by III-4.10. Therefore this spectral sequence degenerates to the isomorphism

$$H^n(L^*) \approx H^n(X;\mathscr{A}).$$

In the first spectral sequence $E_r^{p,q}(\mathfrak{U}) = {}'E_r^{p,q}$ we have

$$E_1^{p,q}(\mathfrak{U}) = \check{C}^p(\mathfrak{U}; H^q(\bullet;\mathscr{A}))$$

by III-4.1, whence

$$E_2^{p,q}(\mathfrak{U}) = \check{H}^p(\mathfrak{U}; H^q(\bullet;\mathscr{A})) \Longrightarrow H^n(X;\mathscr{A}).$$

In the situation of III-Exercise 15, $E_2^{p,q}(\mathfrak{U}) = \check{H}^p(\mathfrak{U}; H^q(\bullet;\mathscr{A})) = 0$ for $p < m$ and $0 \neq q < n$, which implies that the edge homomorphism

$$\check{H}^k(\mathfrak{U};\mathscr{A}) = E_2^{k,0} \twoheadrightarrow E_\infty^{k,0} \rightarrowtail H^k(X;\mathscr{A})$$

is isomorphic for $k < \min(n,m)$ and monomorphic for $k = \min(n,m)$.

31. Put $K_0 = Y$, $K_1 = B$, and $K_2 = K_3 = F$. Then, for the Fary spectral sequence, we have $A_0 = Y - B$, $A_1 = B - F$, $A_2 = \varnothing$, and $A_3 = F$. Thus

$$E_2^{p,q} = H_c^p(Y - B; \mathscr{H}^q(f)) \oplus H_c^{p+1}(B - F; \mathscr{H}^{q-1}(f)) \oplus H^{p+3}(F; \mathscr{H}^{q-3}(f)),$$

where the coefficient sheaves are constant and either 0 or $\mathbb{Z}$. The nonzero cases are

$$\begin{aligned}
E_2^{p,3} &= H_c^p(Y - B) \oplus H_c^{p+1}(B - F) \oplus H^{p+3}(F), \\
E_2^{p,1} &= H_c^{p+1}(B - F), \\
E_2^{p,0} &= H_c^p(Y - B).
\end{aligned}$$

Also, $d_2 = {}^t d_2 + {}^t\Delta$, where ${}^t d_2 = 0$ since $E_2^{p,1}$ and $E_2^{p,0}$ do not involve a common A_t. Thus $d_2 : E_2^{p-1,1} \to E_2^{p+1,0}$ is the connecting homomorphism ${}^t\Delta = \delta$ in the exact sequence

$$H_c^p(Y-F) \to H_c^p(B-F) \xrightarrow{\delta} H_c^{p+1}(Y-B) \to H_c^p(Y-F).$$

But $H_c^*(Y-F) = H^*(Y,F) = 0$ since Y and F are both acyclic. Consequently, d_2 is an isomorphism, killing everything except the $E_2^{p,3}$ terms. Therefore the spectral sequence degenerates from $E_3^{p,q}$ on, giving

$$H^p(X) \approx E_3^{p-3,3} \approx E_2^{p-3,3} \approx H_c^{p-3}(Y-B) \oplus H_c^{p-2}(B-F) \oplus H^p(F).$$

Now, $H_c^{p-3}(Y-B) \approx H^{p-3}(Y,B) \approx \tilde{H}^{p-4}(B)$ and $H_c^{p-2}(B-F) \approx H^{p-2}(B,F) \approx \tilde{H}^{p-2}(B)$, yielding the claimed isomorphism.

32. By IV-2.5, η^k is an isomorphism for $k < n$ and a monomorphism for $k = n$. By the solution of IV-Exercise 5, the edge homomorphism ξ : $H^n(\Gamma_\Phi(\mathscr{L}^*)) \twoheadrightarrow E_\infty^{0,n} \rightarrowtail E_2^{0,n} = \Gamma_\Phi(\mathscr{H}^n(\mathscr{L}^*))$ is just the canonical map $\theta : H^n(\Gamma_\Phi(\mathscr{L}^*)) \to \Gamma_\Phi(\mathscr{H}^n(\mathscr{L}^*))$. The assumption that $H^n(\Gamma_\Phi(\mathscr{L}^*)) \to H^n(\Gamma_{\Phi\cap U}(\mathscr{L}^*|U))$ is zero for small U implies that $\theta = 0$. This implies that $E_\infty^{0,n} = 0$, whence $E_\infty^{n,0} \rightarrowtail H^n(\Gamma_\Phi(\mathscr{L}^*))$ is onto, so that η^n is onto.

    For the application to singular cohomology take, for example, $\mathscr{L}^* = \mathscr{S}^*(X;M) = \mathscr{Sheaf}(U \mapsto S^*(U;M))$, which is Φ-soft when Φ is paracompactifying. Then the result, together with IV-2.5, says (for X paracompact): If every neighborhood U of $x \in X$ contains a neigborhood V of x with ${}_\Delta\tilde{H}^k(U;M) \to {}_\Delta\tilde{H}^k(V;M)$ zero for all $k < n$, then η^k : $H^k(X;M) \to {}_\Delta H^k(X;M)$ is an isomorphism for $k < n$ and a monomorphism for $k = n$. If in addition, each point has a neighborhood V with ${}_\Delta H^n(X;M) \to {}_\Delta H^n(V;M)$ zero, then η^n is an isomorphism. (Actually, as the proof shows, it is enough that for each $\gamma \in {}_\Delta H^n(X;M)$ and each $x \in X$ there is a $V = V(\gamma,x)$ with $\gamma|V = 0$.)

33. Let ${}'E_r^{p,q} = E_r^{p,q}(\mathscr{L}^*)$ and $E_r^{p,q} = E_r^{p,q}(\mathscr{M}^*)$. By hypothesis, ${}'E_2^{0,N} \rightarrowtail E_2^{0,N}$. But ${}'E_\infty^{0,N} \rightarrowtail {}'E_2^{0,N}$ and $E_\infty^{0,N} \rightarrowtail E_2^{0,N}$, so that ${}'E_\infty^{0,N} \rightarrowtail E_\infty^{0,N}$. The proof of IV-2.2 showed that ${}'E_\infty^{p,N-p} \xrightarrow{\approx} E_\infty^{p,N-p}$ for $p > 0$. Thus, for the filtrations of the total terms, we have $H^N(\Gamma_\Phi(\mathscr{L}^*)) = {}'F_N \supset {}'F_{N-1} \supset \cdots$, and we deduce that ${}'F_{N-1} \xrightarrow{\approx} F_{N-1}$ in both cases. In the first case, we have that ${}'F_N/{}'F_{N-1} = {}'E_\infty^{0,N} \rightarrowtail E_\infty^{0,N} = F_N/F_{N-1}$, which implies that ${}'F_N \rightarrowtail F_N$ as claimed.

    For the second case, note that the proof of IV-2.2 also showed that ${}'E_r^{r,N-r+1} \xrightarrow{\approx} E_r^{r,N-r+1}$ since

$$k_r(r) = N - r + \#(S_r \cap [0,r]) = N - r + 2 > N - r + 1.$$

Hence, if we have proved that ${}'E_r^{0,N} \twoheadrightarrow E_r^{0,N}$, then the diagram

$$\begin{array}{ccc} {}'E_r^{0,N} & \twoheadrightarrow & E_r^{0,N} \\ \big\downarrow {}'d_N^{0,N} & & \big\downarrow d_N^{0,N} \\ {}'E_r^{r,N-r+1} & \xrightarrow{\approx} & E_r^{r,N-r+1} \end{array}$$

shows that ${}'E_{r+1}^{0,N} = \operatorname{Ker} {}'d_N^{0,N} \twoheadrightarrow \operatorname{Ker} d_N^{0,N} = E_{r+1}^{0,N}$. Thus ${}'F_N/{}'F_{N-1} = {}'E_\infty^{0,N} \twoheadrightarrow E_\infty^{0,N} = F_N/F_{N-1}$, and we conclude that ${}'F_N \twoheadrightarrow F_N$, as claimed.

Solutions for Chapter V:

3. The solution is given in VI-1.4.
4. The solution is given in VI-12.1.
5. The solution is given in VI-3.2 and VI-3.3.
6. The Mayer-Vietoris sequence (coefficients in $\mathscr{A}$)

$$H_c^n(U\cap V)\to H_c^n(U)\oplus H_c^n(V)\to H_c^n(U\cap V)\to H_c^{n+1}(U\cap V)=0$$

shows that $\mathfrak{H}_c^n(X;\mathscr{A})$ satisfies condition (a) of V-Exercise 3, and continuity II-14.5 applied to the direct system $\{\mathscr{A}_{U_\alpha}\}$ shows that it satisfies condition (b).

7. Such a map h induces a homomorphism $h':\mathscr{A}\to f^*\mathscr{B}$ (and conversely). Thus h_* can be defined as the composition

$$H_p^{\Psi(c)}(X;\mathscr{A})\xrightarrow{h'_*}H_p^{\Psi(c)}(X;f^*\mathscr{B})\xrightarrow{f_*}H_p^{\Psi}(Y;\mathscr{B})$$

of h'_* with the homomorphism f_* of (26) on page 301.

Let $g:Y\to Z$ and let $k:\mathscr{B}\to\mathscr{C}$ be a g-homomorphism. Let Φ be a family of supports on Z, and put $\Psi=\Phi(c)$ and $\Theta=\Psi(c)$. Then we have the diagram

$$\begin{array}{ccccc} H_p^{\Theta}(X;\mathscr{A}) & \xrightarrow{h'_*} & H_p^{\Theta}(X;f^*\mathscr{B}) & \xrightarrow{k''_*} & H_p^{\Theta}(X;f^*g^*\mathscr{C}) \\ & {\scriptstyle h_*}\searrow & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ & & H_p^{\Psi}(Y;\mathscr{B}) & \xrightarrow{k'_*} & H_p^{\Psi}(Y;g^*\mathscr{C}) \\ & & & {\scriptstyle k_*}\searrow & \downarrow{\scriptstyle g_*} \\ & & & & H_p^{\Phi}(Z;\mathscr{C}) \end{array}$$

in which the square commutes by the naturality of f_* and the triangles commute by the definitions of h_* and k_*. This shows that

$$k_*h_*=g_*f_*k''_*h'_*=(gf)_*(kh)'_*=(kh)_*,$$

the last equality being by the definition of $(kh)_*$.

8. The first sequence is induced by the exact coefficient sequence $0\to\mathscr{A}_V\to\mathscr{A}_{U_1}\oplus\mathscr{A}_{U_2}\to\mathscr{A}_U\to 0$ via (34) on page 306 and (8) on page 292. The second is similarly induced by the coefficient sequence $0\to\mathscr{A}_G\to\mathscr{A}_{F_1}\oplus\mathscr{A}_{F_2}\to\mathscr{A}_F\to 0$, where $G=X-V$, $F_i=X-U_i$, and $F=X-U$, and using (35) on page 306.

9. Let $\mathscr{L}^*=\mathscr{I}^*(X;L)$, and plug the exact sequence $0\to\mathscr{L}_V^*\to\mathscr{L}_{U_1}^*\oplus\mathscr{L}_{U_2}^*\to\mathscr{L}^*\to 0$ into the functor $\Gamma_\Phi(\mathscr{D}(\bullet)\otimes\mathscr{M})$, which is exact on replete sheaves by V-3.10. Then use that $\Gamma_\Phi(\mathscr{D}(\mathscr{L}_U^*)\otimes\mathscr{M})\approx\Gamma_{\Phi\cap U}(\mathscr{D}(\mathscr{L}^*|U)\otimes\mathscr{M}|U)\approx C_*^{\Phi\cap U}(U;\mathscr{M})$ by V-5.4. This gives the first sequence. The second follows similarly from the sequence $0\to\mathscr{L}^*\to\mathscr{L}_{F_1}^*\oplus\mathscr{L}_{F_2}^*\to\mathscr{L}_F^*\to 0$, V-5.4 [showing that $\Gamma_\Phi(\mathscr{D}(\mathscr{L}_F^*)\otimes\mathscr{M})\approx\Gamma_\Phi((\mathscr{D}(\mathscr{L}^*|F)\otimes\mathscr{M}|F)^X)\approx\Gamma_{\Phi|F}(\mathscr{D}(\mathscr{L}^*|F)\otimes\mathscr{M}|F)$], V-3.3, and V-5.6.

11. This is an immediate consequence of V-5.6 when $(\mathscr{A},\Phi)$ is elementary, and of V-5.3 when Φ is paracompactifying.

17. (a) $\Phi \subset \Psi \Rightarrow \Psi^{\#} = \{K \mid L \in \Psi \Rightarrow K \cap L \in c\} \subset \{K \mid L \in \Phi \Rightarrow K \cap L \in c\} = \Phi^{\#}$.

(b) $\Phi^{\#\#} = \{K \mid L \in \Phi^{\#} \Rightarrow K \cap L \in c\} \supset \Phi$ by the definition of $\Phi^{\#}$. Hence also $(\Phi^{\#})^{\#\#} \supset \Phi^{\#}$, and $(\Phi^{\#\#})^{\#} \subset \Phi^{\#}$ by (a) applied to (b).

(c) Recall that $\Phi^{\#}(c) = \{K \in c(Y) \mid \overline{f(K)} \in \Phi^{\#}\} = \{K \in c(Y) \mid f(K) \in \Phi^{\#}\}$ since $K \in c(Y)$ implies that $f|K$ is closed and proper. We have

$$\begin{aligned}(f^{-1}\Phi)^{\#} &= \{K \subset X \mid K' \in f^{-1}(\Phi) \Rightarrow K' \cap K \in c\} \\ &= \{K \subset X \mid L \in \Phi \Rightarrow f^{-1}(L) \cap K \in c\}.\end{aligned}$$

Since $\Phi \supset c$, such sets K are in $c(Y)$. Also note that $f(f^{-1}(L) \cap K) = L \cap f(K)$, and so $f^{-1}(L) \cap K$ is compact $\Leftrightarrow L \cap f(K)$ is compact, for $K \in c(Y)$, since $f|K$ is proper. Thus

$$\begin{aligned}(f^{-1}\Phi)^{\#} &= \{K \in c(Y) \mid L \in \Phi \Rightarrow f^{-1}(L) \cap K \in c\} \\ &= \{K \in c(Y) \mid L \in \Phi \Rightarrow L \cap f(K) \in c\} \\ &= \{K \in c(Y) \mid f(K) \in \Phi^{\#}\} \\ &= \Phi^{\#}(c).\end{aligned}$$

19. We omit the details concerning the sheaf $\mathscr{H}_2(X)$, which is easy to understand and is isomorphic to $\mathbb{Z}_{\mathbb{S}^2}$. Consequently, $E_2^{p,-2} = H^p(X; \mathbb{Z}_{\mathbb{S}^2}) \approx H^p(\mathbb{S}^2; \mathbb{Z})$. The sheaf $\mathscr{H}_1(X)$ is more difficult. It is clear that it vanishes on $\mathbb{S}^2 - \{x\}$, where x is the common point of the two spheres. On $\mathbb{S}^1 - \{x\}$ it is constant with stalks $\mathbb{Z}$. At the point x we have $\mathscr{H}_1(X)_x \approx H_1(N, \partial N) \approx \tilde{H}_0(\partial N) \approx \tilde{H}_0(\mathbb{S}^1 + \mathbb{S}^0) \approx \mathbb{Z} \oplus \mathbb{Z}$, where N is a neighborhood of x having the structure of a 2-disk punctured at x with an arc. To understand the way the stalks fit together around x, it is useful to think in terms of the singular homology of $(N, N - \{x\})$ and note that this has a basis consisting of the class s of the singular cycle given by a singular 1-simplex running from the $\mathbb{S}^1$ portion of $N - \{x\}$ through x and to the $\mathbb{S}^2$ portion. Similarly let t be the class of such a 1-simplex running from the $\mathbb{S}^2$ portion through x and to the $\mathbb{S}^1$ portion on the "opposite" side of x from that side along which s runs. Then s induces a local section of $\mathscr{H}_1(X)$ near x that gives a generator at points in $\mathbb{S}^1$ to one side of x and 0 at points on the other side. The section given by t is 0 on the first of these sides and a generator on the other. Neither of these local sections extends to a global section, but $s + t$ does come from a global section (since we arranged for s and t to travel along $\mathbb{S}^1$ in the same direction). There is only one topology consistent with these facts. Let $\mathscr{L} = \mathscr{H}_1(X)|\mathbb{S}^1$. Since $\mathscr{H}_1(X)$ vanishes outside $\mathbb{S}^1$, we have $H^p(X; \mathscr{H}_1(X)) \approx H^p(\mathbb{S}^1; \mathscr{L})$. To compute this, consider $U = \mathbb{S}^1 - \{y\}$, where $y \neq x$, and let A and B be the two rays from x making up U. (Then $A \cap B = \{x\}$.) Note that $\mathscr{L}|U \approx \mathbb{Z}_A \oplus \mathbb{Z}_B$. (We remark that $\mathscr{L}$ could be described as $\mathbb{Z}_A \oplus \mathbb{Z}_B$ patched at the missing point y in an essentially unique manner.) Now, $H_c^p(U; \mathbb{Z}_A) \approx H_c^p(A; \mathbb{Z}) \approx H^p([0,1], \{0\}; \mathbb{Z}) = 0$. Consequently, $H_c^p(U; \mathscr{L}|U) = 0$. The exact sequence of the pair $(\mathbb{S}^1, \{y\})$ with coefficients in $\mathscr{L}$ then shows that $H^p(\mathbb{S}^1; \mathscr{L}) \approx \mathbb{Z}$ for $p = 0$ and is zero for $p \neq 0$. Therefore we have $E_2^{0,-1} = H^0(X; \mathscr{H}_1(X)) \approx H^0(\mathbb{S}^1; \mathscr{L}) \approx \mathbb{Z}$, $E_2^{0,-2} = H^0(X; \mathscr{H}_2(X)) \approx H^0(\mathbb{S}^2; \mathbb{Z}) \approx \mathbb{Z}$, $E_2^{2,-2} = H^2(X; \mathscr{H}_2(X)) \approx H^2(\mathbb{S}^2; \mathbb{Z}) \approx \mathbb{Z}$, and all other terms are zero.

21. We have

$$\begin{aligned}
\varinjlim H_p^c(X;\mathscr{A}_\lambda) &= \varinjlim H_p(\Gamma_c(\mathscr{C}_*(X;L)\otimes\mathscr{A}_\lambda)) && \text{by definition}\\
&\approx H_p(\varinjlim \Gamma_c(\mathscr{C}_*(X;L)\otimes\mathscr{A}_\lambda)) && \text{since } \varinjlim \text{ is exact}\\
&\approx H_p(\Gamma_c(\varinjlim (\mathscr{C}_*(X;L)\otimes\mathscr{A}_\lambda))) && \text{by II-14.5}\\
&\approx H_p(\Gamma_c(\mathscr{C}_*(X;L)\otimes \varinjlim \mathscr{A}_\lambda)) && \text{by (13) on page 20}\\
&= H_p^c(X;\mathscr{A}) && \text{by definition.}
\end{aligned}$$

23. If $\pi : X \to X/G$, then there are the maps

$$H_p(U^\bullet;\mathbb{Q}) \underset{\mu_*}{\overset{\pi_*}{\rightleftarrows}} H_p(U;\mathbb{Q})$$

for $U \subset X/G$ open. These induce sheaf homomorphisms

$$\pi\mathscr{H}_p(X;\mathbb{Q}) \underset{\check{\mu}_*}{\overset{\check{\pi}_*}{\rightleftarrows}} \mathscr{H}_p(X/G;\mathbb{Q})$$

with $\check{\pi}_*\check{\mu}_* = \operatorname{ord} G$ and $\check{\mu}_*\check{\pi}_* = \sum_{g\in G} g_*$. Since

$$(\pi\mathscr{H}_p(X;\mathbb{Q}))_y \approx \bigoplus_{\pi(x)=y} \mathscr{H}_p(X;\mathbb{Q})_x = 0,$$

for $p \neq n$, we have that $\check{\pi}_*\check{\mu}_*$ is an isomorphism factoring through 0, whence $\mathscr{H}_p(X/G;\mathbb{Q}) = 0$ for $p \neq n$. Also, $\check{\mu}_* : \mathscr{H}_n(X/G;\mathbb{Q})_y \to (\pi\mathscr{H}_p(X;\mathbb{Q}))_y$ is a monomorphism onto the group $(\pi\mathscr{H}_n(X;\mathbb{Q}))_y^G$ of invariant elements, which is the diagonal in $\bigoplus_{\pi(x)=y}\mathbb{Q}$, since G preserves orientation. It follows that $\mathscr{H}_n(X/G;\mathbb{Q})_y \approx \mathbb{Q}$, so that X/G is an n-$hm_{\mathbb{Q}}$ by V-16.8(d), since X/G is locally connected. Also, $\Gamma(\mathscr{H}_n(X/G;\mathbb{Q}) \approx H_n(X/G;\mathbb{Q}) \neq 0$ since $\check{\mu}_*\check{\pi}_* = \operatorname{ord} G \neq 0$, so that X/G is orientable. (Note that the proof applies to any field L of characteristic prime to $\operatorname{ord} G$, in place of $\mathbb{Q}$. Also note that X/G is clc_L^∞ if X is clc_L^∞ by II-19.3.)

26. By II-Exercise 33, $H_c^1(X;L)$ is free, whence $\operatorname{Ext}(H_c^1(X;L),L) = 0$. By (9) on page 292, $H_0(X;L) \approx \operatorname{Hom}(H_c^0(X;L),L) \approx \operatorname{Hom}(\Gamma_c(L),L)$. Since $\Gamma_c(L)$ is the direct sum of copies of L over the compact components of X, this is the direct product of copies of L over the compact components of X, as claimed. For an n-cm_L this is isomorphic to $H^n(X;\mathscr{O})$ by V-9.2. For X orientable it follows similarly from V-12.8 and duality that $H^n(X;M) = \prod_\alpha M_\alpha$ for any L-module M. The consequence that $H^n(U;M) = 0$ for U open in $\mathbb{R}^n$, where this can be regarded as ordinary singular cohomology, seems surprisingly difficult to prove without advanced tools. I know of no way to prove it that is suitable for a first course in algebraic topology.

28. First restrict attention to the case in which X is orientable and has no compact components. By V-Exercise 26 and duality, $H^p(U;L) = 0$ for $p \geq n$. The exact sequence

$$H^{p-1}(U;L) \to H^p(X,U;L) \to H^p(X;L)$$

shows that $H^p(X,U;L) = 0$ for all $p > n$. Since $H^1(X,U;\mathscr{Z}^n(X;L)) \approx H^{n+1}(X,U;L) = 0$ (see II-12.1 and the proof of II-16.2), $\mathscr{Z}^n(X;L)$ is flabby by II-Exercise 21. Since flabbiness is a local property by II-Exercise 10, it follows that $\mathscr{Z}^n(X;L)$ is flabby in the *general case* of an arbitrary separable metric n-cm X. This implies, by the reverse of the reasoning above, that $H^{n+1}(X,U;L) = 0$ in the general case.

32. Under the hypotheses, the spectral sequence of V-8.4 has

$$E_2^{p,q} = H_\Phi^p(X; \mathscr{H}_{-q}(X; \mathscr{A})) \Longrightarrow H^\Phi_{-p-q}(X; \mathscr{A}).$$

By (9) on page 292, $\mathscr{H}_k(X; L) = 0$ for $k > n$ and for $k < -1$. Also, $\mathscr{H}_k(X; \mathscr{A}) = 0$ for $k > n$ by (14) on page 294.

Now $E_2^{p,q} = 0$ for $p+q < -n$, whence $H_k^\Phi(X; \mathscr{A}) = 0$ for $k > n$. For $p+q = -n$, the only nonzero term in $\{E_2^{p,q}\}$ is $E_2^{0,-n} = H_\Phi^0(X; \mathscr{H}_n(X; \mathscr{A})) = \Gamma_\Phi(\mathscr{H}_n(X; \mathscr{A}))$, whence this is isomorphic to the "total" group $H_n^\Phi(X; \mathscr{A})$. If $\mathscr{A}$ is torsion-free, then for $k \in \mathbb{Z}$ there is an exact sequence $0 \to \mathscr{A} \xrightarrow{k} \mathscr{A} \to \mathscr{B} \to 0$. The induced exact sequence

$$0 = H_{n+1}^\Phi(X; \mathscr{B}) \to H_n^\Phi(X; \mathscr{A}) \xrightarrow{k} H_n^\Phi(X; \mathscr{A})$$

shows that $H_n^\Phi(X; \mathscr{A})$ is torsion-free.

33. Assume that $\mathscr{H}_n(X; L) = 0$ for all n. Since the union of an increasing sequence of compact subsets of X is paracompact, there exists an open paracompact subspace of X. Thus we may as well assume that X is paracompact. Let $\Phi = cld$ on X. Let $x \in X$ and construct a sequence of open neighborhoods U_i of x in X with $\overline{U}_{i+1} \subset U_i$ for all i and with $\overline{U}_1$ compact. Let $A = \bigcap U_i = \bigcap \overline{U}_i$, which is compact and nonempty. Then $X - A = \bigcup X - U_i$ is paracompact, and so $\dim_{\Phi|A,L} X \le 1 + \dim_L X < \infty$ by II-16.10. Now the spectral sequence

$$E_2^{p,q} = H^p(X, X - A; \mathscr{H}_{-q}(X; L)) \Longrightarrow H_{-p-q}(A; L)$$

of V-8.5 has $E_2^{p,q} = 0$, so that we have that $H_0(A; L) = 0$ in particular. Hence $\operatorname{Hom}(H^0(A; L), L) = 0$ by (9) on page 292. But $H^0(A; L)$ has L as a direct summand since $A \ne \varnothing$, a contradiction.

Here is another proof in the case $L = \mathbb{Z}$ (the argument also applies to L being a field). If $\dim_L X < \infty$ and $\mathscr{H}_*(X; L) = 0$, then the basic spectral sequence of 8.1 implies that $H_*(U; L) = 0$ for all open $U \subset X$. This implies, by (11) on page 293, that $\operatorname{Ext}(H_c^*(U), L) = 0 = \operatorname{Hom}(H_c^*(U), L)$. This implies, by V-14.7, that $H_c^*(U; L) = 0$. By II-16.14 $\dim_L X = 0$, and by II-16.21, X is totally disconnected. Therefore, x has a *compact* open neighborhood K. But then $H^0(K; L) = H_c^0(K; L) = 0$ implies that $K = \varnothing$, a contradiction.

Note that this, together with V-Exercise 32 shows that the assumption that $\dim_{\Phi,L} Y < \infty$ in V-8.1 cannot be usefully weakened to the hypothesis that $\mathscr{H}_p^\Psi(f, f|A; \mathscr{A}) = 0$ for $p \le n < \infty$.

35. By tautness and the cohomology sequence of $(X, X - A)$, or directly from II-12.1, we see that $H_\Phi^p(X, X - A; \mathscr{M}) \approx \varinjlim H_\Phi^p(X, X - K; \mathscr{M})$ where K ranges over $\Phi|A$. This is isomorphic to $\varinjlim H_{n-p}(K; \mathscr{M} \otimes \mathscr{O}^{-1})$ by V-9.3 since $K \in \Phi$. By V-Exercise 32 this vanishes for $n - p > \dim_L K$ and hence for $n - p > \underline{\dim}_{\Phi,L}^X A$ by the definition of the latter.

37. Consider the f-cohomomorphism

$$\mathscr{C}_*(Y) \otimes f\mathscr{A} \rightsquigarrow f^*\mathscr{C}_*(Y) \otimes \mathscr{A} \approx \mathscr{C}_*(X) \otimes \mathscr{A},$$

which corresponds to a homomorphism $\mathscr{C}_*(Y)\otimes f\mathscr{A}\to f(f^*\mathscr{C}_*(Y)\otimes\mathscr{A})$. First, we claim that this induces a homology isomorphism

$$\mathscr{H}_*(\mathscr{C}_*(Y)\otimes f\mathscr{A})\to\mathscr{H}_*(f(f^*\mathscr{C}_*(Y)\otimes\mathscr{A}))\approx f\mathscr{H}_*(f^*\mathscr{C}_*(Y)\otimes\mathscr{A})$$

since the direct image functor $\mathscr{B}\mapsto f\mathscr{B}$ is exact here by IV-7.5. It suffices to prove this on the stalks at $y\in Y$. Let $K_p=\mathscr{C}_p(Y)_y$ and let A_α be the stalk at $x_\alpha\in f^{-1}(y)$ of $\mathscr{A}$. Then the map on the stalks at y is

$$H_p(K_*\otimes\prod_\alpha A_\alpha)\to\prod_\alpha H_p(K_*\otimes A_\alpha).$$

There is the commutative diagram

$$\begin{array}{ccccccccc}
0\to & H_p(K_*)\otimes\prod A_\alpha & \to & H_p(K_*\otimes\prod A_\alpha) & \to & H_{p-1}(K_*)*\prod A_\alpha & \to 0\\
& \downarrow\approx & & \downarrow & & \downarrow\approx & \\
0\to & \prod(H_p(K_*)\otimes A_\alpha) & \to & \prod(H_p(K_*\otimes A_\alpha)) & \to & \prod(H_{p-1}(K_*)*A_\alpha) & \to 0
\end{array}$$

in which the isomorphisms on the ends are due to $H_p(K_*)$ being finitely generated. By the five-lemma the map in the center is also an isomorphism, proving the contention. Since Y is finite-dimensional and Φ is paracompactifying, the induced maps

$$H_p(\Gamma_\Phi(\mathscr{C}_*(Y)\otimes f\mathscr{A}))\to H_p(\Gamma_\Phi(f(\mathscr{C}_*(X)\otimes\mathscr{A})))$$

are isomorphisms by IV-2.2. The term on the left is $H_p^\Phi(Y;f\mathscr{A})$ by definition, and the one on the right is

$$H_p(\Gamma_\Phi(f\mathscr{C}_*(X;\mathscr{A})))\approx H_p(\Gamma_{\Phi(cld)}(\mathscr{C}_*(X;\mathscr{A})))=H_p^{f^{-1}\Phi}(X;\mathscr{A})$$

since $\Phi(cld)=f^{-1}\Phi$ by IV-5.4.

38. Let $H^q(F)$ denote the stalk of $\mathscr{H}^q(f;L)$. For $\Rightarrow$ consider the spectral sequence $\mathscr{E}^2_{p,q}=\mathscr{H}om(H^q(F),\mathscr{H}_p(Y;L))\Longrightarrow\mathscr{H}_{p+q}(f;L)$ of V-18.4. Assuming, as we may, that Y is orientable and that $\mathscr{H}^*(f;L)$ is constant, this degenerates into the isomorphism $\mathscr{H}om(H^q(F),L)\approx\mathscr{H}_{n+q}(f)$, which shows that the latter sheaf is locally constant.

For $\Leftarrow$ let $H_q(F)$ denote the common stalk of $\mathscr{H}_q(f;L)$ and assume that this sheaf is constant. Also assume that the Leray sheaf $\mathscr{H}^p(f;L)$ is constant with stalks $H^p(F)$. Suppose that $H_q(F)=0$ unless $s\le q\le t$ and that it is nonzero on the two ends of this interval. Then in the basic spectral sequence $E_2^{p,q}=H_c^p(U;\mathscr{H}_{-q}(f;L))\Longrightarrow H^c_{-p-q}(U^\bullet)$ the term $E_2^{n,-s}$ survives and shows that $H^c_{s-n}(U^\bullet)\neq 0$. Consequently, $s\ge n$. However, if $s>n$, the same spectral sequence shows that $H_0^c(U^\bullet)=0$, an absurdity. Thus $s=n$.

Now, using the basic spectral sequence V-8.1 in the same way as the Leray spectral sequence was used in the proof of V-18.4, and using V-12.8, we derive the spectral sequence

$$\mathscr{E}^2_{p,q}=\mathscr{H}om(H_q(F),\mathscr{H}_{-p}(Y;L))\Longrightarrow\mathscr{H}^{p+q}(f;L)=H^{p+q}(F).$$

The term $\mathscr{E}^2_{-n,n}=\mathscr{H}om(H_n(F),\mathscr{H}_n(Y;L))$ survives and is isomorphic to the limit term $\mathscr{H}^0(f;L)=H^0(F)$, which is constant. This can happen

only if $\mathscr{H}_n(Y;L)$ is constant. Let its common stalk be denoted by $S \neq 0$. Now let $r = \min\{p \mid \mathscr{H}_p(Y;L) \neq 0\}$ and note that $r \le n = \dim_L Y$. Then $\mathscr{H}_r(Y;L)_y \neq 0$ for some $y \in Y$. The $\mathscr{E}^2_{-r,t}$ term of the spectral sequence survives to give the isomorphism $\mathscr{H}om(H_t(F), \mathscr{H}_r(Y;L)) \approx \mathscr{H}^{t-r}(f;L)$. In particular, $H^{t-r}(F) \approx \mathscr{H}^{t-r}(f;L)_y \approx \mathscr{H}om(H_t(F), \mathscr{H}_r(Y;L)_y) \neq 0$. It follows that $t - r \le k$. Now, in the spectral sequence of V-18.4, the term $\mathscr{E}^2_{n,k} = \mathscr{H}om(H^k(F), \mathscr{H}_n(Y))$ survives and shows that $H_{n+k}(F) \neq 0$, whence $n + k \le t$. From the last two inequalities we get $n + t - r \le n + k \le t$, whence $n \le r$. But we had that $r \le n$, and so $n = r$. Therefore $\mathscr{H}_p(Y;L) = 0$ for $p \neq n$.

Now, both homological spectral sequences degenerate to the isomorphisms $\operatorname{Hom}(H_q(F), S) \approx H^{q-n}(F)$ and $\operatorname{Hom}(H^{q-n}(F), S) \approx H_q(F)$, from which it follows that all these are all finite-dimensional and S has rank one. Thus Y is an n-hm_L, and $\operatorname{rank} H_q(F) = \operatorname{rank} H^{q-n}(F)$.

We remark that X need not be a homology manifold in this situation. For example, f could be the projection to the y-axis Y of $X = Y \cup ([0,1] \times \{0\})$. Another example is the one-point union of a manifold Y with the Hilbert cube $\mathbb{I}^\infty$ with f being the quotient map identifying $\mathbb{I}^\infty$ to a point. This shows that X need not even be finite-dimensional.

39. Condition (a) implies (b) by V-18.6. Thus we need only show that (b) implies (c). Since X is orientable over $\mathbb{Z}_2$, we know that Y is an $(n-k)$-$hm_{\mathbb{Z}_2}$ by V-18.5. Note that $\dim_{\mathbb{Z}} Y \le n - k$ by IV-8.12. Suppose that $y \in Y$ is such that $f^{-1}(y)$ has no orientable neighborhood. Let U be a connected open neighborhood of y and consider the Leray spectral sequence of $f^{-1}(U) \to U$. We have

$$H_c^{n-k}(U;\mathbb{Z}) = E_2^{n-k,k} = E_\infty^{n-k,k} = H_c^n(f^{-1}(U);\mathbb{Z}) \approx \mathbb{Z}_2$$

by V-16.16(g) since $f^{-1}(U)$ is not orientable. Moreover, for $V \subset U$ another connected open neighborhood of y we have that $H_c^n(f^{-1}(V);\mathbb{Z}) \to H_c^n(f^{-1}(U);\mathbb{Z})$ is an isomorphism by V-16.16(d). Applying $\operatorname{Hom}(\bullet, \mathbb{Z}_2)$ to II-15.3 and using that $\operatorname{Hom}(H_c^i(U;\mathbb{Z}_2), \mathbb{Z}_2) \approx H_i(U;\mathbb{Z}_2)$, we get the exact commutative diagram

$$\begin{array}{ccccc} 0 & \to & \operatorname{Hom}(H_c^{n-k}(U;\mathbb{Z}) * \mathbb{Z}_2, \mathbb{Z}_2) & \to & H_{n-k-1}(U;\mathbb{Z}_2) \\ & & \downarrow \approx & & \downarrow \\ 0 & \to & \operatorname{Hom}(H_c^{n-k}(V;\mathbb{Z}) * \mathbb{Z}_2, \mathbb{Z}_2) & \to & H_{n-k-1}(V;\mathbb{Z}_2). \end{array}$$

The groups in the middle are isomorphic to $\mathbb{Z}_2$ and so it follows that $\mathscr{H}_{n-k-1}(Y;\mathbb{Z}_2)_y \neq 0$, contrary to Y being an $(n-k)$-$hm_{\mathbb{Z}_2}$.

41. If $m > n$ and $U \subset X$ is open and paracompact, then

$$\operatorname{Ext}(H_c^{m+1}(U;\mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Hom}(H_c^m(U;\mathbb{Z}), \mathbb{Z}) \approx H_m(U;\mathbb{Z}) \approx H^{n-m}(U;\mathscr{O}) = 0,$$

whence $\operatorname{Ext}(H_c^{m+1}(U;\mathbb{Z}), \mathbb{Z}) = 0 = \operatorname{Hom}(H_c^m(U;\mathbb{Z}), \mathbb{Z})$. By V-14.7 we deduce that $H_c^k(U;\mathbb{Z}) = 0$ for $k > n+1$, and the result follows from II-16.16.

42. This follows immediately from the spectral sequence ${}''E_r^{p,q}$ of V-8.4.

Solutions for Chapter VI:

4. As the proof shows, the diagram (4) on page 433 is valid for $k = n+1$, except that the horizontal maps on the right side are only epimorphisms. The result follows. [Note that this does not imply that $H_{n+1}(\mathfrak{A}_*(X)) \to \check{H}_{n+1}(X; \mathfrak{H}_0(\mathfrak{A}_*))$ is surjective, and indeed, that is not generally true.]

5. Let $\mathfrak{M} = \mathfrak{Cosheaf}(M)$. Then there is the local isomorphism $h : \mathfrak{M} \to M$. Let $\mathfrak{K} = \operatorname{Ker} h$ and $\mathfrak{C} = \operatorname{Coker} h$. Since $\mathfrak{M}$ is a cosheaf and M is an epiprecosheaf, $\mathfrak{C}$ is an epiprecosheaf by VI-Exercise 1. By VI-Exercise 2, $\mathfrak{C} = 0$. Thus we have the natural exact sequence $0 \to \mathfrak{K}(U) \to \mathfrak{M}(U) \xrightarrow{j_U} M \to 0$. In particular, $j_U \neq 0$ whenever $U \neq \varnothing$. Let U be an open neighborhood of the point $x \in X$ and let $V \subset U$ be an open neighborhood of x so small that $\mathfrak{K}(V) \to \mathfrak{K}(U)$ is zero. Suppose that $U = U_0 + U_1$ is a decomposition of U into two disjoint open sets with $x \in U_0$. Let $W \subset U_0$ be an open neighborhood of x so small that $\mathfrak{K}(W) \to \mathfrak{K}(V)$ is zero. Let $V_i = V \cap U_i$. Then the commutative diagram

$$\begin{array}{ccccc}
\mathfrak{K}(U) & \rightarrowtail & \mathfrak{M}(U_0) \oplus \mathfrak{M}(U_1) & \twoheadrightarrow & M \\
\uparrow{\scriptstyle 0} & & \uparrow & & \uparrow{\scriptstyle =} \\
\mathfrak{K}(V) & \rightarrowtail & \mathfrak{M}(V) & \twoheadrightarrow & M \\
\uparrow{\scriptstyle 0} & & \uparrow & & \uparrow{\scriptstyle =} \\
\mathfrak{K}(W) & \rightarrowtail & \mathfrak{M}(W) & \twoheadrightarrow & M
\end{array}$$

shows that the image of $\mathfrak{M}(V) \to \mathfrak{M}(U)$ equals the image of $\mathfrak{M}(W) \to \mathfrak{M}(U)$. But the component of this image in $\mathfrak{M}(U_1)$ is zero since $W \subset U_0$. It follows that $\mathfrak{M}(V_1) \to \mathfrak{M}(U_1)$ is zero. However, this contradicts the fact that $j_{V_1} : \mathfrak{M}(V_1) \to M$ is nonzero unless $V_1 = \varnothing$. Therefore $V \subset U_0$. Since this is independent of the choice of the decomposition $U = U_0 \cup U_1$, it follows that the psuedo-component of U containing x is open. This being true for all x and U implies that X is locally connected.

6. Let $\gamma \in \tilde{H}_0^c(X; L)$. Since $\tilde{H}_0^c(X; L) = \varinjlim \tilde{H}_0(K; L)$, where K ranges over compact subsets of X, there is a compact set K such that γ is in the image from $\tilde{H}_0(K; L)$. Cover K by sets $U_0, \dots, U_n$ such that each $\tilde{H}_0^c(U_i; L) \to \tilde{H}_0^c(X; L)$ is zero. Adding sets to this collection, if necessary, we can assume that $U_0 \cup \cdots \cup U_n$ is connected, since X is connected. We can then reindex this collection such that $(U_0 \cup \cdots \cup U_k) \cap U_{k+1} \neq \varnothing$ for all $0 \le k < n$. Then, with $U = U_0 \cup \cdots \cup U_k$ and $V = U_{k+1}$ we have the exact commutative Mayer-Vietoris diagram (coefficients in L)

$$\begin{array}{ccccccc}
\tilde{H}_0^c(U \cap V) & \to & \tilde{H}_0^c(U) \oplus \tilde{H}_0^c(V) & \to & \tilde{H}_0^c(U \cup V) & \to & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle g} & & \\
\tilde{H}_0^c(X) & \to & \tilde{H}_0^c(X) \oplus \tilde{H}_0^c(X) & \to & \tilde{H}_0^c(X) & \to & 0,
\end{array}$$

by V-Exercise 9. By induction on k, this shows that $g = 0$. Consequently, $\tilde{H}_0^c(U_0 \cup \cdots \cup U_n) \to \tilde{H}_0^c(X)$ is zero. Since by construction, γ is in the image of this map, $\gamma = 0$.

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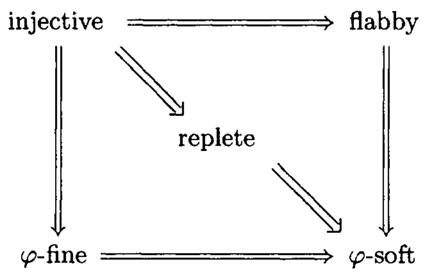
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List of Symbols

List of Selected Facts

The relationships among the various classes of acyclic sheaves are many and varied. Since they are also scattered throughout the text, we have appended this list of many of them for the convenience of the reader. The cross-references provide justifications for the statements. For the full statements, see these references. In statements involving the properties Φ-soft and Φ-fine, we tacitly assume that Φ is paracompactifying.

(a) We have the following implications:



II-5.3, II-9.6, II-9.16, II-Exercise 17, V-3.2.

(b) $\mathscr{I}$ injective $\Rightarrow f\mathscr{I}$ injective; II-3.1.

(c) $\mathscr{A}$ flabby $\Rightarrow f\mathscr{A}$ flabby; II-5.7.

(d) $\mathscr{A}$ flabby, $\mathscr{M}$ locally constant with finitely generated stalks, $\mathscr{A} * \mathscr{M} = 0 \Rightarrow \mathscr{A} \otimes \mathscr{M}$ flabby; II-5.13.

(e) $\mathscr{A}$ flabby $\Rightarrow f_\Psi \mathscr{A}$ Φ-soft; IV-3.3.

(f) $\mathscr{A}_\lambda$ flabby $\Rightarrow \prod_\lambda \mathscr{A}_\lambda$ flabby; II-5.9.

(g) $\mathscr{A}_\lambda$ Φ-soft $\Rightarrow \prod_\lambda \mathscr{A}_\lambda$ Φ-soft; II-Exercise 43.

(h) $\mathscr{A}$ $\Phi(\Psi)$-soft $\Rightarrow f_\Psi \mathscr{A}$ Φ-soft; IV-Exercise 1.

(i) $\mathscr{A}$ Φ-soft $\Rightarrow \mathscr{A}|A$ $(\Phi|A)$-soft; II-9.2.

(j) $\mathscr{A}$ Φ-soft $\Rightarrow \mathscr{A}_A$ Φ-soft; II-9.13.

(k) $\mathscr{B}$ $(\Phi|A)$-soft $\Rightarrow \mathscr{B}^X$ Φ-soft; II-9.12.

(l) $E(\Phi) \supset E(\Psi)$, $\mathscr{A}$ Φ-soft $\Rightarrow \mathscr{A}$ Ψ-soft; II-16.5.

(m) $\mathscr{R}$ a sheaf of rings with unit, $\mathscr{A}$ an $\mathscr{R}$-module, $\mathscr{R}$ Φ-fine $\Leftrightarrow$ $\mathscr{R}$ Φ-soft $\Rightarrow$ $\mathscr{A}$ Φ-fine; II-9.16.

(n) $\mathscr{A}$ a $\mathscr{C}^0(X;L)$-module $\Rightarrow$ $f_\Psi\mathscr{A}$ a $\mathscr{C}^0(Y;L)$-module; IV-3.4.

(o) $\mathscr{I}$ injective $\Rightarrow$ $\mathscr{H}\!om(\mathscr{A},\mathscr{I})$ flabby; II-Exercise 17.

(p) $\mathscr{A}$ Φ-fine $\Rightarrow$ $\mathscr{A}\otimes\mathscr{B}$ Φ-fine; II-9.18.

(q) $\mathscr{A}$ Φ-soft, torsion-free $\Rightarrow$ $\mathscr{A}\otimes\mathscr{B}$ Φ-soft; II-16.31.

(r) $\mathscr{A}_\lambda$ Φ-soft $\Rightarrow$ $\varinjlim \mathscr{A}_\lambda$ Φ-soft; II-16.30.

(s) $\mathscr{A},\mathscr{B}$ c-fine, X locally compact, Hausdorff $\Rightarrow$ $\mathscr{A}\widehat{\otimes}\mathscr{B}$ c-fine; II-Exercise 14.

(t) $\mathscr{A},\mathscr{B}$ c-soft, $\mathscr{A}\widehat{*}\mathscr{B}=0$, X locally compact, Hausdorff $\Rightarrow$ $\mathscr{A}\widehat{\otimes}\mathscr{B}$ c-soft; II-Exercise 14.

(u) $\mathscr{A}_U$ Φ-acyclic, all open U $\Leftrightarrow$ $\mathscr{A}$ Φ-soft $\Leftrightarrow$ $\mathscr{A}|U$ $(\Phi|U)$-acyclic, all U; II-16.1.

(v) $\mathscr{A}$ Φ-acyclic, all Φ $\Leftrightarrow$ $\mathscr{A}$ flabby; II-Exercise 21.

(w) $\mathscr{A}$ flabby, $A\subset X$ Φ-taut $\Rightarrow$ $\mathscr{A}|A$ $(\Phi\cap A)$-acyclic; II-10.5.

(x) $f: X\to Y$ locally c-Vietoris, $\mathscr{A}$ c-fine on Y $\Rightarrow$ $f^*\mathscr{A}$ c-fine; IV-6.7.

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